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The Zariski–Lipman conjecture for log canonical spaces

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Abstract

In this paper, we prove the Zariski–Lipman conjecture for log canonical spaces.

Contents

1. *Introduction*

The Zariski–Lipman conjecture asserts that a complex variety X with a locally free tangent sheaf T_X is necessarily smooth [**13**]. The conjecture has been shown in special cases: for hypersurfaces or homogeneous complete intersections [**8**, **15**], for local complete intersections [**9**], for isolated singularities in higher dimensional varieties [**17**, Section 1.6], and more generally, for varieties whose singular locus has codimension at least 3 (see [**4**]).

The Minimal Model Program was initiated in the early 1980s as an attempt to extend the birational classification of surfaces to higher dimensions. It became clear that singularities are unavoidable in the birational classification of higher dimensional complex projective varieties; this led to the development of a powerful theory of singularities of pairs (see Definition 3.4 for basic notions, such as klt and log canonical singularities). The class of log canonical singularities is the largest class of singularities where the conjectures of the Minimal Model Program are expected to hold.

The Zariski–Lipman conjecture has been shown for klt spaces in [**7**] (see also [**1**, Corollary 5.7]). In this paper, we prove the conjecture for log canonical spaces. Note that log canonical spaces in general have singularities in codimension 2.

Theorem 1.1 (Zariski–Lipman conjecture for log canonical spaces). *Let X be a log canonical space such that the tangent sheaf T_X is locally free. Then X is smooth.*

We remark that the results hold as well for singularities of complex analytic spaces, and algebraic varieties defined over a field of characteristic zero.

After this paper was posted, Graf [**5**] presented a different proof of Theorem 1.1, based on extensions theorems for 1-forms on log canonical spaces, in the spirit of [**7**].

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The paper is organized as follows. In Section 2, we study entire solutions of a particular system of polynomial equations. In Section 3, we review basic definitions of singularities of pairs, and the notion of canonical desingularization. In Section 4, we recall the Camacho–Sad formula, and provide applications to surfaces with trivial logarithmic tangent sheaf (see Proposition 4.8). The proof of Theorem 1.1 occupies Section 5.

Notation and conventions. Throughout this paper, we work over the field of complex numbers. Varieties are always assumed to be irreducible and reduced. We denote by $\mathrm{Sing}(X)$ the singular locus of a variety X. If X is a variety, then we denote by T_X the tangent sheaf of X.

2. *Preliminaries*

Let r be a positive integer, and let $e_1, \ldots, e_r$ be integers. We define the rational function $\Phi_{e_1,\ldots,e_r}$ by the formula

$$\Phi_{e_1,\ldots,e_r}(x) = \cfrac{1}{e_r - \cfrac{1}{e_{r-1} - \cfrac{1}{\ldots - \cfrac{1}{e_2 - \cfrac{1}{e_1 - x}}}}}.$$

The proof of Theorem 5.2 makes use of the following elementary result.

Lemma 2.1. *Let r be a positive integer, and let $e_1, \ldots, e_r$ be negative integers. Then $\Phi_{e_1,\ldots,e_r}(x) = x$ as rational functions if and only if $e_1 = \ldots = e_r = -1$, and $r \equiv 0 \pmod 3$.*

Proof. Let S and U be the rational functions defined by $S(x) = -1/x$, and $U(x) = (x-1)/x$. Recall that the modular group $\mathrm{PSL}_2(\mathbb{Z})$ is the free product of the cyclic group $\langle S \rangle$ of order 2 and the cyclic group $\langle U \rangle$ of order 3. Set $\Psi_{e_i}(x) := 1/(e_i - x)$ for $i \in \{1, \ldots, r\}$. Then $\Psi_{e_i} = S \circ (U \circ S)^{-e_i}$, and

$$\begin{aligned}\Phi_{e_1,\ldots,e_r} &= \Psi_{e_r} \circ \ldots \circ \Psi_{e_1} \\ &= S \circ (U \circ S)^{-e_r - 1} \circ U \circ (U \circ S)^{-e_{r-1}-1} \circ \ldots \circ U \circ (U \circ S)^{-e_2 - 1} \circ U \circ (U \circ S)^{-e_1}.\end{aligned}$$

