J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2014-0110

Journal für die reine und angewandte Mathematik


# Codimension 1 Mukai foliations on complex projective manifolds

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**Abstract.** In this paper we classify codimension 1 Mukai foliations on complex projective manifolds.

**Contents**



## 1. Introduction

This paper is concerned with codimension 1 holomorphic foliations on complex projective manifolds. When the ambient space is $\mathbb{P}^n$, the problem of classifying and describing such foliations is classical. The degree $\deg(\mathscr{F})$ of a codimension 1 holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^n$ is defined as the number of tangencies of a general line with $\mathscr{F}$. When $\deg(\mathscr{F})$ is low, $\mathscr{F}$ presents very special behavior. In particular, those with $\deg(\mathscr{F}) \leqslant 2$ have been classified. Codimension 1 foliations on $\mathbb{P}^n$ with $\deg(\mathscr{F}) \leqslant 1$ were classified in [31]. If $\deg(\mathscr{F}) = 0$, then $\mathscr{F}$ is induced by a pencil of hyperplanes, i.e., it is the relative tangent sheaf to a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$. If $\deg(\mathscr{F}) = 1$, then either of the following holds:

- $\mathscr{F}$ is induced by a pencil of hyperquadrics containing a double hyperplane,
- $\mathscr{F}$ is the linear pullback of a foliation on $\mathbb{P}^2$ induced by a global holomorphic vector field.

The first named author was partially supported by CNPq and Faperj Research Fellowships. The second named author was partially supported by the CLASS project of the ANR.

Codimension 1 foliations of degree 2 on $\mathbb{P}^n$ were classified in [12]. The space of such foliations has six irreducible components, and much is known about them. In particular, when $n \geqslant 4$, the leaves of these foliations are always covered by rational curves.

In this paper we extend this classification to arbitrary complex projective manifolds. In order to do so, we reinterpret the degree of a foliation $\mathscr{F}$ on $\mathbb{P}^n$ as a numerical invariant defined in terms of its *canonical class* $K_{\mathscr{F}} := -c_1(\mathscr{F})$. For a codimension 1 foliation $\mathscr{F}$ on $\mathbb{P}^n$, $\deg(\mathscr{F}) = n - 1 + \deg(K_{\mathscr{F}})$. So, foliations with low degree are precisely those with $-K_{\mathscr{F}}$ most positive.

**Definition 1** ([3]). A *Fano foliation* is a holomorphic foliation $\mathscr{F}$ on a complex projective manifold $X$ such that $-K_{\mathscr{F}}$ is ample. The *index* $\iota_{\mathscr{F}}$ of $\mathscr{F}$ is the largest integer dividing $-K_{\mathscr{F}}$ in $\mathrm{Pic}(X)$.

It follows from [8, Theorem 0.1] that the leaves of a Fano foliation $\mathscr{F}$ are always covered by positive dimensional rationally connected algebraic subvarieties of $X$. Recent results suggest that the higher the index of $\mathscr{F}$ is, the higher the dimension of these subvarieties is. In order to state this precisely, we define the *algebraic* and *transcendental* parts of a holomorphic foliation.

**Definition 2.** Let $\mathscr{F}$ be a holomorphic foliation of rank $r_{\mathscr{F}}$ on a normal variety $X$. There exist a normal variety $Y$, unique up to birational equivalence, a dominant rational map with connected fibers $\varphi : X \dashrightarrow Y$, and a holomorphic foliation $\mathscr{G}$ on $Y$ of rank

$$r_{\mathscr{G}} = r_{\mathscr{F}} - (\dim(X) - \dim(Y))$$

such that the following hold (see [37, Section 2.4]):

(1) $\mathscr{G}$ is purely transcendental, i.e., there is no positive-dimensional algebraic subvariety through a general point of $Y$ that is tangent to $\mathscr{G}$,

(2) $\mathscr{F}$ is the pullback of $\mathscr{G}$ via $\varphi$ (see Paragraph 12 for this notion).

The foliation on $X$ induced by $\varphi$ is called the *algebraic part* of $\mathscr{F}$, and its rank is the *algebraic rank* of $\mathscr{F}$, which we denote by $r^a_{\mathscr{F}}$. When $r^a_{\mathscr{F}} = r_{\mathscr{F}}$, we say that $\mathscr{F}$ is *algebraically integrable*.

**Theorem 3.** *Let $\mathscr{F}$ be a Fano foliation of rank $r_{\mathscr{F}}$ and index $\iota_{\mathscr{F}}$ on a complex projective manifold $X$. Then $\iota_{\mathscr{F}} \leqslant r_{\mathscr{F}}$, and equality holds only if $X \cong \mathbb{P}^n$* ([6, Theorem 1.1]). *In this case, by* [14, Théorème 3.8], *$\mathscr{F}$ is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r_{\mathscr{F}}}$. In particular, one has $r^a_{\mathscr{F}} = r_{\mathscr{F}}$.*

In analogy with the case of Fano manifolds, we define *del Pezzo foliations* to be Fano foliations $\mathscr{F}$ with index $\iota_{\mathscr{F}} = r_{\mathscr{F}} - 1 \geqslant 1$. Del Pezzo foliations were investigated in [3, 4]. By [3, Theorem 1.1], if $\mathscr{F}$ is a del Pezzo foliation on a complex projective manifold $X$, then $r^a_{\mathscr{F}} = r_{\mathscr{F}}$, except when $X \cong \mathbb{P}^n$ and $\mathscr{F}$ is the pullback under a linear projection of a transcendental foliation on $\mathbb{P}^{n-r_{\mathscr{F}}+1}$ induced by a global vector field, in which case $r^a_{\mathscr{F}} = r_{\mathscr{F}} - 1$.

The following is the complete classification of codimension 1 del Pezzo foliations on complex projective manifolds. For manifolds with Picard number 1, the classification was obtained in [38, Proposition 3.7], while [4, Theorem 1.3] deals with mildly singular varieties of arbitrary Picard number.

**Theorem 4** ([4, Theorem 1.3]). *Let $\mathscr{F}$ be a codimension 1 del Pezzo foliation on an $n$-dimensional complex projective manifold $X$.*

(1) *Suppose that $\rho(X) = 1$. Then one of the following holds:*

- *$X \cong \mathbb{P}^n$ and $\mathscr{F}$ is a degree 1 foliation,*
- *$X \cong Q^n \subset \mathbb{P}^{n+1}$ and $\mathscr{F}$ is induced by a pencil of hyperplane sections.*

(2) *Suppose that $\rho(X) \geqslant 2$. Then $n \in \{3, 4\}$ and there exist*

- *an exact sequence of vector bundles on $\mathbb{P}^1$,*
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$
*with $\mathscr{E}$ ample of rank $n$ and $\operatorname{rank}(\mathscr{K}) = 2$,*
- *a foliation by curves $\mathscr{C}$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$, generically transverse to the natural projection $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$, induced by a nonzero global section of $T_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})} \otimes q^* \det(\mathscr{V})^*$,*

*such that $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$, and $\mathscr{F}$ is the pullback of $\mathscr{C}$ via the induced relative linear projection $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. Moreover, one of the following holds:*

(a) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some positive integer $a$,*

(b) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some positive integer $a$,*

(c) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b), \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b))$ *for distinct positive integers $a$ and $b$.*

Next we define *Mukai foliations* as Fano foliations $\mathscr{F}$ with index $\iota_{\mathscr{F}} = r_{\mathscr{F}} - 2 \geqslant 1$. When $X = \mathbb{P}^n$ and $r_{\mathscr{F}} = n - 1 \geqslant 3$, the Mukai condition is equivalent to $\deg(\mathscr{F}) = 2$. One checks from the classification in [12] that $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 2$.

The aim of this paper is to classify codimension 1 Mukai foliations on complex projective manifolds $X \not\cong \mathbb{P}^n$. The classification is summarized in Theorems 5 and 8, distinguishing the cases when $\rho(X) = 1$ and $\rho(X) > 1$.

