CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS, AND RELATED TOPICS.

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Contents

1. Introduction

$X =$ a smooth complex projective variety, $n := \dim(X)$, K_X a canonical divisor on X (any divisor whose associated line bundle is $\det(\Omega^1_X)$).

Conjecturally, and very roughly, the « building bricks » of algebraic geometry are varieties with

- $-K_X$ ample (Fano varieties),
- K_X trivial,
- K_X ample and

mild singularities.

Fano manifolds are quite rare in nature. Let X = Fano manifold.

- $\dim(X) = 1$ iff $X \simeq \mathbf{P}^1$;
- $\dim(X) = 2$ iff either $X \simeq \mathbf{P}^1 \times \mathbf{P}^1$, or $X \simeq$ the blow-up of $\mathbf{P}^2$ at $\leqslant 8$ points in general position = 10 deformations types (usually called del Pezzo surfaces);
- $\dim(X) = 3$: 17 deformations types with Picard number $\rho(X) = 1$ (Iskovskikh '77, '78, Shokurov '79) and 88 deformations types with Picard number $\rho(X) \geqslant 2$ (Mori-Mukai '82, '03);
- There are at most $(n+2)^{(n+2)^{n2^{3n}}}$ deformations types of complex Fano varieties of dimension n (Campana '91, Nadel '91, Kollár-Miyaoka-Mori '92).

Definition 1. The index of a Fano manifold X is the largest positive integer ι_X such that $-K_X \sim \iota_X H$ for a divisor H on X.

Theorem 2 (Kobayashi-Ochiai '73)**.** *X = Fano manifold of index ι_X. Then*

- $\iota_X \leqslant n+1$*;*
- $\iota_X = n+1$ *iff* $X \simeq \mathbf{P}^n$*;*
- $\iota_X = n$ *iff* $X \simeq Q_n$ *(smooth quadric in* $\mathbf{P}^{n+1}$*).*

Definition 3. Fano manifolds of dimension n and index $\iota_X = (n-1)k$ for some integer $k \geqslant 1$ are usually called del Pezzo manifolds.

Fano manifolds of dimension $\geqslant 3$ and index $i = n-3$ were classified by Fujita.

Theorem 4 (Fujita '80, '81 and '84)**.** *Let X be Fano manifold of dimension $n \geqslant 3$ and index $i = n-3$. Then X is either*

(1) *an hypersurface of degree 6 in the weighted projective space $\mathbf{P}(3,2,1,\dots,1)$, or*

(2) *an hypersurface of degree 4 in the weighted projective space $\mathbf{P}(2,1,\dots,1)$, or*

(3) *a cubic hypersurface, or*

(4) *a complete intersection of two quadrics, or*

(5) *a linear section of the Grassmannian $G(2,5)$ ($n \leqslant 6$), or*

(6) *the blow-up of $\mathbf{P}^3$ in 1 point, or*

(7) $\mathbf{P}_{\mathbf{P}^2}(T_{\mathbf{P}^2})$, *or*

(8) $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, *or*

(9) $\mathbf{P}^2 \times \mathbf{P}^2$.

($4_n + 1_{n\leqslant 6}$ deformations types with Picard number $\rho(X) = 1$ and 4 deformations types with Picard number $\rho(X) \geqslant 2$).

Similar ideas can be applied in the context of foliations on complex projective manifolds.

2. Foliations

Definition 5. A (singular) foliation on X = sheaf of $\mathscr{O}_X$-modules $\mathscr{F} \subset T_X$, closed under the Lie bracket, saturated in T_X ($= T_X/\mathscr{F}$ is torsion free) of rank $1 \leqslant r \leqslant n-1$.
$K_{\mathscr{F}}$ = a canonical divisor of $\mathscr{F}$ = any divisor whose associated line bundle is $\mathcal{O}_X(-K_{\mathscr{F}}) \simeq \det(\mathscr{F}) := (\wedge^r \mathscr{F})^{**}$.

In analogy, we define a

Definition 6. Fano foliation = foliation $\mathscr{F}$ with $-K_{\mathscr{F}}$ ample.

As in the case of Fano manifolds, we expect Fano foliations to present very special behavior. This is the case for instance if the rank of $\mathscr{F}$ is 1, i.e., $\mathscr{F}$ is an ample invertible subsheaf of T_X. By Wahl's Theorem, this can only happen if $(X, \mathscr{F}) \simeq \big(\mathbf{P}^n, \mathscr{O}(1)\big)$.

Theorem 7 (Mori '79, Wahl '83, Andreatta-Wiśniewski '01)**.** *Let X be a smooth complex projective n-dimensional variety. Assume that the tangent bundle T_X contains an ample locally free subsheaf $\mathscr{E}$ of rank $1 \leqslant r \leqslant n$. Then $X \simeq \mathbf{P}^n$ and either $\mathscr{E} \simeq \mathscr{O}_{\mathbf{P}^n}(1)^{\oplus r}$ ($\rightsquigarrow$ degree 0 foliation on $\mathbf{P}^n$) or $\mathscr{E} = T_X$.*

Definition 8. The index of a Fano foliation $\mathscr{F}$ on X is the largest positive integer $\iota_{\mathscr{F}}$ such that $-K_{\mathscr{F}} \sim \iota_{\mathscr{F}} H$ for a divisor H on X.

In analogy with the first part of Kobayachi-Ochiai's theorem, we have the following.

Theorem 9 (Araujo,–, Kovács '08)**.** *Let $\mathscr{F}$ be a Fano foliation of rank r on an n-dimensional complex projective manifold X, with $1 \leq r \leq n-1$. Then $\iota_{\mathscr{F}} \leq r$ and equality holds only if $X \cong \mathbf{P}^n$.*

Foliations as q-forms. Let X be a smooth variety of dimension $n \geqslant 2$, and $\mathscr{F} \subsetneq T_X$ a foliation of rank r on X. Set $N^*_{\mathscr{F}} := (T_X/\mathscr{F})^*$, and $N_{\mathscr{F}} := (N^*_{\mathscr{F}})^*$. These are called the *conormal* and *normal* sheaves of the foliation $\mathscr{F}$, respectively. The conormal sheaf $N^*_{\mathscr{F}}$ is a saturated subsheaf of Ω^1_X of rank $q := n - r$. The q-th wedge product of the inclusion $N^*_{\mathscr{F}} \subset \Omega^1_X$ gives rise to a nonzero twisted differential q-form ω with coefficients in the line bundle $\mathscr{L} := \det(N_{\mathscr{F}})$, which is *locally decomposable* and *integrable*. To say that $\omega \in H^0(X, \Omega^q_X \otimes \mathscr{L})$ is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for $i \in \{1, \ldots, q\}$. Conversely, given a twisted q-form $\omega \in H^0(X, \Omega^q_X \otimes \mathscr{L}) \setminus \{0\}$ which is locally decomposable and integrable, we define a foliation of rank r on X as the kernel of the morphism $T_X \to \Omega^{q-1}_X \otimes \mathscr{L}$ given by the contraction with ω.