Therefore, $\Phi_{e_1,\ldots,e_r}(x) = x$ (as rational functions) if and only if $e_r = \ldots = e_{r-1} = -1$, and $S \circ U^r \circ S \circ (U \circ S)^{-e_1 - 1}(x) = x$. Now $S \circ U^r \circ S \circ (U \circ S)^{-e_1-1}(x) = x$ if and only if $e_1 = -1$, and $r \equiv 0 \pmod 3$. This completes the proof of Lemma 2.1. □

3. *Canonical desingularizations, and the Zariski–Lipman conjecture for klt spaces*

3.1 (Logarithmic tangent sheaf). Let Y be a nonsingular variety of dimension $n \geqslant 1$, and $\Delta \subset Y$ be a *divisor with simple normal crossings*. That is, Δ is an effective divisor and its local equation at an arbitrary point $y \in Y$ decomposes in the local ring $\mathscr{O}_y$ into a product $y_1 \cdots y_k$, where $y_1, \ldots, y_k$ form part of a regular system of parameters $(y_1, \ldots, y_n)$ of $\mathscr{O}_y$. Let

$$T_Y(-\log \Delta) \subseteq T_Y = \mathrm{Der}_{\mathbf{C}}(\mathscr{O}_Y)$$

be the subsheaf consisting of those derivations that preserve the ideal sheaf $\mathscr{O}_Y(-\Delta)$. One easily checks that the *logarithmic tangent sheaf* $T_Y(-\log \Delta)$ is a locally free sheaf of Lie subalgebras of T_Y, having the same restriction to $Y \setminus \Delta$, and hence the same rank n.

If Δ is defined at y by the equation $y_1 \cdots y_k = 0$ as above, then a local basis of $T_Y(-\log \Delta)$ (after localization at y) consists of

$$y_1\partial_1, \ldots, y_k\partial_k, \quad \partial_{k+1}, \ldots, \partial_n,$$

where $(\partial_1, \ldots, \partial_n)$ is the local basis of T_Y dual to the local basis $(dy_1, \ldots, dy_n)$ of Ω^1_Y.

A local computation shows that $T_Y(-\log \Delta)$ can be identified with the subsheaf of T_Y containing those vector fields that are tangent to Δ at smooth points of Δ.

The dual of $T_Y(-\log \Delta)$ is the sheaf $\Omega^1_Y(\log \Delta)$ of logarithmic differential 1-forms, that is, of rational 1-forms α on Y such that α and $d\alpha$ have at most simple poles along Δ.

The top exterior power $\wedge^n T_Y(-\log \Delta)$ is the invertible sheaf $\mathscr{O}_Y(-K_Y - \Delta)$, where K_Y denotes a canonical divisor.

We will need the following observation (see also [**10**, Lemma 2.14]).

Lemma 3.2. *Let Y be a smooth variety of dimension at least 2, and $\Delta \subset Y$ be a divisor with simple normal crossings. If $H \subsetneq Y$ is a smooth hypersurface such that $\Delta \cap H$ is a divisor in H with simple normal crossings, then there is an exact sequence*

$$0 \to \mathscr{N}^*_{H/Y} \to \Omega^1_Y(\log \Delta)_{|H} \to \Omega^1_H(\log \Delta_{|H}) \to 0.$$

Proof. Consider the composite map $\alpha\colon \mathscr{N}^*_{H/Y} \to \Omega^1_{Y|H} \to \Omega^1_Y(\log \Delta)_{|H}$, and the morphism $\beta\colon \Omega^1_Y(\log \Delta)_{|H} \to \Omega^1_H(\log \Delta_{|H})$ induced by the restriction map. A local computation shows that α and β yield an exact sequence as claimed. □

3.3 (Singularities of pairs). We recall some definitions of singularities of pairs, developed in the context of the Minimal Model Program.