**Theorem 5.** *Let $\mathscr{F}$ be a codimension 1 Mukai foliation on an $n$-dimensional complex projective manifold $X \not\cong \mathbb{P}^n$ with $\rho(X) = 1$, $n \geqslant 4$. Then the pair $(X, \mathscr{F})$ satisfies one of the following conditions:*

(1) *$X \cong Q^n \subset \mathbb{P}^{n+1}$ and $\mathscr{F}$ is one of the following:*

(a) *$\mathscr{F}$ is cut out by a pencil of hyperquadrics of $\mathbb{P}^{n+1}$ containing a double hyperplane. In this case, $r^a_{\mathscr{F}} = r_{\mathscr{F}}$.*

(b) *$\mathscr{F}$ is the pullback under the restriction to $X$ of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$ of a foliation on $\mathbb{P}^2$ induced by a global vector field. In this case, $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$.*

(2) *$X$ is a Fano manifold with $\rho(X) = 1$ and index $\iota_X = n - 1$, and $\mathscr{F}$ is induced by a pencil in $|\mathscr{O}_X(1)|$, where $\mathscr{O}_X(1)$ is the ample generator of $\operatorname{Pic}(X)$. In this case, $r^a_{\mathscr{F}} = r_{\mathscr{F}}$.*

**Remark 6.** In case (2), by Fujita's classification (see Section 3.1), $X$ is isomorphic to one of the following:

- a cubic hypersurface in $\mathbb{P}^{n+1}$.
- an intersection of two hyperquadrics in $\mathbb{P}^{n+2}$.

- a linear section of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ under the Plücker embedding.
- a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(2,1,\dots,1)$.
- a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,\dots,1)$.

The classification of codimension 1 Mukai foliation on complex projective manifolds $X$ with $\rho(X) > 1$ in Theorem 8 is much longer and intricate. For the reader's convenience, before stating it, we present some of its immediate consequences.

**Theorem 7.** *Let $X$ be an $n$-dimensional complex projective manifold with $\rho(X) > 1$, and $n \geqslant 4$. Let $\mathscr{F}$ be a codimension 1 Mukai foliation on $X$. Then the following hold:*

(1) $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$.

(2) *If $r^a_{\mathscr{F}} = r_{\mathscr{F}} - 1$, then $X$ is a projective space bundle over a curve or a surface, and $\mathscr{F}$ is the pullback of a codimension 1 foliation on a surface or a threefold.*

(3) *If $n \geqslant 7$, then $X$ is a $\mathbb{P}^{n-1}$-bundle over a curve.*

(4) *If $X$ is not rationally connected, then $X$ is a $\mathbb{P}^{n-1}$-bundle over a curve.*

**Theorem 8.** *Let $X$ be an $n$-dimensional complex projective manifold with $\rho(X) > 1$, and $n \geqslant 4$. Let $\mathscr{F}$ be a codimension 1 Mukai foliation on $X$. Then one of the following holds:*

(1) *$X$ admits a $\mathbb{P}^{n-1}$-bundle structure $\pi : X \to \mathbb{P}^1$, $r^a_{\mathscr{F}} = r_{\mathscr{F}}$, and the restriction of $\mathscr{F}$ to a general fiber of $\pi$ is induced by a pencil of hyperquadrics of $\mathbb{P}^{n-1}$ containing a double hyperplane.*

(2) *There exist*

- *a complete smooth curve $C$, together with an exact sequence of vector bundles on $C$,*

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

*with $\mathscr{E}$ ample of rank $n$, and $r := \mathrm{rank}(\mathscr{K}) \in \{2,3\}$,*

- *a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}_C(\mathscr{K})$, generically transverse to the natural projection $p : \mathbb{P}_C(\mathscr{K}) \to C$, satisfying*

$$\det(\mathscr{G}) \cong p^*(\det(\mathscr{V})) \otimes \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(r-3) \quad \textit{and} \quad r^a_{\mathscr{G}} \geqslant r_{\mathscr{G}} - 1,$$

*such that $X \cong \mathbb{P}_C(\mathscr{E})$, and $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the induced relative linear projection $\mathbb{P}_C(\mathscr{E}) \dashrightarrow \mathbb{P}_C(\mathscr{K})$. In this case, $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$.*

(3) *$X$ admits a $Q^{n-1}$-bundle structure $\pi : X \to \mathbb{P}^1$, $n \in \{4,5\}$, $r^a_{\mathscr{F}} = r_{\mathscr{F}}$, and the restriction of $\mathscr{F}$ to a general fiber of $\pi$ is induced by a pencil of hyperplane sections of $Q^{n-1}$. More precisely, there exist*

- *an exact sequence of vector bundles on $\mathbb{P}^1$,*

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

*with* $\mathrm{rank}(\mathscr{E}) = n+1$, $\mathrm{rank}(\mathscr{K}) = 2$, *and natural projections $\pi : \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}^1$ and $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$,*

- *an integer $b$ and a foliation by rational curves $\mathscr{G} \cong q^*(\det(\mathscr{V}) \otimes \mathscr{O}_{\mathbb{P}^1}(b))$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$,* *such that $X \in |\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2) \otimes \pi^*\mathscr{O}(b)|$, and $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the restriction to $X$ of the relative linear projection $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. Moreover, one of the following holds:*

(a) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$, and $b = 2$ (with $n = 4$).*

(b) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$, and $b = 1$ (with $n = 4$).*

(c) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$, and $b = 0$ (with $n = 4$).*

(d) $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}$ *for some integer $a$, and $\mathscr{E}$ is an ample vector bundle of rank 5 or 6 with $\deg(\mathscr{E}) = 2 + 2a - b$ (with $n \in \{4, 5\}$).*

(e) $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(c)$ *for distinct integers $a$ and $c$, and $\mathscr{E}$ is an ample vector bundle of rank 5 or 6 with $\deg(\mathscr{E}) = 1 + a + c - b$ (with $n \in \{4, 5\}$).*

(4) *There exist*

- *a smooth projective surface $S$, together with an exact sequence of $\mathscr{O}_S$-modules,*

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0,$$

*where $\mathscr{K}$, $\mathscr{E}$, and $\mathscr{V} := \mathscr{Q}^{**}$ are vector bundles on $S$, $\mathscr{E}$ is ample of rank $n-1$, and* $\operatorname{rank}(\mathscr{K}) = 2$,

- *a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}_S(\mathscr{K})$, generically transverse to the natural projection $q : \mathbb{P}_S(\mathscr{K}) \to S$, satisfying $\det(\mathscr{G}) \cong q^*\det(\mathscr{V})$ and $r^a_{\mathscr{G}} \geqslant 1$,*

*such that $X \cong \mathbb{P}_S(\mathscr{E})$, and $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the induced relative linear projection $\mathbb{P}_S(\mathscr{E}) \dashrightarrow \mathbb{P}_S(\mathscr{K})$. In this case, $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$. Moreover, one of the following holds:*

(a) $S \cong \mathbb{P}^2$, $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{P}^2}(i)$ *for some $i \in \{1, 2, 3\}$, and $4 \leqslant n \leqslant 3 + i$.*

(b) *$S$ is a del Pezzo surface $\not\cong \mathbb{P}^2$, $\det(\mathscr{V}) \cong \mathscr{O}_S(-K_S)$, and $4 \leqslant n \leqslant 5$.*

(c) $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, *$\det(\mathscr{V})$ is a line bundle of type $(1, 1)$, $(2, 1)$ or $(1, 2)$, and $n = 4$.*

(d) $S \cong \mathbb{F}_e$ *for some integer $e \geqslant 1$, $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{F}_e}(C_0 + (e + i)f)$, where $i \in \{1, 2\}$, $C_0$ is the minimal section of the natural morphism $\mathbb{F}_e \to \mathbb{P}^1$, $f$ is a general fiber, and $n = 4$.*

(5) *$n = 5$, $X$ is the blowup of one point $P \in \mathbb{P}^5$, and $\mathscr{F}$ is induced by a pencil of hyperplanes in $\mathbb{P}^5$ containing $P$ in its base locus.*

(6) *$n = 4$, $X$ is the blowup of $\mathbb{P}^4$ at $m \leqslant 8$ points in general position on a plane $\mathbb{P}^2 \cong S \subset \mathbb{P}^4$, and $\mathscr{F}$ is induced by the pencil of hyperplanes in $\mathbb{P}^4$ with base locus $S$.*

(7) *$n = 4$, $X$ is the blowup of a smooth quadric $Q^4$ at $m \leqslant 7$ points in general position on a codimension 2 linear section $Q^2 \cong S \subset Q^4$, and $\mathscr{F}$ is induced by the pencil of hyperplane sections of $Q^4 \subset \mathbb{P}^5$ with base locus $S$.*

**Remark 9.** Foliations $\mathscr{G}$ that appear in Theorem 8 (2) when $r = 3$ are classified in Proposition 46. When $r = 2$, we construct examples in Example 48.

Foliations $\mathscr{G}$ that appear in Theorem 8 (4) are classified in Remark 59, Proposition 61, Proposition 66, and Proposition 69.