Proposition 10 (Cerveau-Déserti '05)**.** *Let $\mathscr{F}$ be a Fano foliation on $\mathbf{P}^n$ with index equal to the rank (usually called degree 0 foliations on $\mathbf{P}^n$). Then $\mathscr{F}$ is induced by a linear projection $\mathbf{P}^n \dashrightarrow \mathbf{P}^{n-r}$ ($\rightsquigarrow$ leaves are algebraic).*

Proof. Set $q := n - r$. Let $\omega \in H^0(\Omega^q_{\mathbf{P}^n}(q+1))$ be a twisted differential q-form defining $\mathscr{F}$. Then ω is induced by a q-form

$$\Omega = \sum_{I=(i_1<\cdots<i_q)} \ell_I \, dz_{i_1} \wedge \cdots \wedge dz_{i_q}$$

on $\mathbf{C}^{n+1}$ such that $i_R\Omega = 0$ where $R = \sum_{1\leqslant i\leqslant n+1} z_i \frac{\partial}{\partial z_i}$ and ℓ_I is a linear form.

Let us assume for simplicity that $q = 1$. Let $x \in \mathbf{C}^{n+1}$ be a general point. By Frobenius' theorem, there exists a local holomorphic function f defined in an open neighborhood of x such that $\Omega \wedge df = 0$. Thus $d\Omega \wedge df = 0$. We may assume that $f = z_0 + g$ where $g \in \mathfrak{m}_x^2$. Since $d\Omega$ has constant coefficients, we must have $d\Omega \wedge dz_0 = 0$. Thus $d\Omega = dz_0 \wedge d\ell = \frac{1}{2} d(z_0 d\ell - \ell dz_0)$ for some linear form ℓ on $\mathbf{C}^{n+1}$. Finally, $\Omega = \frac{1}{2} d(z_0 d\ell - \ell dz_0) + dq$ for some quadratic form q on $\mathbf{C}^{n+1}$. Since $i_R\Omega = 0$, we must have $q = 0$. The claim follows. □

Singularities of pairs. Let X be a normal (projective) variety, and $\Delta = \sum a_i \Delta_i$ an effective $\mathbf{Q}$-divisor on X, where the Δ_i's are distinct prime divisors. Suppose that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Let $f : \tilde{X} \to X$ be a log resolution of the pair (X, Δ). This means that $\tilde{X}$ is a smooth (projective) variety, f is a birational projective morphism whose exceptional locus is the union of prime divisors E_i's, and the divisor $\sum E_i + f_*^{-1}\Delta$ has simple normal crossing support. There are uniquely defined rational numbers $a(E_i, X, \Delta)$'s such that

$$K_{\tilde{X}} + f_*^{-1}\Delta \sim_{\mathbf{Q}} f^*(K_X + \Delta) + \sum_{E_i} a(E_i, X, \Delta) E_i.$$

The $a(E_i, X, \Delta)$'s do not depend on the log resolution f, but only on the valuations associated to the E_i's.

We say that (X, Δ) is *klt* (resp. *lc*) if $0 \leqslant a_i < 1$ (resp. $0 \leqslant a_i \leqslant 1$) and, for some log resolution $f : \tilde{X} \to X$ of (X, Δ), $a(E_i, X, \Delta) > -1$ (resp. $a(E_i, X, \Delta) \geqslant -1$) for every f-exceptional prime divisor E_i. If this condition holds for some log resolution of (X, Δ), then it holds for every log resolution of (X, Δ).

Singularities of Foliations.

Definition 11. Let $\mathscr{F}$ be a foliation on a smooth n-dimensional variety X. Suppose that $\mathscr{F}$ has algebraic leaves. Let $Y \subset X$ be the closure of a general leaf. Then there is a commutative diagram

$$\begin{tikzcd} \Omega^r_X \arrow[r] \arrow[d] & \mathscr{O}_X(K_{\mathscr{F}}) \arrow[d] \\ \Omega^r_Y \arrow[r, "\eta_Y"] & \mathscr{O}_X(K_{\mathscr{F}})_{|Y} \end{tikzcd}$$

whose vertical maps are the standard maps.

Let $n : \widetilde{Y} \to Y$ the normalization morphism. Then η_Y extends uniquely to $\Omega^r_{\tilde{Y}} \to n^*\mathscr{O}_X(K_{\mathscr{F}})_{|Y}$ (follows from a theorem of Seidenberg). We obtain a map $\tilde{\eta}_Y : \mathscr{O}_{\tilde{Y}}(K_{\tilde{Y}}) \to n^*\mathscr{O}_X(K_{\mathscr{F}})_{|Y}$. Let Δ_Y be the (Weil) divisor on $\tilde{Y}$ of zeroes of $\tilde{\eta}_Y$. Note that Δ_Y is an effective integral Weil divisor such that $\mathscr{O}_{\tilde{Y}}(K_{\tilde{Y}} + \Delta_Y) \simeq n^*\mathscr{O}_X(K_{\mathscr{F}})_{|Y}$. The pair (Y, Δ_Y) is called a (general) *log leaf*.

Definition 12. Let X be a smooth complex variety, r an integer with $1 \leqslant r \leqslant \dim(X) - 1$, and $\mathscr{F}$ a foliation of rank r on X with algebraic leaves. Then $\mathscr{F}$ is said to have log terminal (resp. log canonical) singularities along a general leaf if (Y, Δ_Y) has log terminal (resp. log canonical) singularities where (Y, Δ_Y) is a general log leaf.

Foliations of rank r and index r on $\mathbb{P}^n$. Let $\mathscr{F} \subsetneq T_{\mathbb{P}^n}$ be a Fano foliation of rank r and index $\iota_{\mathscr{F}} = r$ on $\mathbb{P}^n$. These are classically known as *degree 0 foliations on* $\mathbb{P}^n$. Recall that $\mathscr{F}$ is defined by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$. The singular locus of $\mathscr{F}$ is a linear subspace S of dimension $r-1$. The closure of the leaf through a point $p \notin S$ is the r-dimensional linear subspace L of $\mathbb{P}^n$ containing both p and S. Let $p_1, \ldots, p_r \in S$ be r linearly independent points in S, and $v_i \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$ a nonzero section vanishing at p_i. Then the v_i's define an injective map $\mathscr{O}_{\mathbb{P}^n}(1)^{\oplus r} \to T_{\mathbb{P}^n}$ whose image is $\mathscr{F}$. Thus the restricted map $\mathscr{F}|_L \to T_L$ is induced by the sections $v_i|_L \in H^0(L, T_L(-1)) \subset H^0(L, T_{\mathbb{P}^n}(-1)|_L)$. In particular, the zero locus of the map $\det(\mathscr{F})|_L \to \det(T_L)$ is the codimension one linear subspace $S \subset L$. Thus the log leaf $(\tilde{F}, \tilde{\Delta}) = (L, S)$ is log canonical, and $\mathscr{F}$ has log canonical singularities along a general leaf.

Fano foliations on Grassmannians. Let m and n be nonnegative integers, and V a complex vector space of dimension $n+1$. Let $G = G(m+1, V)$ be the Grassmannian of $(m+1)$-dimensional linear subspaces of V, with tautological exact sequence

$$0 \to \mathscr{K} \to V \otimes \mathscr{O}_G \to \mathscr{Q} \to 0.$$

Let k be an integer such that $0 \leqslant k \leqslant n-m-1$, and W a $(k+1)$-dimensional linear subspace of V. Set

$$\mathscr{F} := W \otimes \mathscr{K}^* \subset V \otimes \mathscr{K}^*.$$

The map $V \otimes \mathscr{K}^* \to \mathscr{Q} \otimes \mathscr{K}^*$ induced by $V \otimes \mathscr{O}_G \to \mathscr{Q}$ yields a map $\mathscr{F} \to \mathscr{Q} \otimes \mathscr{K}^* \simeq T_G$. For a general point $[L] \in G$, $L \cap W = \{0\}$ since $k+m \leqslant n-1$. Thus the map $\mathscr{F} \to T_G$ is injective at $[L]$. Since $\mathscr{F}$ is locally free, $\mathscr{F} \hookrightarrow T_G$ is injective. Let P be the linear span of L and W in V. It has dimension $m+k+2 \leqslant n+1$. Notice that the Grassmannian $G(m+1, P) \subset G$ is tangent to $\mathscr{F}$ at a general point of $G(m+1, P)$.