Definition 3.4 (see [**12**, Section 2.3]). Let X be a normal variety, and $B = \sum a_i B_i$ be an effective $\mathbb{Q}$-divisor on X, that is, B is a nonnegative $\mathbb{Q}$-linear combination of distinct prime Weil divisors B_i on X. Suppose that $K_X + B$ is $\mathbb{Q}$-Cartier, that is, some nonzero multiple of it is a Cartier divisor.

Let $\pi : Y \to X$ be a log resolution of the pair (X, B). This means that Y is a smooth variety, π is a birational projective morphism whose exceptional set $\mathrm{Exc}(\pi)$ is of pure codimension 1, and the divisor $\sum E_i + \pi_*^{-1} B$ has simple normal crossings, where the E_i are the irreducible components of $\mathrm{Exc}(\pi)$. There are uniquely defined rational numbers $a(E_i, X, B)$ such that

$$K_Y + \pi_*^{-1} B = \pi^*(K_X + B) + \sum_{E_i} a(E_i, X, B) E_i.$$

The rational numbers $a(E_i, X, B)$ do not depend on the log resolution π, but only on the valuations associated to the exceptional divisors E_i's ; $a(E_i, X, B)$ is called the discrepancy of E_i with respect to (X, B).

Let

$$\mathrm{discrep}(X, B) = \inf_E \{a(E, X, B)\},$$

where E runs through all the prime exceptional divisors of all projective birational morphisms. Then, either $\mathrm{discrep}(X, B) = -\infty$, or $-1 \leqslant \mathrm{discrep}(X, B) \leqslant 1$. If X is smooth, then $\mathrm{discrep}(X, 0) = 1$.

We say that (X, B) is *log terminal* (or *klt*) if all $a_i < 1$, and, for some log resolution $\pi : Y \to X$ of (X, B), $a(E_i, X, B) > -1$ for every π-exceptional prime divisor E_i. We say that (X, B) is *log canonical* if all $a_i \leqslant 1$, and, for some log resolution $\pi : Y \to X$ of (X, B), $a(E_i, X, B) \geqslant -1$ for every π-exceptional prime divisor E_i. If these conditions hold for some log resolution of

(X,B), then they hold for every log resolution of (X,B). Moreover, (X,B) is log canonical (respectively, klt) if and only if $\operatorname{discrep}(X,B) \geqslant -1$ (respectively, $\operatorname{discrep}(X,B) > -1$ and all $a_i < 1$).

We say that X is klt (respectively, log canonical) if so is $(X,0)$.

Note that if the tangent sheaf T_X is locally free, then K_X is obviously $\mathbb{Q}$-Cartier, so that discrepancies make sense.

3.5 Canonical desingularization. In the proofs of Theorems 1.1 and 3.8, we will consider a suitable resolution of singularities, whose existence is guaranteed by the following theorem.

Theorem 3.6 [**6**, Corollary 4.7; **11**, Theorems 3.35 and 3.45]. *Let X be a normal variety. Then there exists a log resolution $\pi : Y \to X$ of $(X,0)$ such that*

(1) *π is an isomorphism over $X \setminus \operatorname{Sing}(X)$ and*
(2) *$\pi_* T_Y(-\log \Delta) \simeq T_X$ where Δ is the largest reduced divisor contained in $\pi^{-1}(\operatorname{Sing}(X))$.*

Note that $\operatorname{Supp}(\Delta) = \operatorname{Exc}(\pi)$. In particular, Δ has simple normal crossings. We call a resolution π as in Theorem 3.6 a *canonical desingularization* of X.

3.7 Zariski–Lipman conjecture for klt spaces. Recall from [**7**] that the Zariski–Lipman conjecture holds for klt spaces (see also [**1**, Corollary 5.7] for related results). We reproduce the proof from [**1**, Corollary 5.7] for the reader's convenience.