This paper is organized as follows.

In Section 2, we review basic definitions and results about holomorphic foliations and Fano foliations.

Section 3 is devoted to Mukai foliations on manifolds with Picard number 1. First we show that if $X$ is a manifold with $\rho(X) = 1$ admitting a codimension 1 Mukai foliation $\mathscr{F}$, then $X$ is a Fano manifold with index $\iota_X \geqslant \dim(X) - 1$ (Lemma 28). The proof of Theorem 5 relies on the classification of such manifolds, which is reviewed in Section 3.1. The proof distinguishes two cases, depending on whether or not $\mathscr{F}$ contains a codimension 2 del Pezzo subfoliation $\mathscr{G}$. Such foliations are classified in Section 3.2 (Theorem 29).

Section 4 is devoted to Mukai foliations on manifolds with Picard number $> 1$. The existence of a codimension 1 Mukai foliation $\mathscr{F}$ on a manifold $X$ with $\rho(X) > 1$ implies the existence of an extremal ray in $\overline{\mathrm{NE}}(X)$ with large length. We use adjunction theory to classify all possible contractions of such extremal rays (Theorem 40). In order to prove Theorem 8, we analyze the behavior of the foliation $\mathscr{F}$ with respect to the contraction of a large extremal ray. This is done separately for each type of contraction. Section 4.2 deals with projective space bundles over curves. Section 4.3 deals with quadric bundles over curves. Section 4.4 deals with projective space bundles over surfaces. Birational contractions are treated in Section 4.5.

**Notation and conventions.** We always work over the field $\mathbb{C}$ of complex numbers. Varieties are always assumed to be irreducible. We denote by $\mathrm{Sing}(X)$ the singular locus of a variety $X$.

Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on a variety $X$.

- We denote by $\mathscr{F}^*$ the sheaf $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.
- If $r$ is the generic rank of $\mathscr{F}$, we denote by $\det(\mathscr{F})$ the sheaf $(\wedge^r \mathscr{F})^{**}$.
- If $\mathscr{G}$ is another sheaf of $\mathscr{O}_X$-modules on $X$, we denote by $\mathscr{F}\,[\otimes]\,\mathscr{G}$ the sheaf $(\mathscr{F} \otimes \mathscr{G})^{**}$.

If $\mathscr{E}$ is a locally free sheaf of $\mathscr{O}_X$-modules on a variety $X$, we denote by $\mathbb{P}_X(\mathscr{E})$ the Grothendieck projectivization $\mathrm{Proj}_X(\mathrm{Sym}(\mathscr{E}))$, and by $\mathscr{O}_{\mathbb{P}}(1)$ its tautological line bundle.

Suppose $X$ is a normal variety and let $X \to Y$ be any morphism. We denote by $T_{X/Y}$ the sheaf $(\Omega^1_{X/Y})^*$. In particular, $T_X = (\Omega^1_X)^*$.

If $X$ is a smooth variety and $D$ is a reduced divisor on $X$ with simple normal crossings support, we denote by $\Omega^1_X(\log D)$ the sheaf of differential 1-forms with logarithmic poles along $D$, and by $T_X(-\log D)$ its dual sheaf $\Omega^1_X(\log D)^*$. Notice that

$$\det(\Omega^1_X(\log D)) \cong \mathscr{O}_X(K_X + D).$$

We denote by $Q^n$ a (possibly singular) quadric hypersurface in $\mathbb{P}^{n+1}$.

Given line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on two varieties $X$ and $Y$, we denote by $\mathscr{L}_1 \boxtimes \mathscr{L}_2$ the line bundle $\pi_1^*\mathscr{L}_1 \otimes \pi_2^*\mathscr{L}_2$ on $X \times Y$, where $\pi_1$ and $\pi_2$ are the projections onto $X$ and $Y$, respectively.

Let $L$ be a Cartier divisor on a projective variety. We denote by $\mathrm{Bs}(L)$ the base locus of the complete linear system $|L|$.

**Acknowledgement.** Much of this work was developed during the authors' visits to IMPA and Institut Fourier. We would like to thank both institutions for their support and hospitality. We also thank the referee for their thoughtful suggestions on how to improve the presentation of some of the results.

## 2. Preliminaries

**2.1. Foliations.** We start with some basic definitions.

**Definition 10.** A *foliation* on a normal variety $X$ is a (possibly zero) coherent subsheaf $\mathscr{F} \subsetneq T_X$ such that

- $\mathscr{F}$ is closed under the Lie bracket,
- $\mathscr{F}$ is saturated in $T_X$ (i.e., $T_X/\mathscr{F}$ is torsion free).

The *rank* $r_{\mathscr{F}}$ of $\mathscr{F}$ is the generic rank of $\mathscr{F}$. The *codimension* of $\mathscr{F}$ is defined as

$$q_{\mathscr{F}} := \dim(X) - r_{\mathscr{F}} \geqslant 1.$$

The inclusion $\mathscr{F} \hookrightarrow T_X$ induces a nonzero map

$$\eta : \Omega_X^{r_{\mathscr{F}}} = \wedge^{r_{\mathscr{F}}}(\Omega_X^1) \to \wedge^{r_{\mathscr{F}}}(T_X^*) \to \wedge^{r_{\mathscr{F}}}(\mathscr{F}^*) \to \det(\mathscr{F}^*).$$

The *singular locus of* $\mathscr{F}$ is the singular scheme of this map, i.e., it is the closed subscheme of $X$ whose ideal sheaf is the image of the induced map $\Omega_X^{r_{\mathscr{F}}}[\otimes]\det(\mathscr{F}) \to \mathscr{O}_X$.

A closed subvariety $Y$ of $X$ is said to be *invariant* by $\mathscr{F}$ if it is not contained in the singular locus of $\mathscr{F}$, and the restriction $\eta_{|Y} : \Omega_X^{r_{\mathscr{F}}}{}_{|Y} \to \det(\mathscr{F}^*)_{|Y}$ factors through the natural map $\Omega_X^{r_{\mathscr{F}}}{}_{|Y} \to \Omega_Y^{r_{\mathscr{F}}}$.

**11** (Foliations defined by $q$-forms). Let $\mathscr{F}$ be a codimension $q$ foliation on an $n$-dimensional normal variety $X$. The *normal sheaf* of $\mathscr{F}$ is

$$N_{\mathscr{F}} := (T_X/\mathscr{F})^{**}.$$

The $q$-th wedge product of the inclusion $N_{\mathscr{F}}^* \hookrightarrow (\Omega_X^1)^{**}$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega_X^q[\otimes]\det(N_{\mathscr{F}}))$ whose zero locus has codimension at least 2 in $X$. Such an $\omega$ is *locally decomposable* and *integrable*. To say that $\omega$ is locally decomposable means that, in a neighborhood of a general point of $X$, $\omega$ decomposes as the wedge product of $q$ local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for $\omega$ is equivalent to the condition that $\mathscr{F}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on $X$, $q \geqslant 1$, and $\omega \in H^0(X, \Omega_X^q[\otimes]\mathscr{L})$ be a global section whose zero locus has codimension at least 2 in $X$. Suppose that $\omega$ is locally decomposable and integrable. Then one defines a foliation of rank $r = n - q$ on $X$ as the kernel of the morphism $T_X \to \Omega_X^{q-1}[\otimes]\mathscr{L}$ given by the contraction with $\omega$. These constructions are inverse of each other.

**12** (Foliations described as pullbacks). Let $X, Y$ be normal varieties and $\varphi : X \dashrightarrow Y$ be a dominant rational map that restricts to a morphism $\varphi^\circ : X^\circ \to Y^\circ$, where $X^\circ \subset X$ and $Y^\circ \subset Y$ are smooth open subsets.

Let $\mathscr{G}$ be a codimension $q$ foliation on $Y$ defined by a twisted $q$-form

$$\omega \in H^0(Y, \Omega_Y^q[\otimes]\det(N_{\mathscr{G}})).$$

Then $\omega$ induces a nonzero twisted $q$-form

$$\omega_{X^\circ} \in H^0(X^\circ, \Omega_{X^\circ}^q \otimes (\varphi^\circ)^*(\det(N_{\mathscr{G}})_{|Y^\circ})),$$

which in turn defines a codimension $q$ foliation $\mathscr{F}^\circ$ on $X^\circ$. We say that the saturation $\mathscr{F}$ of $\mathscr{F}^\circ$ in $T_X$ *is the pullback of* $\mathscr{G}$ *via* $\varphi$, and write $\mathscr{F} = \varphi^{-1}\mathscr{G}$.