Suppose that $k \leqslant n-m-2$ (or equivalently that $\dim(P) < \dim(V)$). Then $\mathscr{F}$ is a subbundle of T_G in codimension one, and thus saturated in T_G (easy). In particular $\mathscr{F}$ is a Fano foliation on G of rank $r = (m+1)(k+1)$. Its singular locus S is the set of points $[L] \in G$ such that $\dim(L \cap W) \geqslant 1$.

Recall that $\operatorname{Pic}(G) = \mathbf{Z}[\mathscr{O}_G(1)]$ where $\mathscr{O}_G(1) \simeq \det(\mathscr{Q})$ is the pullback of $\mathscr{O}_{\mathbb{P}(\wedge^{m+1} V)}(1)$ under the Plücker embedding. It follows that $\mathscr{F}$ has index $\iota_{\mathscr{F}} = k+1$. In particular, $\iota_{\mathscr{F}} = r-1$ if and only if $m=1$ and $k=0$. In this case, $G = G(2, V)$ and $\mathscr{F}$ is the rank 2 foliation on G whose general leaf is the $\mathbb{P}^2$ of 2-dimensional linear subspaces of a general 3-plane containing the line W.

Finally, observe that $S \cap G(m+1, P)$ is irreducible and has codimension one in $G(m+1, P)$. Moreover, $\det(T_{G(m+1,P)}) \simeq \mathscr{O}_{G(m+1,P)}(m+k+2)$, and $\det(\mathscr{F})|_{G(m+1,P)} \simeq \mathscr{O}_{G(m+1,P)}(k+1)$. It follows that the map $\det(\mathscr{F})|_{G(m+1,P)} \to \det(T_{G(m+1,P)})$ vanishes at order $m+1$ along $S \cap G(m+1, P)$. So the general log leaf of $\mathscr{F}$ is

$$(\tilde{F}, \tilde{\Delta}) = \Big(G(m+1, P), (m+1) \cdot \big(S \cap G(m+1, P) \big) \Big).$$

In particular, $\mathscr{F}$ has log canonical singularities along a general leaf if and only if $m = 0$, i.e., $G = \mathbb{P}^n$, and $\mathscr{F}$ is the foliation described above.

When $m = 1$ and $k = 0$, we obtain a rank 2 del Pezzo foliation on $G = G(2, V)$ with general log leaf $(\tilde{F}, \tilde{\Delta}) \simeq (\mathbb{P}^2, 2H)$, where H is a line in $\mathbb{P}^2$.

Theorem 13 (Fujita)**.** *Let Δ be an integral Weil divisor on a normal projective variety F of dimension $\geqslant 1$, and let $\mathscr{L}$ be an ample line bundle. Suppose that $-(K_F + \Delta) \sim_{\mathbf{Z}} ic_1(\mathscr{L})$ with $i \in \mathbf{Z}$.*

(1) *If $i \geqslant \dim(X) + 1$, then $\Delta = 0$, $i = \dim(X) + 1$ and $(X, \mathscr{L}) \simeq (\mathbf{P}^{\dim(X)}, \mathscr{O}_{\mathbf{P}^{\dim(X)}}(1))$.*

(2) *If $i = \dim(X)$, then either $(X, \mathscr{L}, \mathscr{O}_X(\Delta)) \simeq (\mathbf{P}^{\dim(X)}, \mathscr{O}_{\mathbf{P}^{\dim(X)}}(1), \mathscr{O}_{\mathbf{P}^{\dim(X)}}(1))$ or $\Delta = 0$ and $(X, \mathscr{L}) \simeq (Q_{\dim(X)}, \mathscr{O}_{Q_{\dim(X)}}(1))$ where $Q_{\dim(X)}$ is a possibly singular quadric hypersurface in $\mathbf{P}^{\dim(X)}$.*

3. Foliations and rational curves

If a smooth projective variety X admits a Fano foliation $\mathscr{F}$, then it is uniruled by a result of Miyaoka. In order to study the pair $(X, \mathscr{F})$, it is useful to understand the behavior of $\mathscr{F}$ with respect to families of rational curves on X.

We start by recalling some definitions and results from the theory of rational curves on smooth projective varieties.

Minimal dominating families of rational curves. Let X be a smooth projective variety, and H a family of rational curves on X, i.e., an irreducible component of $\mathrm{RatCurves}^n(X)$. If C is a curve from the family H, with normalization morphism $f : \mathbb{P}^1 \to C \subset X$, then we denote by $[C]$ or $[f]$ any point of H corresponding to C. We denote by $Locus(H)$ the locus of X swept out by curves from H. We say that H is *unsplit* if it is proper, and *minimal* if, for a general point $x \in Locus(H)$, the closed subset H_x of H parametrizing curves through x is proper. We say that H is *dominating* if $\overline{Locus(H)} = X$. In this case we say that a curve C parametrized by H is a *moving curve* on X, and that any curve from H is a deformation of C.

Let X be a smooth projective uniruled variety. Then X always carries a minimal dominating family of rational curves. Fix one such family H, and let $[f] \in H$ be a general point. Then $f^*T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$, where $d = \deg(f^*T_X) - 2 \geq 0$.

$d = \deg(f^*T_X) - 2$.

Rationally connected quotients. Let $H_1, \ldots, H_k$ be families of rational curves on X. For each i, let $\overline{H}_i$ denote the closure of H_i in $\mathrm{Chow}(X)$. Two points $x, y \in X$ are said to be $(H_1, \ldots, H_k)$-equivalent if they can be connected by a chain of 1-cycles from $\overline{H}_1 \cup \cdots \cup \overline{H}_k$. This defines an equivalence relation on X. By a reuslt due to Campana ('92), there exists a proper surjective equidimensional morphism $\pi_0 : X_0 \to T_0$ from a dense open subset of X onto a normal variety whose fibers are $(H_1, \ldots, H_k)$-equivalence classes. We call this map the $(H_1, \ldots, H_k)$-*rationally connected quotient of* X. When T_0 is a point we say that X is $(H_1, \ldots, H_k)$-rationally connected.

Lemma 14 (Andreatta-Wiśniewski '01)**.** *Let H be a dominating family of rational curves on X. Suppose that H is unsplit. Then $\rho(X) = 1$ iff $\dim(T_0) = 0$.*

Lemma 15 (Araujo,–'11)**.** *Let X be a smooth projective variety, $H_1, \cdots, H_k$ unsplit dominating families of rational curves on X, and $\mathscr{F}$ a foliation on X. Denote by $\pi_0 : X_0 \to T_0$ the $(H_1, \cdots, H_k)$-rationally connected quotient of X. If $T_{\mathbb{P}^1} \subset f^*\mathscr{F}$ for general $[f] \in H_i$, $1 \leq i \leq k$, then there is an inclusion $T_{X_0/T_0} \subset \mathscr{F}|_{X_0}$.*

Proof. Notice that a general curve from each of the families H_i's is contained in a leaf of $\mathscr{F}$.