Theorem 3.8 [**7**]. *Let X be a klt space such that the tangent sheaf T_X is locally free. Then X is smooth.*

Proof. We assume to the contrary that $\operatorname{Sing}(X) \neq \emptyset$. Let $\pi : Y \to X$ be a canonical desingularization of X, and let Δ be the largest reduced divisor contained in $\pi^{-1}(\operatorname{Sing}(X))$. Note that $\Delta \neq 0$ since $\operatorname{Sing}(X) \neq \emptyset$. Consider the morphism of vector bundles

$$\pi^* T_X \simeq \pi^*(\pi_* T_Y(-\log \Delta)) \to T_Y(-\log \Delta),$$

where $\pi^*(\pi_* T_Y(-\log \Delta)) \to T_Y(-\log \Delta)$ is the evaluation map. It induces an injective map of sheaves

$$\pi^* \mathscr{O}_X(-K_X) \simeq \pi^* \det(T_X) \hookrightarrow \det(T_Y(-\log \Delta)) \simeq \mathscr{O}_Y(-K_Y - \Delta).$$

This implies that $a(\Delta_i, X) \leqslant -1$ for any irreducible component Δ_i of Δ, yielding a contradiction and completing the proof of Theorem 3.8. □

Remark 3.9. We have the following reformulation of Theorem 3.8. Let X be a variety such that the tangent sheaf T_X is locally free. If X is not smooth, then $\operatorname{discrep}(X) \in \{-\infty, -1\}$.

4. *The Camacho–Sad formula*

4.1 Foliations. A (singular) foliation on a smooth complex analytic surface S is a locally free subsheaf $\mathscr{L} \subsetneq T_S$ of rank 1 such that the corresponding twisted vector field $\vec{v} \in H^0(X, T_S \otimes \mathscr{L}^{\otimes -1})$ has isolated zeroes. Its singular locus $\operatorname{Sing}(\mathscr{L})$ is the zero locus of $\vec{v}$. Considering the natural perfect pairing $\Omega^1_S \otimes \Omega^1_S \to \omega_S$, we see that $\mathscr{L} \subsetneq T_S$ gives rise to a twisted 1-form with isolated zeroes $\omega \in H^0(X, \Omega^1_S \otimes \mathscr{M})$ with $\mathscr{M} = \omega_S \otimes \mathscr{L}$. Conversely, given a twisted 1-form $\omega \in H^0(X, \Omega^1_S \otimes \mathscr{M})$ with isolated zeroes, we define a foliation as the kernel of the morphism $T_S \to \mathscr{L}$ given by the contraction with ω.

4.2 Camacho–Sad formula. A curve $C \subset S$ is said to be $\mathscr{L}$-invariant if any of its irreducible components is the closure of a leaf of $\mathscr{L}$. Let $C \subset S$ be a compact (not necessarily irreducible) $\mathscr{L}$-invariant curve, and let $p \in C \cap \mathrm{Sing}(\mathscr{L})$. Let ω be a local 1-form defining $\mathscr{L}$ in a neighborhood of p, and let f be a local equation of C at p. Then there exist nonzero local functions g and h, and a local 1-form η such that f and h are relatively prime and $g\omega = hdf + f\eta$ (see [**16**, Lemma 1.1]). Following [**16**], we set

$$\mathrm{CS}(\mathscr{L}, C, p) = -\mathrm{Res}_p \frac{1}{h} \eta_{|C}.$$

The right-hand side depends only on $\mathscr{L}$ and C.

Example 4.3. Let (x, y) be local coordinates at p. Suppose that $\mathscr{L}$ is given by the local 1-form $\omega = \lambda x(1 + o(1))\, dy - \mu y(1 + o(1))\, dx$ with $\mu \neq 0$. Set $p = (0, 0)$, and let C be the invariant curve defined by $x = 0$. Then $\mathrm{CS}(\mathscr{L}, C, p) = \lambda/\mu$.

We can now state the Camacho–Sad formula (see [**16**, Theorem 2.1], see also [**3**]).

Theorem 4.4 (Camacho–Sad formula). *Let $\mathscr{L}$ be a foliation on a smooth complex analytic surface S, and let $C \subset S$ be a compact (not necessarily irreducible) $\mathscr{L}$-invariant curve. Then*

$$C^2 = \sum_{p \in C \cap \mathrm{Sing}(\mathscr{L})} \mathrm{CS}(\mathscr{L}, C, p).$$

4.5. Here we give some applications of the Camacho–Sad formula. The following two are immediate consequences of Theorem 4.4, of independent interest.