Suppose that $X^\circ$ is such that $\varphi^\circ$ is an equidimensional morphism. Let $(B_i)_{i\in I}$ be the (possibly empty) set of hypersurfaces in $Y^\circ$ contained in the set of critical values of $\varphi^\circ$ and invariant by $\mathscr{G}$. A straightforward computation shows that

$$\det(N_{\mathscr{F}^\circ}) \cong (\varphi^\circ)^*\det(N_{\mathscr{G}_{|Y^\circ}}) \otimes \mathscr{O}_{X^\circ}\Big(\sum_{i\in I}((\varphi^\circ)^*B_i)_{\mathrm{red}} - (\varphi^\circ)^*B_i\Big). \tag{2.1}$$

Conversely, let $\mathscr{F}$ be a foliation on $X$, and suppose that the general fiber of $\varphi$ is tangent to $\mathscr{F}$. This means that, for a general point $x$ on a general fiber $F$ of $\varphi$, the linear subspace $\mathscr{F}_x \subset T_xX$ determined by the inclusion $\mathscr{F} \subset T_X$ contains $T_xF$. Suppose moreover that $\varphi^\circ$ is smooth with connected fibers. Then, by [3, Lemma 6.7], there is a holomorphic foliation $\mathscr{G}$ on $Y$ such that $\mathscr{F} = \varphi^{-1}\mathscr{G}$. Suppose that $X^\circ$ can be taken so that $\mathrm{codim}_X(X \setminus X^\circ) \geqslant 2$. Denote by $T_{X/Y}$ the saturation of $T_{X^\circ/Y^\circ}$ in $T_X$, and by $\varphi^*\mathscr{G}$ an extension of $(\varphi^\circ)^*\mathscr{G}_{|Y^\circ}$ to $X$. Then (2.1) gives

$$\det(\mathscr{F}) \cong \det(T_{X/Y})\,[\otimes]\,\det(\varphi^*\mathscr{G}). \tag{2.2}$$

**Definition 13.** Let $\mathscr{F}$ be a foliation on a normal projective variety $X$. The *canonical class* $K_{\mathscr{F}}$ of $\mathscr{F}$ is any Weil divisor on $X$ such that $\mathscr{O}_X(-K_{\mathscr{F}}) \cong \det(\mathscr{F})$.

**14** (Restricting foliations to subvarieties). Let $X$ be a smooth variety, and let $\mathscr{F}$ be a codimension $q$ foliation on $X$ defined by a twisted $q$-form $\omega \in H^0(X, \Omega_X^q \otimes \det(N_{\mathscr{F}}))$. Let $Z$ be a smooth subvariety with normal bundle $N_{Z/X}$. Suppose that the restriction of $\omega$ to $Z$ is nonzero. Then it induces a nonzero twisted $q$-form $\omega_Z \in H^0(Z, \Omega_Z^q \otimes \det(N_{\mathscr{F}})_{|Z})$, and a codimension $q$ foliation $\mathscr{F}_Z$ on $Z$. There is a maximal effective divisor $B$ on $Z$ such that

$$\omega_Z \in H^0(Z, \Omega_Z^q \otimes \det(N_{\mathscr{F}})_{|Z}(-B)).$$

A straightforward computation shows that

$$\mathscr{O}_Z(K_{\mathscr{F}_Z}) \cong \det(N_{Z/X})(K_{\mathscr{F}|Z} - B).$$

**Definition 15.** Let $X$ be a normal variety. A foliation $\mathscr{F}$ on $X$ is said to be *algebraically integrable* if the leaf of $\mathscr{F}$ through a general point of $X$ is an algebraic variety. In this situation, by abuse of notation we often use the word *leaf* to mean the closure in $X$ of a leaf of $\mathscr{F}$.

**16** ([25, Theorem 3], [3, Lemma 3.2]). Let $X$ be normal projective variety, and let $\mathscr{F}$ be an algebraically integrable foliation on $X$. There is a unique irreducible closed subvariety $W$ of $\mathrm{Chow}(X)$ whose general point parametrizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in $X$). In other words, if $U \subset W \times X$ is the universal cycle, with universal morphisms $\pi : U \to W$ and $e : U \to X$, then $e$ is birational, and, for a general point $w \in W$, $e(\pi^{-1}(w)) \subset X$ is the closure of a leaf of $\mathscr{F}$.

We call the normalization $\tilde{W}$ of $W$ the *space of leaves* of $\mathscr{F}$, and the induced rational map $X \dashrightarrow \tilde{W}$ a *rational first integral for* $\mathscr{F}$.

We end this subsection with a useful criterion of algebraic integrability for foliations.

**Theorem 17** ([8, Theorem 0.1], [32, Theorem 1]). *Let $X$ be a normal complex projective variety, and let $\mathscr{F}$ be a foliation on $X$. Let $C \subset X$ be a complete curve disjoint from the singular loci of $X$ and $\mathscr{F}$. Suppose that the restriction $\mathscr{F}_{|C}$ is an ample vector bundle on $C$. Then the leaf of $\mathscr{F}$ through any point of $C$ is an algebraic variety, and the leaf of $\mathscr{F}$ through a general point of $C$ is rationally connected.*

### 2.2. Fano foliations.

**Definition 18.** Let $\mathscr{F}$ be a foliation on a normal projective variety $X$. We say that $\mathscr{F}$ is a *Fano foliation* (respectively $\mathbf{Q}$-*Fano foliation*) if $-K_{\mathscr{F}}$ is an ample Cartier (respectively $\mathbf{Q}$-Cartier) divisor on $X$.

The *index* $\iota_{\mathscr{F}}$ of a Fano foliation $\mathscr{F}$ on $X$ is the largest integer dividing $-K_{\mathscr{F}}$ in $\mathrm{Pic}(X)$. We say that a Fano foliation $\mathscr{F}$ is a *del Pezzo foliation* if $\iota_{\mathscr{F}} = r_{\mathscr{F}} - 1$. We say that it is a *Mukai foliation* if $\iota_{\mathscr{F}} = r_{\mathscr{F}} - 2$.

The existence of a $\mathbf{Q}$-Fano foliation on a variety $X$ imposes strong restrictions on $X$.

**Theorem 19** ([4, Theorem 1.4]). *Let $X$ be a klt projective variety, and let $\mathscr{F} \subsetneq T_X$ be a $\mathbf{Q}$-Fano foliation. Then $K_X - K_{\mathscr{F}}$ is not pseudo-effective.*

Suppose that a complex projective manifold $X$ admits a Fano foliation $\mathscr{F}$. By Theorem 19, $K_X$ is not pseudo-effective, and hence $X$ is uniruled by [9]. So we can consider a *minimal dominating family of rational curves* on $X$. This is an irreducible component $H$ of $\mathrm{RatCurves}^n(X)$ such that

- the curves parametrized by $H$ sweep out a dense subset of $X$,
- for a general point $x \in X$, the subset of $H$ parametrizing curves through $x$ is proper.

To compute the intersection number $-K_{\mathscr{F}} \cdot \ell$, where $\ell$ is a general curve from the family $H$, we will use the following observations.

**Lemma 20.** *Let $X$ be a complex projective manifold, and let $\mathscr{F}$ be a codimension 1 foliation on $X$. Let $C \subset X$ be a curve not contained in the singular locus of $\mathscr{F}$, and denote by $g$ its geometric genus. If $C$ is not tangent to $\mathscr{F}$, then $-K_{\mathscr{F}} \cdot C \leqslant -K_X \cdot C + 2g - 2$.*

*Proof.* Set $n := \dim(X)$, and let $\omega \in H^0(X, \Omega^1_X \otimes \det(N_{\mathscr{F}}))$ be a 1-form defining $\mathscr{F}$, as in Paragraph 11. Consider the normalization morphism $f : \tilde{C} \to C \subset X$. The pullback of $\omega$ to $\tilde{C}$ yields a nonzero 1-form

$$\tilde{\omega} \in H^0(\tilde{C}, \Omega^1_{\tilde{C}} \otimes f^* \det(N_{\mathscr{F}})).$$

Thus

$$\deg(\Omega^1_{\tilde{C}} \otimes f^* \det(N_{\mathscr{F}})) \geqslant 0,$$

proving the lemma. □

**Lemma 21.** *Let $X$ be a uniruled complex projective manifold, and let $\mathscr{F}$ be a foliation on $X$. Let $\ell \subset X$ be a general member of a minimal dominating family of rational curves on $X$. If $\ell$ is not tangent to $\mathscr{F}$, then $-K_{\mathscr{F}} \cdot \ell \leqslant -K_X \cdot \ell - 2$.*