Let $x \in X$ be a general point. We define inductively a sequence of (irreducible) subvarieties of X as follows. Set $V_0(x) := \{x\}$, and let $V_{j+1}(x)$ be the closure of the union of curves from the families H_i, $1 \leq i \leq k$, that pass through a general point of $V_j(x)$.

Then $\dim V_{j+1}(x) \geq \dim V_j(x)$, and equality holds if and only if $V_{j+1}(x) = V_j(x)$. In particular, there exists j_0 such that $V_j(x) = V_{j_0}(x)$ for every $j \geq j_0$. We set $V(x) = V_{j_0}(x)$. Since x is general, $V(x)$ is smooth at x. Notice also that $V(x)$ is irreducible, and that $V(x)$ is contained in the leaf of $\mathscr{F}$ through x by construction.

We define the subfoliation $\mathscr{V} \subset \mathscr{F}$ by setting $\mathscr{V}_x = T_x V(x)$ for general $x \in X$. The leaf of $\mathscr{V}$ through x is precisely $V(x)$. In particular $\mathscr{V}$ is an algebraically integrable foliation of X. Moreover, by construction, a general curve from each of the families H_i's is contained in a leaf of $\mathscr{V}$, and avoids the singular locus of $\mathscr{V}$. The result then follows from Lemma 16 below. $\square$

Lemma 16 (Araujo,–'11)**.** *Let X be a smooth projective uniruled variety, $H_1, \cdots, H_k$ unsplit families of rational curves on X, and $\mathscr{F}$ an algebraically integrable foliation on X. Denote by $\pi_0 : X_0 \to T_0$ the $(H_1, \cdots, H_k)$-rationally connected quotient of X. Suppose that a general curve from each of the families H_i's is contained in a leaf of $\mathscr{F}$ and avoids the singular locus of $\mathscr{F}$. Then there is an inclusion $T_{X_0/T_0} \subset \mathscr{F}|_{X_0}$.*

Proof. Let W be the closure in $\mathrm{Chow}(X)$ of the subvariety parametrizing general leaves of $\mathscr{F}$, with universal family morphisms:

$$\begin{array}{ccc} U & \xrightarrow{q} & X. \\ {\scriptstyle p}\downarrow & & \\ W & & \end{array}$$

Let A_W be a general very ample effective divisor on W, and set $A = q_*(p^*(A_W))$. By assumption, a general curve $\ell \subset X$ parametrized by each H_i is contained in a leaf of $\mathscr{F}$, and avoids the singular locus of $\mathscr{F}$. Thus $A \cdot \ell = 0$.

Let $X_t = (\pi_0)^{-1}(t)$ be a general fiber of π_0. Observe that every proper curve $C \subset X_t$ is numerically equivalent in X to a linear combination of curves from the families H_i's , and so $A \cdot C = 0$. This shows that $A|_{X_t} \equiv 0$, and thus $X_t \subset q(p^{-1}(w))$ for some $w \in W$, i.e., X_t is contained in a leaf of $\mathscr{F}$. We conclude that $T_{X_0/T_0} \subset \mathscr{F}|_{X_0}$ by Lemma 17 below. $\square$

Lemma 17 (Araujo,–'11)**.** *Let $\mathscr{F}$ be a foliation of rank $r_{\mathscr{F}}$ on a normal variety X, and $\pi : X \to Y$ an equidimensional morphism with connected fibers onto a normal variety. Suppose that the general fiber of π is contained in a leaf of $\mathscr{F}$. Then $\mathscr{F}$ induces a foliation*

$\mathscr{G}$ of rank $r_{\mathscr{G}} = r_{\mathscr{F}} - \big(\dim(X) - \dim(Y)\big)$ on Y, together with an exact sequence

$$0 \to T_{X/Y} \to \mathscr{F} \to (\pi^*\mathscr{G})^{**}.$$

Definition 18. Under the hypothesis of Lemma 17, we say that $\mathscr{F}$ is the *pullback via π of the foliation $\mathscr{G}$*.

4. The relative anticanonical bundle of a fibration and applications

Miyaoka proved ('93) that the anticanonical bundle of a smooth projective morphism $f : X \to C$ onto a smooth proper curve cannot be ample. We generalized this result by dropping the smoothness assumption, and replacing $-K_{X/C}$ with $-(K_{X/C} + \Delta)$, where Δ is an effective Weil divisor on X such that (X, Δ) is log canonical over the generic point of C. In this section we also provide applications to the theory of Fano foliations.

Theorem 19 (Araujo,–, Kovács '08)**.** *Let X be a normal projective variety, and $f : X \to C$ a surjective morphism with connected fibers onto a smooth curve. Let $\Delta \subseteq X$ be an effective Weil $\mathbf{Q}$-divisor. Assume that $K_X + \Delta$ is $\mathbf{Q}$-Cartier.*

1. *If (X, Δ) is log canonical over the generic point of C, then $-(K_{X/C}+\Delta)$ is not ample.*
2. *If (X, Δ) is klt over the generic point of C, then $-(K_{X/C} + \Delta)$ is not nef and big.*

Proof. To prove (1), we assume to the contrary that (X, Δ) is log canonical over the generic point of C, and $-(K_{X/C} + \Delta)$ is ample. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities of (X, Δ), A an ample divisor on C, and $m \gg 0$ such that $D = -m(K_{X/C} + \Delta) - f^*A$ is very ample. Then

$$K_{\tilde{X}} + \pi_*^{-1}\Delta = \pi^*(K_X + \Delta) + E_+ - E_-,$$

where E_+ and E_- are effective π-exceptional divisors with no common components and the support of $\pi_*^{-1}\Delta + E_+ + E_-$ is a snc divisor.

Set $\tilde{f} := f \circ \pi$ and let $\tilde{D} \in |\pi^*D|$ be a general member. Setting $\tilde{\Delta} = \pi_*^{-1}\Delta + \frac{1}{m}\tilde{D} + E_-$, we obtain that $(\tilde{X}, \tilde{\Delta})$ is log canonical over the generic point of C and that

$$K_{\tilde{X}} + \tilde{\Delta} \sim_{\mathbf{Q}} \tilde{f}^*K_C + E_+ - \frac{1}{m}\tilde{f}^*A.$$

Furthermore, since E_+ is effective and π-exceptional, $\pi_*\mathscr{O}_{\tilde{X}}(lE_+) = \mathscr{O}_X$ for any $l \in \mathbf{N}$. Then for any $l \in \mathbf{N}$,

$$\tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C} + \tilde{\Delta})) \simeq \tilde{f}_*\mathscr{O}_{\tilde{X}}(l(mE_+ - \tilde{f}^*A)) \simeq \mathscr{O}_C(-lA).$$

Finally, observe that $\tilde{f}_*\mathscr{O}_{\tilde{X}}(lm(K_{\tilde{X}/C} + \tilde{\Delta}))$ is semi-positive (Campana '04), but that contradicts the fact that A is ample. This proves (1).

To prove (2), we assume to the contrary that (X, Δ) is klt over the generic point of C, and $-(K_{X/C} + \Delta)$ is nef and big. There exists an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor N on X such that $-(K_{X/C} + \Delta) - \varepsilon N$ is ample for $0 < \varepsilon \ll 1$. Let $0 < \varepsilon \ll 1$ be such that $(X, \Delta + \varepsilon N)$ is klt over the generic point of C. Set $\Delta' := \Delta + \varepsilon N$. Then

$$-(K_{X/C} + \Delta') = -(K_{X/C} + \Delta) - \varepsilon N$$

is ample, contradicting part (1) above. This proves (2). □

As a first application of Theorem 19, we derive a special property of Fano foliations with mild singularities. This property will play a key role in our study of Fano foliations.