Lemma 4.6. *Let S be smooth surface, $\mathscr{Q}$ be a line bundle on S, and $T_S \twoheadrightarrow \mathscr{Q}$ be a surjective map of sheaves. Let $C \subset X$ be a smooth complete connected curve of genus g. If $\deg_C(\mathscr{Q}) < 2 - 2g$, then $g = 0$, and $\deg_C(\mathscr{Q}) = 0$.*

Proof. Let $\mathscr{L}$ be the kernel of $T_S \twoheadrightarrow \mathscr{Q}$. Suppose that $\deg_C(\mathscr{Q}) < 2 - 2g$. Then the composite map $T_C \to T_{S|C} \to \mathscr{Q}_{|C}$ is the zero map, and hence $T_C = \mathscr{L}_{|C} \subsetneq T_{S|C}$, and $\mathscr{N}_{C/S} \simeq \mathscr{Q}_{|C}$. In particular, C is a leaf of the regular foliation by curves $\mathscr{L} \subsetneq T_S$. By the Camacho–Sad formula (see Theorem 4.4), we must have $\deg_C(\mathscr{Q}) = C^2 = 0$. This completes the proof of Lemma 4.6. $\square$

Corollary 4.7. *Let S be smooth surface, $\mathscr{Q}$ be a line bundle on S, and $T_S \twoheadrightarrow \mathscr{Q}$ be a surjective map of sheaves. Let $C \subset X$ be a smooth complete connected curve of genus $g \leqslant 1$. Then $\deg_C(\mathscr{Q}) \geqslant 0$.*

The next result is crucial for the proof of Theorem 1.1.

PROPOSITION 4.8. *Let S be smooth surface, and let $C \subset S$ be a possibly reducible complete curve with simple normal crossings. If $T_S(-\log C) \simeq \mathscr{O}_S \oplus \mathscr{O}_S$, then the intersection matrix of irreducible components of C is not negative definite.*

Proof. Let C' be a connected component of C, and set $C'' = C \setminus C'$. Then, up to replacing S by $S \setminus C''$, we may assume that C is a connected curve.

We denote the irreducible components of C by $C_1, \ldots, C_r$ ($r \geqslant 1$). Recall that the dual graph Γ of C is defined as follows. The vertices of Γ are the curves C_i, and for $i \neq j$, the vertices C_i and C_j are connected by $C_i \cdot C_j$ edges.

We first show that either C is a smooth curve of genus 1, or C is a cycle of rational curves (see [**14**, Theorem 4.6.28] for related results). Note that $\mathscr{O}_S(K_S) \simeq \mathscr{O}_S(-C)$ since $\det(T_S(-\log C)) \simeq \mathscr{O}_S(-K_S - C)$, and $T_S(-\log C) \simeq \mathscr{O}_S \oplus \mathscr{O}_S$. By the adjunction formula, for $1 \leqslant i \leqslant r$, we have

$$\mathscr{O}_{C_i}(K_{C_i}) \simeq \mathscr{O}_S(K_S + C_i)_{|C_i} \simeq \mathscr{O}_{C_i}\left(-\sum_{j \neq i} C_{j\,|C_i}\right),$$

and hence

$$\deg_{C_i}(\mathscr{O}_{C_i}(K_{C_i})) = 2g(C_i) - 2 = -\sum_{j \neq i} \deg_{C_i}(\mathscr{O}_{C_i}(C_{j\,|C_i})) \leqslant 0.$$

Thus, either C is irreducible, and $g(C) = 1$, or $r \geqslant 2$, $C_i \simeq \mathbb{P}^1$ for all $1 \leqslant i \leqslant r$ and the dual graph of C is a cycle. This proves our claim.

We argue by contradiction and assume that the intersection matrix $\{C_i \cdot C_j\}_{i,j}$ is negative definite.