*Proof.* Set $n := \dim(X)$, and consider the normalization morphism $f : \mathbb{P}^1 \to \ell \subset X$. By [34, Corollary IV.2.9],

$$f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)},$$

where $0 \leqslant d = -K_X \cdot \ell - 2 \leqslant n-1$. Write

$$f^*\mathscr{F} \cong \bigoplus_{i=1}^{r_{\mathscr{F}}} \mathscr{O}_{\mathbb{P}^1}(a_i).$$

Since $\ell$ is general, $f^*\mathscr{F}$ is a subbundle of $f^*T_X$, and since $\ell$ is not tangent to $\mathscr{F}$, the inclusion $f^*\mathscr{F} \hookrightarrow f^*T_X$ induces an inclusion

$$\bigoplus_{i=1}^{r_{\mathscr{F}}} \mathscr{O}_{\mathbb{P}^1}(a_i) \cong f^*\mathscr{F} \hookrightarrow f^*T_X/T_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}.$$

Thus $a_i \leqslant 1$ for $1 \leqslant i \leqslant r_{\mathscr{F}}$, and

$$-K_{\mathscr{F}} \cdot \ell = \sum_{i=1}^{r_{\mathscr{F}}} a_i \leqslant d = -K_X \cdot \ell - 2.$$

This completes the proof of the lemma. □

**Definition 22.** Let $\mathscr{F}$ be an algebraically integrable foliation on a complex projective manifold $X$. Let $i : \tilde{F} \to X$ be the normalization of the closure of a general leaf of $\mathscr{F}$. There is an effective divisor $\tilde{\Delta}$ on $\tilde{F}$ such that $K_{\tilde{F}} + \tilde{\Delta} \sim i^*K_{\mathscr{F}}$ ([3, Definition 3.4]). The pair $(\tilde{F}, \tilde{\Delta})$ is called a *general log leaf* of $\mathscr{F}$.

In [3], we applied the notions of singularities of pairs, developed in the context of the minimal model program, to the log leaf $(\tilde{F}, \tilde{\Delta})$. The case when $(\tilde{F}, \tilde{\Delta})$ is *log canonical* is specially interesting. We refer to [35, Section 2.3] for the definition of log canonical pairs. Here we only remark that if $\tilde{F}$ is smooth and $\tilde{\Delta}$ is a reduced simple normal crossing divisor, then $(\tilde{F}, \tilde{\Delta})$ is log canonical.

**Proposition 23** ([3, Proposition 5.3]). *Let $\mathscr{F}$ be an algebraically integrable Fano foliation on a complex projective manifold $X$. Suppose that the general log leaf of $\mathscr{F}$ is log canonical. Then there is a common point contained in the closure of a general leaf of $\mathscr{F}$.*

**2.3. Fano foliations with large index on $\mathbb{P}^n$ and $Q^n$.** Jouanolou's classification of codimension 1 foliations on $\mathbb{P}^n$ of degree 0 and 1 has been generalized to arbitrary rank in [14] and [38], respectively. The *degree* $\deg(\mathscr{F})$ of a foliation $\mathscr{F}$ on $\mathbb{P}^n$ is defined as the degree of the locus of tangency of $\mathscr{F}$ with a general linear subspace $\mathbb{P}^{n-r_{\mathscr{F}}} \subset \mathbb{P}^n$. By Paragraph 11, a codimension $q$ foliation on $\mathbb{P}^n$ of degree $d$ is given by a twisted $q$-form

$$\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(q+d+1)).$$

One easily checks that

$$\deg(\mathscr{F}) = \deg(K_{\mathscr{F}}) + r_{\mathscr{F}}.$$

**24** ([14, Théorème 3.8]). A codimension $q$ foliation of degree 0 on $\mathbb{P}^n$ is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^q$.

**25** ([38, Theorem 6.2]). A codimension $q$ foliation $\mathscr{F}$ of degree 1 on $\mathbb{P}^n$ satisfies one of the following conditions:

- $\mathscr{F}$ is induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$, defined by $q$ linear forms $L_1, \dots, L_q$ and one quadratic form $Q$,
- $\mathscr{F}$ is the linear pullback of a foliation on $\mathbb{P}^{q+1}$ induced by a global holomorphic vector field.

In the first case, $\mathscr{F}$ is induced by the $q$-form on $\mathbb{C}^{n+1}$

$$\begin{aligned}\Omega &= \sum_{i=1}^{q}(-1)^{i+1} L_i dL_1 \wedge \cdots \wedge \widehat{dL_i} \wedge \cdots \wedge dL_q \wedge dQ + (-1)^q 2Q dL_1 \wedge \cdots \wedge dL_q \\ &= (-1)^q \left(\sum_{i=q+1}^{n+1} L_j \frac{\partial Q}{\partial L_i}\right) dL_1 \wedge \cdots \wedge dL_q \\ &\qquad + \sum_{i=1}^{q} \sum_{j=q+1}^{n+1} (-1)^{i+1} L_i \frac{\partial Q}{\partial L_j} dL_1 \wedge \cdots \wedge \widehat{dL_i} \wedge \cdots \wedge dL_q \wedge dL_j,\end{aligned}$$

where $L_{q+1}, \dots, L_{n+1}$ are linear forms such that $L_1, \dots, L_{n+1}$ are linearly independent. The singular locus of $\mathscr{F}$ is the union of the quadric

$$\{L_1 = \cdots = L_q = Q = 0\} \cong Q^{n-q-1}$$

and the linear subspace

$$\left\{\frac{\partial Q}{\partial L_{q+1}} = \cdots = \frac{\partial Q}{\partial L_{n+1}} = 0\right\}.$$

In the second case, the singular locus of $\mathscr{F}$ is the union of linear subspaces of codimension at least 2 containing the center $\mathbb{P}^{n-q-2}$ of the projection.

**26** ([5, Proposition 3.18]). A codimension $q$ del Pezzo foliation on a smooth quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$ is induced by the restriction of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^q$.

## 3. Codimension 1 Mukai foliations on Fano manifolds with $\rho = 1$

**3.1. Fano manifolds of high index.** A *Fano manifold* $X$ is a complex projective manifold whose anti-canonical class $-K_X$ is ample. The index $\iota_X$ of $X$ is the largest integer dividing $-K_X$ in $\mathrm{Pic}(X)$. By Kobayachi–Ochiai's theorem ([33]), $\iota_X \leqslant n + 1$, equality holds if and only if $X \cong \mathbb{P}^n$, and $\iota_X = n$ if and only if $X \cong Q^n \subset \mathbb{P}^{n+1}$.

Fano manifolds with $\iota_X = \dim X - 1$ were classified by Fujita in [18,19,21]. Those with Picard number 1 are isomorphic to one of the following:

(1) a cubic hypersurface in $\mathbb{P}^{n+1}$.

(2) an intersection of two hyperquadrics in $\mathbb{P}^{n+2}$.

(3) a linear section of the Grassmannian $G(2, 5) \subset \mathbb{P}^9$ under the Plücker embedding.

(4) a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(2, 1, \dots, 1)$.

(5) a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \dots, 1)$.

A Fano manifold $X$ such that $(\dim X - 1)$ divides $\iota_X$ is called a *del Pezzo* manifold. In this case, either $X \cong \mathbb{P}^3$, or $\iota_X = \dim X - 1$.

Fano manifolds with $\iota_X = \dim X - 2$ are called *Mukai* manifolds. Their classification was first announced in [40]. We do not include it here. Instead, we refer to [2, Theorem 7] for the full list of Mukai manifolds with Picard number 1, and state below the property that we need. This property can be checked directly for each Mukai manifold in the list.

**Remark 27.** Let $X$ be a Mukai manifold with $\rho(X) = 1$ and $\dim X \geqslant 4$. Denote by $\mathscr{L}$ the ample generator of $\mathrm{Pic}(X)$. Then $X$ is covered by rational curves having degree 1 with respect to $\mathscr{L}$.

If a complex projective manifold $X$ with $\rho(X) = 1$ admits a Fano foliation $\mathscr{F}$, then $X$ is a Fano manifold with index $\iota_X > \iota_{\mathscr{F}}$ by Theorem 19. When $\iota_{\mathscr{F}}$ is high, we can improve this bound.