Proposition 20. *Let X be a normal projective variety, and $\mathscr{F} \subsetneq T_X$ an algebraically integrable Fano foliation on X. If $(X, \mathscr{F})$ has log canonical singularities along a general leaf, then there is a common point in the closure of a general leaf of $\mathscr{F}$.*

Proof. Let W be the normalization of the closure in $\mathrm{Chow}(X)$ of the subvariety parametrizing general leaves of $\mathscr{F}$, and U the normalization of the universal cycle over W, with universal family morphisms:

$$\begin{array}{ccc} U & \xrightarrow{e} & X\,. \\ {\scriptstyle\pi}\big\downarrow & & \\ W & & \end{array}$$

Denote by U_w the fiber of π over a point $w \in W$.

For every $x \in X$, $\pi|_{e^{-1}(x)} : e^{-1}(x) \to W$ is finite. If we show that $\dim(e^{-1}(x)) \geqslant \dim(W)$ for some $x \in X$, then we conclude that $\pi\big(e^{-1}(x)\big) = W$, and thus $x \in e\big(U_w\big)$ for every $w \in W$, i.e., x is contained in the closure of a general leaf of $\mathscr{F}$.

Suppose to the contrary that $\dim\big(e^{-1}(x)\big) < \dim(W)$ for every $x \in X$. Let $C \subset W$ be a general complete intersection curve, and let U_C be the normalization of $\pi^{-1}(C)$, with natural morphisms $\pi_C : U_C \to C$ and $e_C : U_C \to X$. Since C is general, C is not contained in $\pi\big(e^{-1}(x)\big)$ for any $x \in X$, and thus the morphism $e_C : U_C \to X$ is finite onto its image. In particular, $e_C^*(-K_{\mathscr{F}})$ is ample.

Notice that $\mathscr{F}$ induces a generically surjective morphism $\Omega^{r_{\mathscr{F}}}_{U_C/C} \to e_C^* \det(\mathscr{F})^*$. By Lemma 21 below, after replacing C with a finite cover if necessary, we may assume that π_C has reduced fibers. This implies that $\det(\Omega^1_{U_C/C}) \simeq \mathscr{O}_{U_C}(K_{U_C/C})$. Thus there exists an effective integral divisor Δ_C on U_C such that

$$-(K_{U_C/C} + \Delta_C) = e_C^*(-K_{\mathscr{F}}).$$

Since $(X, \mathscr{F})$ has log canonical singularities along a general leaf, the pair (U_C, Δ_C) is log canonical over the generic point of C. But this contradicts Theorem 19, and the result follows. □

Lemma 21 (Bosch, Siegfried and Lütkebohmert '95)**.** *Let X be a quasi-projective variety, and $f : X \to C$ a flat surjective morphism onto a smooth curve with reduced general fiber. Then there exists a finite morphism $C' \to C$ such that $f' : X' \to C'$ is flat with reduced fibers. Here X' denotes the normalization of $C' \times_C X$ and $f' : X' \to C'$ is the morphism induced by the projection $C' \times_C X \to C'$.*

Proposition 22. *Let $\mathscr{F}$ be an algebraically integrable foliation on a smooth projective variety X. Suppose that $\mathscr{F}$ has log terminal singularities along a general leaf. Then* $\det(\mathscr{F})$ *is not nef and big.*

Proof. We let $C \subset \mathrm{Chow}(X)$ be a general complete curve contained in the closure of the subvariety parametrizing general leaves of $\mathscr{F}$. We denote by U the normalization of the universal cycle over C, with natural morphisms $\pi : U \to C$ and $e : U \to X$. Since C is general, $e : U \to X$ is birational onto its image. Thus if $-K_{\mathscr{F}}$ is nef and big, then so is $e^*(-K_{\mathscr{F}})$.

Note that $\mathscr{F}$ induces a Pfaff field $\Omega^r_{U/C} \to e^*\mathscr{O}_C(-K_{\mathscr{F}})$, where r denotes the rank of $\mathscr{F}$. By Lemma 21, after replacing C with a finite cover if necessary, we may assume that π has reduced fibers. This implies that $\det(\Omega^r_{U/C}) \simeq \mathscr{O}_U(K_{U/C})$. Thus there exists a canonically defined effective divisor Δ on U such that

$$-(K_{U/C} + \Delta) = e^*(-K_{\mathscr{F}}).$$

By assumption, (U, Δ) is log terminal over the generic point of C. So, by Theorem 19, $e^*(-K_{\mathscr{F}})$ cannot be nef and big. $\square$

5. Proof of Theorem 9

Step 1. Algebraic integrability.

Theorem 23 (Bost '01, Bogomolov-Mcquillan '01)**.** *Let X be a normal complex projective variety, and $\mathscr{F}$ a foliation on X. Let $C \subset X$ be a complete curve disjoint from the singular loci of X and $\mathscr{F}$. Suppose that the restriction $\mathscr{F}|_C$ is an ample vector bundle on C. Then the leaf of $\mathscr{F}$ through any point of C is an algebraic variety, and the leaf of $\mathscr{F}$ through a general point of C is moreover rationally connected.*

Proof. We show that the leaf of $\mathscr{F}$ through any point of C is an algebraic variety. Suppose that C is smooth for simplicity. Let $\mathscr{L}$ be an ample line bundle on X. Let $\hat{X}$ be the formal completion of X along C. By Frobenius' Theorem, there is a smooth formal subscheme $\hat{V}$ of dimension $r+1$ of $\hat{X}$ tangent to $\hat{\mathscr{F}}$. By replacing X with the Zariski closure of $\hat{V}$ in X, we may assume that $\hat{V}$ is dense in X. Thus the restriction map induces an inclusion

$$H^0(X, \mathscr{L}^{\otimes D}) \hookrightarrow H^0(\hat{X}, \hat{\mathscr{L}}^{\otimes D}).$$

Observe that it is enough to prove the estimate that $H^0(\hat{X}, \hat{\mathscr{L}}^{\otimes D}) = O(D^{\dim(\hat{V})})$ where D is an integer which goes to $+\infty$ since $\dim(X)$ is the degree of the Hilbert polynomial of $\mathscr{L}$.

To prove the estimate, one filters $H^0(\hat{X}, \hat{\mathscr{L}}^{\otimes D})$ by the order of vanishing along C :

$$rk\, H^0(\hat{X}, \hat{\mathscr{L}}^{\otimes D}) \leqslant \sum_{i \geqslant 0} rk\, H^0(C, S^i(N^*_{C/\hat{V}}) \otimes \mathscr{L}^{\otimes D}).$$

Finally, $rk\, H^0(C, S^i(N^*_{C/\hat{V}}) \otimes \mathscr{L}^{\otimes D})$ grows at most like $(i+D)^{\dim(C) + rk\, N_{C/\hat{V}} - 1}$, and vanishes when $\frac{i}{D}$ is large enough since $N_{C/\hat{V}} \simeq \mathscr{F}_{|C}$ is ample. □

Proposition 24 (Araujo,–'11). *Let $\mathscr{F}$ be a Fano foliation of rank r on an n-dimensional complex projective manifold X, with $1 \leq r \leq n-1$. If $\iota_{\mathscr{F}} \geqslant r$ then $\mathscr{F}$ has algebraic leaves.*

Proof. Note that X is uniruled. Let H be a minimal dominating family of rational curves on X, and let $\pi_0 : X_0 \to T_0$ be the H-rationally connected quotient of X.