Suppose first that C is irreducible with $g(C) = 1$. Recall that there is a surjective map of sheaves $T_S(-\log C) \twoheadrightarrow T_C$. On the other hand, $T_S(-\log C) \simeq \mathscr{O}_S \oplus \mathscr{O}_S$. Hence, there exists a nonzero global vector field $\vec{v} \in H^0(S, T_S(-\log C)) \subseteq H^0(S, T_S)$ such that $\vec{v}_{|C} \not\equiv 0$. Since $T_C \simeq \mathscr{O}_C$, $\vec{v}(s) \neq 0$ for any $s \in C$. Set $\mathscr{L} = \mathscr{O}_S\vec{v} \subsetneq T_S$. Then, C is a complete $\mathscr{L}$-invariant curve, disjoint from the singular locus $\mathrm{Sing}(\mathscr{L})$. Thus, by the Camacho–Sad formula (see Theorem 4.4), we must have $C^2 = 0$, yielding a contradiction.

Suppose that $r \geqslant 2$, $C_i \simeq \mathbb{P}^1$ for any $1 \leqslant i \leqslant r$ and that the dual graph of C is a cycle. If $r = 2$, then $C_1 \cap C_2 = \{p_1, p_2\}$ with $p_1 \neq p_2$. Suppose that $r \geqslant 3$. By renumbering the irreducible components of C_i if necessary, we may assume that for each $i \in \{1, \ldots, r\}$, C_i meets $\overline{C \setminus C_i}$ in $p_i \in C_{i-1}$ and $p_{i+1} \in C_{i+1}$, where $C_{r+1} = C_1$. Note that $p_{r+1} = p_1$

Let $\vec{v}_k \in H^0(S, T_S(-\log C)) \subseteq H^0(S, T_S)$ for $k \in \{1, 2\}$ such that $T_S(-\log C) \simeq \mathscr{O}_S\vec{v}_1 \oplus \mathscr{O}_S\vec{v}_2$. Let $\lambda \in \mathbf{C}$. Set $\vec{v}_\lambda = \vec{v}_1 + \lambda\vec{v}_2$, and $\mathscr{L}_\lambda = \mathscr{O}_S\vec{v}_\lambda \subseteq T_S$.

Set $C_0 = C_r$. Fix $i \in \{1, \ldots, r\}$, and let (x_i, y_i) be local coordinates at p_i such that x_i (respectively, y_i) is a local equation of C_{i-1} (respectively, C_i) at p_i. Then $x_i\partial_{x_i}$ and $y_i\partial_{y_i}$ are local generators of $T_S(-\log C)$ at p_i. Therefore, there exist local functions a_i, b_i, c_i, d_i at p_i such that the matrix

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

is invertible, and such that

$$\begin{cases} \vec{v}_1 = a_i(x_i, y_i)x_i\partial_{x_i} + b_i(x_i, y_i)y_i\partial_{y_i}, \\ \vec{v}_2 = c_i(x_i, y_i)x_i\partial_{x_i} + d_i(x_i, y_i)y_i\partial_{y_i}. \end{cases}$$

Thus,

$$\vec{v}_\lambda = (a_i(x_i, y_i) + \lambda c_i(x_i, y_i))x_i\partial_{x_i} + (b_i(x_i, y_i) + \lambda d_i(x_i, y_i))y_i\partial_{y_i},$$

and a local generator for $\mathscr{L}_\lambda$ is given by the 1-form

$$\begin{aligned}\omega_\lambda &= (a_i(x_i,y_i)+\lambda c_i(x_i,y_i))x_i\,dy_i - (b_i(x_i,y_i)+\lambda d_i(x_i,y_i))y_i\,dx_i\\ &= (a_i(p_i)+\lambda c_i(p_i))x_i(1+o(1))\,dy_i - (b_i(p_i)+\lambda d_i(p_i))y_i(1+o(1))\,dx_i.\end{aligned}$$

This implies that for $\lambda \in \mathbf{C}\setminus\{-b_1(p_1)/d_1(p_1), -a_1(p_1)/c_1(p_1), \ldots, -b_r(p_r)/d_r(p_r), -a_r(p_r)/c_r(p_r)\}$

(1) $\vec{v}_\lambda$ vanishes exactly at $\{p_1,\ldots,p_r\}$;
(2) $\mathrm{CS}(\mathscr{L}_\lambda, C_{i-1}, p_i) = (a_i(p_i)+\lambda c_i(p_i))/(b_i(p_i)+\lambda d_i(p_i))$ (see Example 4.3);
(3) $\mathrm{CS}(\mathscr{L}_\lambda, C_i, p_i) = (b_i(p_i)+\lambda d_i(p_i))/(a_i(p_i)+\lambda c_i(p_i))$ (see Example 4.3).