**Lemma 28.** *Let $X$ be an $n$-dimensional Fano manifold with Picard number 1, $n \geqslant 4$, and let $\mathscr{F}$ be a Fano foliation on $X$. Suppose that $\iota_{\mathscr{F}} \geqslant n - 3$. Then $\iota_X \geqslant \iota_{\mathscr{F}} + 2$.*

*Proof.* By Theorem 3, we have that $\iota_{\mathscr{F}} \leqslant r_{\mathscr{F}}$, and equality holds only if $X \cong \mathbb{P}^n$. If $\iota_{\mathscr{F}} = r_{\mathscr{F}} - 1 = n - 2$, then either $X \cong \mathbb{P}^n$ or $X \cong Q^n \subset \mathbb{P}^{n+1}$ by Theorem 4. In all these cases, $\iota_X \geqslant \iota_{\mathscr{F}} + 2$. So we may assume from now on that $\iota_{\mathscr{F}} = n - 3$ and $r_{\mathscr{F}} \in \{n-2, n-1\}$.

Let $\mathscr{L}$ be the ample generator of $\mathrm{Pic}(X)$. Let $H$ be a minimal dominating family of rational curves on $X$, and let $\ell$ be a general curve parametrized by $H$. By [13], $-K_X \cdot \ell = n + 1$ if and only if $X \cong \mathbb{P}^n$. So we may assume that $-K_X \cdot \ell \leqslant n$. Set $\lambda := \mathscr{L} \cdot \ell$.

By [29, Proposition 2], $\ell$ is not tangent to $\mathscr{F}$. Hence, by Lemma 21,

$$\lambda(n-3) = -K_{\mathscr{F}} \cdot \ell \leqslant -K_X \cdot \ell - 2 \leqslant n - 2.$$

If $\lambda > 1$, then $\lambda = 2$, $n = 4$ and $\iota_X = 2$, contradicting Remark 27. So we conclude that $\lambda = 1$, and thus

$$\iota_X = -K_X \cdot \ell \geqslant -K_{\mathscr{F}} \cdot \ell + 2 = \iota_{\mathscr{F}} + 2.$$

This completes the proof of the lemma. □

**3.2. Del Pezzo foliations of codimension 2.** When proving Theorem 5, we distinguish two cases, depending on whether or not the codimension 1 Mukai foliation $\mathscr{F}$ contains a codimension 2 del Pezzo subfoliation. The aim of this subsection is to provide a classification of these.

**Theorem 29.** *Let $X$ be an $n$-dimensional Fano manifold with $\rho(X) = 1$, $n \geqslant 4$, and let $\mathscr{G}$ be a codimension 2 del Pezzo foliation on $X$. Then the pair $(X, \mathscr{G})$ satisfies one of the following conditions:*

(1) *$X \cong \mathbb{P}^n$ and $\mathscr{G}$ is the pullback under a linear projection of a foliation on $\mathbb{P}^3$ induced by a global vector field.*

(2) *$X \cong \mathbb{P}^n$ and $\mathscr{G}$ is induced by a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(2,1,1)$ defined by one quadratic form and two linear forms.*

(3) *$X \cong Q^n \subset \mathbb{P}^{n+1}$ and $\mathscr{G}$ is induced by the restriction to the manifold $X$ of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$.*

Throughout this subsection, we will use the following notation.

**Notation.** Let $X$ be an $n$-dimensional Fano manifold with $\rho(X) = 1$ and let $\mathscr{L}$ be an ample line bundle on $X$ such that $\mathrm{Pic}(X) = \mathbb{Z}[\mathscr{L}]$. Given a sheaf of $\mathscr{O}_X$-modules $\mathscr{E}$ on $X$ and an integer $m$, we denote by $\mathscr{E}(m)$ the twisted sheaf $\mathscr{E} \otimes \mathscr{L}^{\otimes m}$.

Under the assumptions of Theorem 29, $\mathscr{G}$ is defined by a nonzero section

$$\omega \in H^0(X, \Omega_X^2(\iota_X - n + 3))$$

as in Paragraph 11. In order to compute these cohomology groups, we will use the knowledge of several cohomology groups of special Fano manifolds, which we gather below.

**30** (Bott's formulas). Let $p, q$ and $k$ be integers, with $p$ and $q$ non-negative. Then

$$h^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n-q}{k}\binom{k-1}{q} & \text{for } p = 0, 0 \leqslant q \leqslant n \text{ and } k > q, \\ 1 & \text{for } k = 0 \text{ and } 0 \leqslant p = q \leqslant n, \\ \binom{-k+q}{-k}\binom{-k-1}{n-q} & \text{for } p = n, 0 \leqslant q \leqslant n \text{ and } k < q - n, \\ 0 & \text{otherwise.} \end{cases}$$

**31** ([17, Satz 8.11]). Suppose $X$ is a smooth $n$-dimensional complete intersection in a weighted projective space. Then

(1) $h^q(X, \Omega_X^q) = 1$ for $0 \leqslant q \leqslant n, q \neq \frac{n}{2}$.

(2) $h^p(X, \Omega_X^q(t)) = 0$ in the following cases:

- $0 < p < n$, $p + q \neq n$ and either $p \neq q$ or $t \neq 0$,
- $p + q > n$ and $t > q - p$,
- $p + q < n$ and $t < q - p$.

**32** ([6, Theorem 1.1]). Let $X$ be an $n$-dimensional Fano manifold with $\rho(X) = 1$. Then $h^0(X, \Omega_X^q(\iota_X - n + q)) = 0$ unless $X \cong \mathbb{P}^n$, or $X \cong Q^n$ and $q = n$. In particular for a smooth hyperquadric $Q = Q^n \subset \mathbb{P}^{n+1}$, $n \geqslant 3$, we have $h^0(Q, \Omega_Q^2(2)) = 0$.

**33** ([3, Lemma 4.5]). Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geqslant 3$. Suppose that $q \geqslant 1$ and $t \leqslant q \leqslant n-2$. Then $h^0(X, \Omega_X^q(t)) = 0$.

**34.** Let $Y$ be an $n$-dimensional Fano manifold with $\rho(Y) = 1$, and let $X \in |\mathscr{O}_Y(d)|$ be a smooth divisor. The following exact sequences will be used to relate foliations on $X$ with foliations on $Y$:

$$0 \to \Omega_Y^q(t-d) \to \Omega_Y^q(t) \to \Omega_Y^q(t)|_X \to 0, \tag{3.1}$$

$$0 \to \Omega_X^{q-1}(t-d) \to \Omega_Y^q(t)|_X \to \Omega_X^q(t) \to 0. \tag{3.2}$$

By [41, Lemma 1.2], if $h^p(Y, \Omega_Y^{q-1}) \neq 0$ and $p + q - 1 < n$, then the map in cohomology induced by the exact sequence (3.2) (with $t = d$)

$$H^p(X, \Omega_X^{q-1}) \to H^p(X, \Omega_Y^q(d)_{|X})$$

is nonzero.

**35.** Let $a_0, \dots, a_n$ be positive integers such that $\gcd(a_0, \dots, \hat{a}_i, \dots, a_n) = 1$ for every $i \in \{0, \dots, n\}$. Denote by $S = S(a_0, \dots, a_n)$ the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ graded by $\deg x_i = a_i$, and by $\mathbb{P} := \mathbb{P}(a_0, \dots, a_n)$ the weighted projective space $\mathrm{Proj}(S(a_0, \dots, a_n))$. For each $t \in \mathbb{Z}$, let $\mathscr{O}_{\mathbb{P}}(t)$ be the $\mathscr{O}_{\mathbb{P}}$-module associated to the graded $S$-module $S(t)$.

Consider the sheaves of $\mathscr{O}_{\mathbb{P}}$-modules $\overline{\Omega}^q_{\mathbb{P}}(t)$ defined in [15, 2.1.5] for $q, t \in \mathbb{Z}$, $q \geqslant 0$. If $U \subset \mathbb{P}$ denotes the smooth locus of $\mathbb{P}$, and $\mathscr{O}_U(t)$ is the line bundle obtained by restricting $\mathscr{O}_{\mathbb{P}}(t)$ to $U$, then

$$\overline{\Omega}^q_{\mathbb{P}}(t)_{|U} = \Omega^q_U \otimes \mathscr{O}_U(t).$$

The cohomology groups $H^p(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t))$ are described in [15, 2.3.2]. We need the following:

- for $J \subset \{0, \dots, n\}$ and $a_J := \sum_{i \in J} a_i$,

$$h^0(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)) = \sum_{i=0}^{q} \Bigg( (-1)^{i+q} \sum_{\# J = i} \dim_{\mathbb{C}}(S_{t-a_J}) \Bigg),$$

- $h^p(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)) = 0$ if $p \notin \{0, q, n\}$.