Let $[f] \in H$ be a general point. Then $f^*\mathscr{F} \subset f^*T_X \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$, where $d = \deg(f^*T_X) - 2 \geq 0$. This implies that either $f^*\mathscr{F}$ is ample or $f^*\mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}$ (and $r - 2 \leqslant d$).

If $f^*\mathscr{F}$ is ample, then conclude using Theorem 23.

Suppose $f^*\mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus r-2} \oplus \mathscr{O}_{\mathbb{P}^1}$. We must have $f^*\mathscr{A} \simeq \mathscr{O}_{\mathbb{P}^1}(1)$. Hence H is unsplit. By Lemma 15, there is an inclusion $T_{X_0/T_0} \subset \mathscr{F}|_{X_0}$. We may assume that $T_{X_0/T_0} \subsetneq \mathscr{F}|_{X_0}$. Observe that a general fiber of π_0 is $\simeq \mathbf{P}^{r-1}$ by the Kobayashi-Ochiai Theorem. By Lemma 25, we may assume that $\mathrm{codim}_X(X \setminus X_0) \geq 2$ and that π_0 has integral fibers. Since $\mathscr{A}_{|F} \simeq \mathscr{O}_{\mathbf{P}^{r-1}}(1)$, π_0 is a $\mathbf{P}^{r-1}$-bundle and $\mathscr{F}_{|X_0}$ is the pullback via π_0 of a foliation by curves $\mathscr{M}_0 \subset T_{T_0}$ on T_0.

Let $C \to T_0$ be a smooth complete curve. Suppose $\deg_C(\mathscr{M}_0) \leqslant 0$. Let $X_C := X_0 \times_{T_0} C$ with morphism π_C onto C. Then $\det(T_{X_C/C}) \simeq \mathscr{A}_{|X_C}^{\otimes k} \otimes \pi_C^*\mathscr{M}_0^{\otimes -1}$ is ample. But this contradicts Theorem 19. Conclude using Theorem 23. □

Lemma 25 (Araujo,–, Kovács '08). *Let X be a smooth complex projective variety and $H_1, \ldots, H_k$ unsplit covering families of rational curves on X. Then there is an open subset*

X_0 *of* X*, with* $\mathrm{codim}_X(X \setminus X_0) \geq 2$*, a smooth variety* T_0*, and a proper surjective equidimensional morphism with irreducible and reduced fibers* $\pi_0 : X_0 \to T_0$ *which is the* $(H_1, \ldots, H_k)$*-rationally connected quotient of* X*.*

Step 2. We denote by F the closure of a general leaf of $\mathscr{F}$, $\tilde{F}$ its normalization, $(\tilde{F}, \tilde{\Delta})$ the corresponding log leaf. Recall that $K_{\tilde{F}} + \tilde{\Delta} = \tilde{e}^* K_{\mathscr{F}}$, where $\tilde{e} : \tilde{F} \to X$ is the natural morphism.

By Theorem 13, either $\tilde{F} \simeq \mathbf{P}^r$ and $\tilde{\Delta}$ is an hyperplane in $\mathbf{P}^r$ or $\tilde{F} \simeq Q_r$ and $\tilde{\Delta} = 0$.

Suppose that $\tilde{F} \simeq Q_r$ and $\tilde{\Delta} = 0$. Note that $\tilde{F}$ has klt singularities, but that contradicts Proposition 22.

Hence, we must have $\tilde{F} \simeq \mathbf{P}^r$, $\tilde{\Delta}$ must be an hyperplane in $\mathbf{P}^r$, and $\tilde{e}^* \mathscr{A} \simeq \mathscr{O}_{\mathbf{P}^r}(1)$.

By Proposition 20, there is a common point $x \in X$ in the closure of a general leaf. Let H be the family of rational curves on X induced by lines in $\tilde{F} \simeq \mathbf{P}^r$.

Then H is unsplit and $Locus(H_x) = X$. Thus $\rho(X) = 1$, $\dim(H_x) = n - 1$ and $-K_X \cdot H = n + 1$. Finally $-K_X = (n+1)c_1(\mathscr{L})$. Conclude using Kobayashi-Ochiai's Theorem.

6. On del Pezzo foliations

In this lecture I shall discuss del Pezzo foliations on complex projective manifolds. As I shall explain, such foliations have algebraic and rationally connected leaves, except for some well understood degree 1 foliations on $\mathbf{P}^n$. I will also discuss the classification of del Pezzo foliations having mild singularities.

Definition 26. del Pezzo foliation on X = Fano foliation of rank r and index $\iota = r - 1$.

Proposition 27 (Loray-Pereira-Touzet '11)**.** *Let* $\mathscr{F}$ *be a del Pezzo foliation on* $\mathbf{P}^n$ *of rank* r *(usually called degree 1 foliation on* $\mathbf{P}^n$*). Then*

(1) *either* $\mathscr{F}$ *is induced by a dominant rational map* $\mathbf{P}^n \dashrightarrow \mathbf{P}(1^{n-r}, 2)$*, defined by* $n - r$ *linear forms and one quadratic form (leaves are algebraic), or*

(2) $\mathscr{F}$ *is the linear pullback of a foliation on* $\mathbf{P}^{n-r-1}$ *induced by a global holomorphic vector field.*

Note that a foliation on $\mathbf{P}^{n-r-1}$ induced by a global holomorphic vector field may or may not have algebraic leaves.

Our first main result shows that this is the only case when a del Pezzo foliation is not algebraically integrable.

Theorem 28. *Let $\mathscr{F}$ be a del Pezzo foliation on a complex projective manifold $X \not\cong \mathbf{P}^n$. Then $\mathscr{F}$ is algebraically integrable, and its general leaves are rationally connected.*

Harder-Narasimhan filtration. Let X be an n-dimensional projective variety, and $\mathscr{A}$ an ample line bundle on X. Let $\mathscr{F}$ be a torsion-free sheaf on X. We define the slope of $\mathscr{F}$ with respect to $\mathscr{A}$ to be $\mu_{\mathscr{A}}(\mathscr{F}) = \frac{c_1(\mathscr{F})\cdot\mathscr{A}^{n-1}}{r_{\mathscr{F}}}$. We say that $\mathscr{F}$ is *$\mu_{\mathscr{A}}$-semistable* if for any subsheaf $\mathscr{E}$ of $\mathscr{F}$ we have $\mu_{\mathscr{A}}(\mathscr{E}) \leq \mu_{\mathscr{A}}(\mathscr{F})$.

Given a torsion-free sheaf $\mathscr{F}$ on X, there exists a filtration of $\mathscr{F}$ by (torsion-free) subsheaves

$$0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \ldots \subsetneq \mathscr{E}_k = \mathscr{F},$$

with $\mu_{\mathscr{A}}$-semistable quotients $\mathscr{Q}_i = \mathscr{E}_i/\mathscr{E}_{i-1}$, and such that $\mu_{\mathscr{A}}(\mathscr{Q}_1) > \mu_{\mathscr{A}}(\mathscr{Q}_2) > \ldots > \mu_{\mathscr{A}}(\mathscr{Q}_k)$. This is called the *Harder-Narasimhan filtration* of $\mathscr{F}$.