In particular, $\mathscr{L}_\lambda \subsetneq T_S$ is a foliation by curves on S, $\mathrm{Sing}(\mathscr{L}_\lambda)=\{p_1,\ldots,p_r\}$, and $\mathrm{CS}(\mathscr{L}_\lambda, C_i, p_i) = 1/\mathrm{CS}(\mathscr{L}_\lambda, C_{i+1}, p_i)$.

Set $e_i = C_i^2 \in \mathbb{Z}$. Note that for each $i\in\{1,\ldots,r\}$, we have

$$\begin{aligned}\mathrm{CS}(\mathscr{L}_\lambda, C_{i+1}, p_{i+1}) &= \frac{1}{\mathrm{CS}(\mathscr{L}_\lambda, C_i, p_{i+1})} \quad \text{by (3)}\\ &= \frac{1}{e_i - \mathrm{CS}(\mathscr{L}_\lambda, C_i, p_i)} \quad \text{by the Camacho–Sad formula.}\end{aligned}$$

Set $x_\lambda = \mathrm{CS}(\mathscr{L}_\lambda, C_1, p_1) = (a_1(p_1)+\lambda c_1(p_1))/(b_1(p_1)+\lambda d_1(p_1))$. Then

$$\begin{aligned}x_\lambda &= \mathrm{CS}(\mathscr{L}_\lambda, C_1, p_1) = \mathrm{CS}(\mathscr{L}_\lambda, C_{r+1}, p_{r+1}) = \frac{1}{e_r - \mathrm{CS}(\mathscr{L}_\lambda, C_r, p_r)}\\ &= \frac{1}{e_r - \frac{1}{e_{r-1}-\mathrm{CS}(\mathscr{L}_\lambda, C_{r-1}, p_{r-1})}} = \ldots = \cfrac{1}{e_r - \cfrac{1}{e_{r-1} - \cfrac{1}{\ldots - \cfrac{1}{e_2 - \cfrac{1}{e_1 - \mathrm{CS}(\mathscr{L}_\lambda, C_1, p_1)}}}}} = \Phi_{e_1,\ldots,e_r}(x_\lambda)\end{aligned}$$

for any $\lambda \in \mathbf{C}\setminus\{-b_1(p_1)/d_1(p_1), -a_1(p_1)/c_1(p_1), \ldots, -b_r(p_r)/d_r(p_r), -a_r(p_r)/c_r(p_r)\}$.

This implies that

$$\Phi_{e_1,\ldots,e_r}(x) = x,$$

as rational functions. By Lemma 2.1, we must have $e_1 = \ldots = e_r = -1$, yielding a contradiction since $(C_1+C_2)^2 = -2+2C_1\cdot C_2 \geqslant 0$. This completes the proof of Proposition 4.8. □

5. *Proof of Theorem* 1.1

We will use the following theorem to reduce to the surface case.

THEOREM 5.1 [4, Corollary]. *Let X be a variety such that the tangent sheaf T_X is locally free. If* $\mathrm{codim}_X \mathrm{Sing}(X) \geqslant 3$, *then X is smooth.*

We are now in position to prove our main result. Note that Theorem 1.1 is a immediate consequence of Theorem 5.2.

THEOREM 5.2 (Zariski–Lipman conjecture for log canonical pairs). *Let (X,B) be a log canonical pair such that the tangent sheaf T_X is locally free. Then X is smooth.*

Proof. Note first that K_X is Cartier since the tangent sheaf T_X is locally free. This implies that X is log canonical as well (see [**12**, Corollary 2.35]).