Now suppose that $\mathbb{P}$ has only isolated singularities, let $d > 0$ be such that $\mathscr{O}_{\mathbb{P}}(d)$ is a line bundle generated by global sections, and let $X \in |\mathscr{O}_{\mathbb{P}}(d)|$ be a smooth hypersurface. We will use the cohomology groups $H^p(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t))$ to compute some cohomology groups $H^p(X, \Omega^q_X(t))$. Note that $X$ is contained in the smooth locus of $\mathbb{P}$, so we have an exact sequence as in (3.2):

$$0 \to \Omega^{q-1}_X(t-d) \to \overline{\Omega}^q_{\mathbb{P}}(t)|_X \to \Omega^q_X(t) \to 0. \tag{3.3}$$

Tensoring the sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(-d) \to \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_X \to 0$$

with the sheaf $\overline{\Omega}^q_{\mathbb{P}}(t)$, and noting that

$$\overline{\Omega}^q_{\mathbb{P}}(t) \otimes \mathscr{O}_{\mathbb{P}}(-d) \cong \overline{\Omega}^q_{\mathbb{P}}(t-d),$$

we get an exact sequence as in (3.1):

$$0 \to \overline{\Omega}^q_{\mathbb{P}}(t-d) \to \overline{\Omega}^q_{\mathbb{P}}(t) \to \overline{\Omega}^q_{\mathbb{P}}(t)|_X \to 0. \tag{3.4}$$

*Proof of Theorem* 29. By Lemma 28, $\iota_X \geqslant n-1$. Recall the classification of Fano manifolds of high index discussed in Section 3.1. We will go through the manifolds in that list, and determine all codimension 2 del Pezzo foliations on them.

Suppose first that $X \cong \mathbb{P}^n$. Then $\mathscr{G}$ is a codimension 2 foliation of degree 1 on $\mathbb{P}^n$. Such foliations are described in Paragraph 25.

Suppose that $X$ is a smooth hyperquadric $Q = Q^n \subset \mathbb{P}^{n+1}$. Codimension 2 del Pezzo foliation on $Q^n$ are described in Paragraph 26.

From now on we suppose that $\iota_X = n-1$. We consider the five possibilities for $X$ described in Section 3.1. If we show that $h^0(X, \Omega^2_X(2)) = 0$, then it follows from Paragraph 11 that $X$ does not admit del Pezzo foliations of codimension 2.

(1) $X$ is a cubic hypersurface in $\mathbb{P}^{n+1}$. The vanishing $h^0(X, \Omega^2_X(2)) = 0$ follows from Paragraph 33 above.

(2) $X$ is the intersection of two hyperquadrics in $\mathbb{P}^{n+2}$. Let $Q$ and $Q'$ be smooth hyperquadrics in $\mathbb{P}^{n+2}$ such that $X = Q \cap Q'$. Consider the exact sequences of Paragraph 34 for $Y = Q$, $d = 2$, $q = 2$ and $t = 2$. They induce maps of cohomology groups:

$$H^0(Q, \Omega_Q^2(2)) \to H^0(X, \Omega_Q^2(2)_{|X}) \to H^0(X, \Omega_X^2(2)).$$

We will show that the composed map $H^0(Q, \Omega_Q^2(2)) \to H^0(X, \Omega_X^2(2))$ is surjective. Since $h^0(Q, \Omega_Q^2(2)) = 0$ by Paragraph 32, the required vanishing

$$h^0(X, \Omega_X^2(2)) = 0$$

will follow.

Surjectivity of the first map

$$H^0(Q, \Omega_Q^2(2)) \to H^0(X, \Omega_Q^2(2)_{|X})$$

follows from the vanishing of $H^1(Q, \Omega_Q^2)$ granted by Paragraph 31. To prove surjectivity of the second map, we consider the long exact sequence in cohomology associated to the sequence in (3.2). By Paragraph 31, $H^1(Q, \Omega_Q^1) \cong \mathbb{C}$. So, as we noted in Paragraph 34 above, the map

$$H^1(X, \Omega_X^1) \to H^1(X, \Omega_Q^2(2)_{|X})$$

is nonzero. Since $H^1(X, \Omega_X^1) \cong \mathbb{C}$ by Paragraph 31, we conclude that the map

$$H^1(X, \Omega_X^1) \to H^1(X, \Omega_Q^2(2)_{|X})$$

is injective, and thus the map

$$H^0(X, \Omega_Q^2(2)_{|X}) \to H^0(X, \Omega_X^2(2))$$

is surjective.

(3) $X$ is a linear section of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ of codimension $c \leqslant 2$. We will show that $X$ does not admit del Pezzo foliations of codimension 2. By Paragraphs 14 and 32, it is enough to prove this in the case $c = 2$.

By [19, Theorem 10.26], $X$ can be described as follows. There is a plane $\mathbb{P}^2 \cong P \subset X$ such that the blowup $f : Y \to X$ of $X$ along $P$ admits a morphism $g : Y \to \mathbb{P}^4$. Moreover, $g$ is the blowup of $\mathbb{P}^4$ along a rational normal curve $C$ of degree 3 contained in an hyperplane $H \subset \mathbb{P}^4$. Denote by $E$ and $F$ the exceptional loci of $f$ and $g$, respectively. Then $q(E) = H$, $f^*\mathscr{O}_X(1) \cong g^*\mathscr{O}_{\mathbb{P}^4}(2) \otimes \mathscr{O}_Y(-F)$, and $g^*\mathscr{O}_{\mathbb{P}^4}(1) \cong \mathscr{O}_Y(E + F)$.

Suppose that $X$ admits a codimension 2 del Pezzo foliation $\mathscr{G}$, which is defined by a twisted 2-form $\omega \in H^0(X, \Omega_X^2(2))$. Then $\omega$ induces a twisted 2-form

$$\alpha \in H^0(Y, \Omega_Y^2 \otimes f^*\mathscr{O}_X(2)) \cong H^0(Y, \Omega_Y^2 \otimes g^*\mathscr{O}_{\mathbb{P}^4}(4) \otimes \mathscr{O}_Y(-2F)).$$

The restriction of $\alpha$ to $Y \setminus F$ induces a twisted 2-form

$$\tilde{\alpha} \in H^0(\mathbb{P}^4, \Omega_{\mathbb{P}^4}^2(4))$$

vanishing along $C$. Denote by $\tilde{\mathscr{G}}$ the foliation on $\mathbb{P}^4$ induced by $\tilde{\alpha}$. There are two possibilities:

- $\tilde{\alpha}$ vanishes along $H$, and hence $\tilde{\mathscr{G}}$ is a degree 0 foliation on $\mathbb{P}^4$,
- $\tilde{\mathscr{G}}$ is a degree 1 foliation on $\mathbb{P}^4$ containing $C$ in its singular locus.

In the first case, $\alpha$ vanishes along $E$, and thus

$$\alpha \in H^0(Y, \Omega^2_Y \otimes f^*\mathscr{O}_X(2) \otimes \mathscr{O}_Y(-E)) \cong H^0(Y, \Omega^2_Y \otimes g^*\mathscr{O}_{\mathbb{P}^4}(3) \otimes \mathscr{O}_Y(-F)).$$

Therefore $C$ must be tangent to $\tilde{\mathscr{G}}$, which is impossible since $\tilde{\mathscr{G}}$ is induced by a linear projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$.

To see that the second case cannot occur either, recall the description of the two types of codimension 2 degree 1 foliations on $\mathbb{P}^4$ from Paragraph 25. In all these foliations, any irreducible component of the singular locus is either a linear subspace of dimension at most 2, or a conic. This proves the claim.

(4)–(5) $X$ is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(2,1,\dots,1)$ or a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,\dots,1)$. Consider the exact sequences of Paragraph 35 for $\mathbb{P} := \mathbb{P}(2,1,\dots,1)$, $d = 4$, $q = 2$ and $t = 2$ (respectively $\mathbb{P} := \mathbb{P}(3,2,1,\dots,1)$, $d = 6$, $q = 2$ and $t = 2$). They induce maps of cohomology groups:

$$H^0(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(2)) \to H^0(X, \overline{\Omega}^2_{\mathbb{P}}(2)_{|X}) \to H^0(X, \Omega^2_X(2)).$$

We will show that the composed map

$$H^0(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(2)) \to H^0(X, \Omega^2_X(2))$$

is surjective. Since $h^0(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(2)) = 0$ by Dolgachev's formulas described in Paragraph 35, the required vanishing $h^0(X, \Omega^2_X(2)) = 0$ follows.