Let X be a normal projective variety, $\mathscr{A}$ an ample line bundle on X, and $\mathscr{F}$ a coherent torsion free sheaf of $\mathscr{O}_X$-modules. Let $m_i \in \mathbf{N}$, $1 \leq i \leq \dim(X) - 1$, be large enough integers, $H_i \in |m_i\mathscr{A}|$ be general members, and set $C := H_1 \cap \cdots \cap H_{\dim(X)-1}$. By the Mehta-Ramanathan Theorem, the Harder-Narasimhan filtration of $\mathscr{F}$ with respect to $\mathscr{A}$ commutes with restriction to C. In this case we say that C is a *general complete intersection curve for $\mathscr{F}$ and $\mathscr{A}$ in the sense of Mehta-Ramanathan*. If $\mathscr{F}$ and $\mathscr{A}$ are clear from the context, we simply say that C is a *general complete intersection curve*.

Proof of Theorem if $\rho(X) = 1$. Note that X is uniruled. Since $\rho(X) = 1$, X is in fact a Fano manifold. Let $\mathscr{A}$ be an ample line bundle on X such that $\mathrm{Pic}(X) = \mathbf{Z}[\mathscr{A}]$. By assumption we have $\det(\mathscr{F}) = \mathscr{A}^{\otimes r-1}$, where $r \geqslant 2$.

Let $C \subset X$ be a general complete intersection curve. Notice that $\mathscr{F}_{|C}$ is locally free. If $\mathscr{F}$ is semi-stable, then $\mathscr{F}_{|C}$ is semi-stable with $\mu(\mathscr{F}_{|C}) > 0$. Hence $\mathscr{F}_{|C}$ is ample, and the claim follows.

Assume $\mathscr{F}$ is not semi-stable and let $\mathscr{F}_1 \subset \mathscr{F}$ be the maximally destabilizing subsheaf. Then $\mathscr{F}_1$ defines a foliation on X. Write $\det(\mathscr{F}_1) = \mathscr{A}^{\otimes k_1}$. Since $\mu(\mathscr{F}_1) > \mu(\mathscr{F})$, we have $\frac{k_1}{r_1} > \frac{r-1}{r}$, and thus $k_1 \geq r_1$. By the foliated version of the Kobayahi-Ochiai's theorem, $(X, \mathscr{A}) \simeq (\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(1))$. Use Loray-Pereira-Touzet '11 to conclude. $\square$

Proposition 29. *Let $n \geqslant 3$ be an integer and let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 3$. Let ι be a positive integer. Then there exists a foliation on X of rank $2 \leqslant r \leqslant n-1$ and index ι if and only if $d-1 \leqslant r-\iota$.*

Let $n \geqslant 3$ be an integer and let $X \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 2$. Then there exists a foliation on X of rank $2 \leqslant r \leqslant n-1$ and index $\iota = r-1$ if and only if $d = 2$.

On del Pezzo foliations on $\mathbf{P}^s$-bundles over $\mathbf{P}^m$. We follow Grothendieck's notation: for a locally free sheaf $\mathscr{E}$ on a variety Y, $\mathbf{P}_Y(\mathscr{E})$ is the space of hyperplanes in fibers of $\mathscr{E}$.

Let $\pi : X \to \mathbb{P}^m$ be a P^s-bundles.

If $s = 1$, then $X \simeq \mathbf{P}^1 \times \mathbf{P}^m$, and $\mathscr{F}$ is the pullback via π of a foliation $\mathscr{O}(1)^{\oplus i} \subset T_{\mathbf{P}^m}$ for some $i \in \{1, 2\}$. For $s \geq 2$, we have the following result.

Theorem 30. *Let $\mathscr{F} \subsetneq T_X$ be a del Pezzo foliation on a $\mathbb{P}^s$-bundle $\pi : X \to \mathbf{P}^m$, with $s \geq 2$. Suppose that $\mathscr{F} \not\subset T_{X/\mathbf{P}^m}$. Then there is an exact sequence of vector bundles $0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$ on $\mathbf{P}^m$ such that $X \simeq \mathbf{P}_{\mathbf{P}^m}(\mathscr{E})$, and $\mathscr{F}$ is the pullback via the relative linear projection $X \dashrightarrow Z = \mathbf{P}_{\mathbf{P}^m}(\mathscr{K})$ of a foliation $q^* \det(\mathscr{Q}) \subset T_Z$. Here $q : Z \to \mathbf{P}^m$ denotes the natural projection. Moreover, one of the following holds.*

1. *$m = 1$, $\mathscr{Q} \simeq \mathscr{O}_{\mathbf{P}^1}(1)$, $\mathscr{K}$ is an ample vector bundle such that $\mathscr{K} \not\simeq \mathscr{O}_{\mathbf{P}^1}(a)^{\oplus s}$ for any integer a, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r = 2)$.*
2. *$m = 1$, $\mathscr{Q} \simeq \mathscr{O}_{\mathbf{P}^1}(2)$, $\mathscr{K} \simeq \mathscr{O}_{\mathbf{P}^1}(a)^{\oplus s}$ for some integer $a \geqslant 1$, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r = 2)$.*

(3) *$m = 1$, $\mathscr{Q} \simeq \mathscr{O}_{\mathbf{P}^1}(1) \oplus \mathscr{O}_{\mathbf{P}^1}(1)$, $\mathscr{K} \simeq \mathscr{O}_{\mathbf{P}^1}(a)^{\oplus s-1}$ for some integer $a \geqslant 1$, and $\mathscr{E} \simeq \mathscr{Q} \oplus \mathscr{K}$ $(r = 3)$.*

(4) *$m \geqslant 2$, $\mathscr{Q} \simeq \mathscr{O}_{\mathbf{P}^m}(1)$, and $\mathscr{K}$ is V-equivariant for some $V \in H^0\big(\mathbb{P}^m, T_{\mathbf{P}^m} \otimes \mathscr{O}_{\mathbf{P}^m}(-1)\big) \setminus \{0\}$ $(r = 2)$.*

Conversely, given $\mathscr{K}$, $\mathscr{E}$ and $\mathscr{Q}$ satisfying any of the conditions above, there exists a del Pezzo foliation of that type.

Let $\mathscr{K}$ be a locally free sheaf on a smooth variety Y, $\mathscr{W}$ a locally free sheaf of rank one on Y, and $V \in H^0(Y, T_Y \otimes \mathscr{W})$ be a twisted vector field on Y. Recall that $\mathscr{K}$ is called V-equivariant if there exists a $\mathbf{C}$-linear map $\tilde{V} : \mathscr{K} \to \mathscr{W} \otimes \mathscr{K}$ lifting the derivation $V : \mathscr{O}_Y \to \mathscr{W}$ (Carrell-Lieberman) $(\tilde{V}(fs) = V(f)s + f\tilde{V}(s))$.

Let $at(\mathscr{K}) \in H^1(Y, \mathscr{E}nd(\mathscr{K}) \otimes \Omega^1_Y)$ be the Atiyah class of $\mathscr{K}$. Then $\mathscr{K}$ is V-equivariant if and only if $V_* at(\mathscr{K}) \in H^1(Y, \mathscr{E}nd(\mathscr{K}) \otimes \mathscr{W})$ vanishes.

Theorem 31. *Let X be a smooth projective variety with Picard number $\rho(X) \geqslant 2$. Let $\mathscr{F} \subset T_X$ be a codimension 1 del Pezzo foliation. Then X is a $\mathbf{P}^s$-bundle over $\mathbf{P}^1$ for some $s \geqslant 2$ and $\mathscr{F} \not\subset T_{X/\mathbf{P}^1}$.*

In particular, $\dim(X) \leqslant 4$.