Let us assume to the contrary that $\mathrm{Sing}(X) \neq \emptyset$. By Theorem 5.1, we have $\mathrm{codim}_X \mathrm{Sing}(X) = 2$. By replacing X with an affine open dense subset, we may assume that X is affine, and that $\mathrm{Sing}(X)$ is irreducible of codimension 2. We may also assume without loss of generality that $T_X \simeq \mathscr{O}_X^{\oplus \dim(X)}$.

Let $\pi : Y \to X$ be a canonical desingularization of X, and let Δ be the largest reduced divisor contained in $\pi^{-1}(\mathrm{Sing}(X))$. Note that $\Delta \neq 0$. As in the proof of Theorem 3.8, we consider the morphism of vector bundles

$$\pi^* T_X \to T_Y(-\log \Delta),$$

and the induced injective map of sheaves

$$\pi^* \mathscr{O}_X(-K_X) \simeq \pi^* \det(T_X) \hookrightarrow \det(T_Y(-\log \Delta)) \simeq \mathscr{O}_Y(-K_Y - \Delta).$$

This yields $a(\Delta_i, X) \leqslant -1$ for any irreducible component Δ_i of Δ. Thus, $a(\Delta_i, X) = -1$ since X has log canonical singularities, and we have an isomorphism

$$\pi^* T_X \simeq T_Y(-\log \Delta).$$

Suppose that $\dim(X) \geqslant 3$. Let $G_1 \subset X$ be a general hyperplane section, and set $H_1 = \pi^{-1}(G_1) \subset Y$. Then G_1 is a normal affine variety (see, for instance, [**2**, Theorem 1.7.1]). Moreover, H_1 is smooth, and $\Delta \cap H_1$ has simple normal crossings by Bertini's Theorem. By Lemma 3.2, there is an exact sequence

$$0 \to \mathscr{N}^*_{H_1/Y} \to \Omega^1_Y(\log \Delta)_{|H_1} \to \Omega^1_{H_1}(\log \Delta_{|H_1}) \to 0.$$

Note that $\mathscr{N}^*_{H_1/Y} \simeq \pi^* \mathscr{N}^*_{G_1/X} \simeq \mathscr{O}_{H_1}$. Thus, there exist regular functions $g_1, \ldots, g_r$ on G_1 such that the map $\mathscr{O}_Y \simeq \mathscr{N}^*_{H_1/Y} \to \Omega^1_Y(\log \Delta)_{|H_1} \simeq \mathscr{O}_Y^{\oplus \dim(Y)}{}_{|H_1}$ is given by $g_1 \circ \pi_{|H_1}, \ldots, g_r \circ \pi_{|H_1}$. Let $i \in \{1, \ldots, r\}$ such that $g_{i|H_1 \cap \mathrm{Sing}(X)} \neq 0$. Then, by replacing X with $X \setminus \{g_i = 0\}$ if necessary, we may assume that $\Omega^1_{H_1}(\log \Delta_{|H_1}) \simeq \mathscr{O}_{H_1}^{\oplus \dim(H_1)}$ (and $\Delta_{|H_1} \neq 0$). Let $G_2, \ldots, G_{\dim(X)-2} \subset X$ be general hyperplane sections, and set $H_i = \pi^{-1}(G_i) \subset Y$, $S = H_1 \cap \ldots \cap H_{\dim(X)-2}$, $C = \Delta \cap H_1 \ldots \cap H_{\dim(X)-2}$, and $T = G_1 \cap \ldots \cap G_{\dim(X)-2} = \pi(S)$. Then S is smooth, and C has simple normal crossings. Proceeding by induction, we conclude that by replacing T with an appropriate open subset, we may assume that $T_S(-\log C) \simeq \mathscr{O}_S^{\oplus 2}$ (and $C \neq 0$). Observe that the induced morphism $\pi_{|S} \colon S \to T$ is birational with exceptional locus C. This implies that the intersection matrix of irreducible components of C is negative definite. But this contradicts Proposition 4.8, completing the proof of Theorem 5.2. □

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