Theorem 32 (Loray-Pereira-Touzet '11)**.** *Let X be a smooth projective variety with Picard number $\rho(X) = 1$. Let $\mathscr{F} \subset T_X$ be a codimension 1 del Pezzo foliation. Then either $X \simeq \mathbf{P}^n$ or $X \simeq Q_n \subset \mathbf{P}^{n+1}$ and $\mathscr{F}$ is induced by a degree 0 foliation on $\mathbf{P}^{n+1}$.*

From Theorem 28, we know that a del Pezzo foliation $\mathscr{F}$ on a complex projective manifold different from $\mathbf{P}^n$ is algebraically integrable. Hence it makes sense to ask that $\mathscr{F}$ has log canonical singularities along a general leaf. Under this restriction we have the following classification result.

Theorem 33. *Let $\mathscr{F}$ be a del Pezzo foliation of rank r on a complex projective manifold X with $\rho(X) \geqslant 2$. Suppose that $\mathscr{F}$ is locally free with log canonical singularities along a general leaf. Then X is a $\mathbf{P}^s$-bundle over $\mathbf{P}^m$.*

Note that $\mathscr{F} \not\subseteq T_{X/\mathbf{P}^m}$ by 20.

Theorem 34. *Let $\mathscr{F} \subsetneq T_X$ be a regular foliation on a complex projective manifold X with Picard number $\rho(X) = 1$. Then $-K_{\mathscr{F}}$ is not ample.*

Proof. Set $\mathscr{Q} = T_X/\mathscr{F}$. Notice that $\det(\mathscr{Q}) \simeq \mathscr{O}_X(-K_{\mathscr{F}}) \otimes \mathscr{L}^{\otimes -1}$ where $\mathscr{L} = \det(\mathscr{F})$. By Baum-Bott ('70), $\det(\mathscr{Q})^{\dim(X)} = 0$. Since $\rho(X) = 1$, we must have $\det(\mathscr{Q}) \equiv 0$. This implies that X is a Fano manifold. Finally, $\det(\mathscr{Q}) \simeq \mathscr{O}_X$, and $h^0(X, \Omega_X^{\dim(X)-r}) \neq 0$. But $h^0(X, \Omega_X^{\dim(X)-r}) = h^{\dim(X)-r}(X, \mathscr{O}_X)$ by Hodge symmetry. By Kodaira vanishing theorem, we must have $r = \dim(X)$, a contradiction. □

Theorem 35. *Let $\mathscr{F}$ be a codimension 1 regular foliation on a complex projective manifold. Then $-K_{\mathscr{F}}$ is not ample.*

Proof. We assume to the contrary that $-K_{\mathscr{F}}$ is not ample. Set $\mathscr{L} := \mathscr{O}_X(-K_{\mathscr{F}})$. The exact sequence

$$0 \to \mathscr{F} \to T_X \to \mathscr{O}_X(-K_X) \otimes \mathscr{L}^{\otimes -1} \to 0$$

gives an injective map

$$\mathscr{O}_X(K_X) \otimes \mathscr{L} \hookrightarrow \Omega_X^1.$$

By Baum-Bott '72, there exists $\alpha \in H^1(X, \mathscr{O}_X(K_X) \otimes \mathscr{L})$ that maps to $c_1(\mathscr{O}(K_X) \otimes \mathscr{L}) \in H^1(X, \Omega_X^1)$. By Kodaira's vanishing theorem, $h^1(X, \mathscr{O}_X(K_X) \otimes \mathscr{L}) = 0$ and we must have $c_1(\mathscr{O}_X(K_X) \otimes \mathscr{L}) \equiv 0$. Thus X is a Fano manifold and $\mathscr{O}_X(-K_X) \simeq \mathscr{L}$. This implies that $h^0(X, \Omega_X^1) \neq 0$, a contradiction. □

Lemma 36. *Let X be a smooth variety, and $\mathscr{F} \subsetneq T_X$ a regular foliation. Set $\mathscr{Q} = T_X/\mathscr{F}$. Then $at(\mathscr{Q}) \in H^1(X, \Omega_X^1 \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q}))$ is in the image of the map $H^1(X, \mathscr{Q}^* \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})) \to H^1(X, \Omega_X^1 \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q}))$.*

Proof. The cohomology of the exact sequence of sheaves on X

$$0 \to \mathscr{Q}^* \to \Omega_X^1 \to \mathscr{F}^* \to 0,$$

yields the exact sequence

$$H^1(X, \mathscr{Q}^* \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})) \to H^1(X, \Omega^1_X \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})) \xrightarrow{\delta} H^1(X, \mathscr{F}^* \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})).$$

Thus it is enough to show that $\delta(at(\mathscr{Q})) = 0$.

Let q denotes the rank of $\mathscr{Q}$. Choose an affine open cover $(U_i)_{i\in I}$ of X such that $\mathscr{Q}$ admits a frame $\alpha_i : \mathscr{O}^q_{U_i} \xrightarrow{\sim} \mathscr{Q}^*|_{U_i}$ for each U_i also viewed as a line vector whose entries are local sections of $\mathscr{Q}^* \subset \Omega^1_X$. By assumption, $\mathscr{F}$ is stable under the Lie bracket. This is equivalent to saying that $d\mathscr{Q}^* \subset \mathscr{Q}^* \wedge \Omega^1_X$. Thus, there exist a matrix β_i whose entries are local sections of $\mathscr{Q}^* \subset \Omega^1_X$ over U_i such that $d\alpha_i = \alpha_i \wedge \beta_i$.

For $i, j \in I$, define $f_{ij} := \alpha_j^{-1}|_{U_{ij}} \circ \alpha_i|_{U_{ij}}$. Then

$$at(\mathscr{Q}) = \big[(-f_j|_{U_{ij}} \circ df_{ij}|_{U_{ij}} \circ f_i^{-1}|_{U_{ij}})_{i,j}\big] \in H^1(X, \Omega^1_X \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})).$$

Note that $\alpha_i = \alpha_j \cdot f_{ij}$ on U_{ij}. Thus

$$d\alpha_i = d\alpha_j \cdot f_{ij} + \alpha_j \wedge df_{ij},$$

and

$$\alpha_i \wedge \beta_i = \alpha_j \cdot f_{ij} \wedge \beta_i = \alpha_j \wedge \beta_j \cdot f_{ij} + \alpha_j \wedge df_{ij}.$$

Let $\vec{v} \in H^0(U_{ij}, \mathscr{F}_{|U_{ij}})$. Then

$$\begin{aligned} \alpha_j \cdot f_{ij} \cdot \beta_i(\vec{v}) &= i_{\vec{v}}(\alpha_j \cdot f_{ij} \wedge \beta_i) \\ &= i_{\vec{v}}(\alpha_j \wedge \beta_j \cdot f_{ij} + \alpha_j \wedge df_{ij}). \\ &= \alpha_j \cdot \beta_j(\vec{v}) \cdot f_{ij} + \alpha_j \cdot df_{ij}(\vec{v}). \end{aligned}$$

This implies that

$$\delta(at(\mathscr{Q})) = \big[(-f_j|_{U_{ij}} \circ df_{ij}|_{U_{ij}} \circ f_i^{-1}|_{U_{ij}})_{i,j}\big] = \big[(\beta_j|_{U_{ij}} - \beta_i|_{U_{ij}})_{i,j}\big] = 0 \in H^1(X, \mathscr{F}^* \otimes \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{Q})).$$

This proves our claim. $\square$