Surjectivity of the first map

$$H^0(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(2)) \to H^0(X, \overline{\Omega}^2_{\mathbb{P}}(2)_{|X})$$

follows from the vanishing of $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(2-d))$, granted by Dolgachev's formulas. Surjectivity of the second map

$$H^0(X, \overline{\Omega}^2_{\mathbb{P}}(2)_{|X}) \to H^0(X, \Omega^2_X(2))$$

follows from the vanishing of $H^1(X, \Omega^1_X(2-d))$, granted by Paragraph 31. □

**3.3. Proof of Theorem 5.** The next result is a slight variation of [38, Proposition 3.5].

**Lemma 36.** *Let $X$ be a Fano manifold with $\rho(X) = 1$, and let $\mathscr{F}$ be a codimension 1 Fano foliation on $X$. Suppose that for each proper Fano subfoliation $\mathscr{G} \subsetneq \mathscr{F}$, we have $\iota_{\mathscr{G}} < \iota_{\mathscr{F}}$. Then $\mathscr{F}$ is induced by a dominant rational map of the form*

$$\varphi = (s_1^{\otimes m_1} : s_2^{\otimes m_2}) : X \dashrightarrow \mathbb{P}^1,$$

*where $m_1$ and $m_2$ are relatively prime positive integers, and $s_1$ and $s_2$ are sections of two line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_1^{\otimes m_1} \cong \mathscr{L}_2^{\otimes m_2}$, and $\mathscr{L}_1 \otimes \mathscr{L}_2 \cong \mathscr{O}_X(-K_X + K_{\mathscr{F}})$.*

*Proof.* By [3, Proposition 7.5], $\mathscr{F}$ is algebraically integrable. Let $\varphi : X \dashrightarrow C$ be a rational first integral, where $C$ is a normal complete curve, and $\varphi$ has irreducible general fibers. As $X$ is rationally connected, we have $C \cong \mathbb{P}^1$. From [38, Theorem 3.3] and [4, Lemma 5.4], we conclude that any fiber of $\varphi$ is irreducible and that $\varphi$ has at most two multiple fibers. The proof of [38, Proposition 3.5] then shows that $\varphi$ satisfies the conclusion of the lemma. □

To prove Theorem 5, let $\mathscr{F}$ be a codimension 1 Mukai foliation on an $n$-dimensional complex projective manifold $X \not\cong \mathbb{P}^n$ with $\rho(X) = 1$, and $n \geqslant 4$. Denote by $\mathscr{L}$ the ample generator of $\mathrm{Pic}(X)$.

Suppose first that $\mathscr{F}$ satisfies the conclusion of Lemma 36. Then there are positive integers $\lambda$ and $a \geqslant b$ such that $a$ and $b$ are relatively prime, $\lambda(a+b) = \iota_X - \iota_{\mathscr{F}}$, and $\mathscr{F}$ is induced by a pencil of hypersurfaces generated by $b \cdot F$ and $a \cdot G$, where $F \in |\mathscr{L}^{\otimes \lambda a}|$ and $G \in |\mathscr{L}^{\otimes \lambda b}|$.

If $X \cong Q^n \subset \mathbb{P}^{n+1}$, then $a = 2$ and $b = 1$. Thus $\mathscr{F}$ is cut out by a pencil of hyperquadrics of $\mathbb{P}^{n+1}$ containing a double hyperplane.

If $X$ is a del Pezzo manifold, then $a = b = 1$, and $\mathscr{F}$ is induced by a pencil in $|\mathscr{L}|$.

From now on, we assume that there exists a Fano subfoliation $\mathscr{G} \subsetneq \mathscr{F}$ with the property that $\iota_{\mathscr{G}} \geqslant \iota_{\mathscr{F}} = n-3$. By Theorem 3, $\iota_{\mathscr{G}} = n-3$, and $r_{\mathscr{G}} = n-2$, i.e., $\mathscr{G}$ is a codimension 2 del Pezzo foliation on $X$. By Theorem 29, $X \cong Q^n \subset \mathbb{P}^{n+1}$, and $\mathscr{G}$ is induced by the restriction to $Q^n$ of a linear projection $\varphi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^2$. By (2.2), $\mathscr{F}$ is the pullback via $\varphi_{|Q^n}$ of a foliation on $\mathbb{P}^2$ induced by a global vector field. This completes the proof of Theorem 5. □

## 4. Codimension 1 Mukai foliations on manifolds with $\rho > 1$

In this section we prove Theorem 8. Our setup is the following.

**Assumptions 37.** Let $X$ be an $n$-dimensional complex projective manifold with Picard number $\rho(X) > 1$, and let $\mathscr{F}$ be a codimension 1 Mukai foliation on $X$ ($n \geqslant 4$). Let $L$ be an ample divisor on $X$ such that $-K_{\mathscr{F}} \sim (n-3)L$, and set $\mathscr{L} := \mathscr{O}_X(L)$.

Under Assumptions 37, Theorem 19 implies that $K_X + (n-3)L$ is not nef. Smooth polarized varieties $(X, L)$ satisfying this condition have been classified. We explain this classification in Section 4.1, and then use it in the following subsections to prove Theorem 8.

**4.1. Adjunction theory.** We will need the following classification of Fano manifolds with large index with respect to the dimension. For $n \geqslant 5$, the list follows from [42]. The classification for $n = 4$ can be found in [30, Table 12.7].

**Theorem 38.** *Let $X$ be an $n$-dimensional Fano manifold with Picard number $\rho(X) > 1$, and $n \geqslant 4$. Let $\mathscr{L}$ be an ample line bundle such that $\mathscr{O}_X(-K_X) \cong \mathscr{L}^{\otimes \iota_X}$.*

- *If $\iota_X = n-1$, then $n = 4$ and*

$$(X, \mathscr{L}) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1) \boxtimes \mathscr{O}_{\mathbb{P}^2}(1)).$$

- *If $\iota_X = n-2$, then $n \in \{4, 5, 6\}$.*

  (1) *If $n = 6$, then*

$$(X, \mathscr{L}) \cong (\mathbb{P}^3 \times \mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1) \boxtimes \mathscr{O}_{\mathbb{P}^3}(1)).$$

  (2) *If $n = 5$, then one of the following holds:*

    (a) $(X, \mathscr{L}) \cong (\mathbb{P}^2 \times Q^3, \mathscr{O}_{\mathbb{P}^2}(1) \boxtimes \mathscr{O}_{Q^3}(1))$.

    (b) $(X, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^3}(T_{\mathbb{P}^3}), \mathscr{O}_{\mathbb{P}(T_{\mathbb{P}^3})}(1))$.

    (c) $(X, \mathscr{L}) \cong (\mathbb{P}_{\mathbb{P}^3}(\mathscr{O}_{\mathbb{P}^3}(2) \oplus \mathscr{O}_{\mathbb{P}^3}(1)^{\oplus 2}), \mathscr{O}_{\mathbb{P}(\mathscr{O}_{\mathbb{P}^3}(2) \oplus \mathscr{O}_{\mathbb{P}^3}(1)^{\oplus 2})}(1))$.

(3) *If $n = 4$, then $X$ is isomorphic to one of the following:*
   (a) $\mathbb{P}^1 \times Y$*, where $Y$ is a Fano* 3*-fold with index* 2*, or* $Y \cong \mathbb{P}^3$.
   (b) *a double cover of* $\mathbb{P}^2 \times \mathbb{P}^2$ *branched along a divisor of bidegree* $(2, 2)$.
   (c) *a divisor of* $\mathbb{P}^2 \times \mathbb{P}^3$ *of bidegree* $(1, 2)$.
   (d) *an intersection of two divisors of bidegree* $(1, 1)$ *on* $\mathbb{P}^3 \times \mathbb{P}^3$.
   (e) *a divisor of* $\mathbb{P}^2 \times Q^3$ *of bidegree* $(1, 1)$.
   (f) *the blowup of* $Q^4$ *along a conic* $C$ *which is not contained in a plane in* $Q^4$.
   (g) $\mathbb{P}_{\mathbb{P}^3}(\mathscr{E})$*, where* $\mathscr{E}$ *is the null-correlation bundle on* $\mathbb{P}^3$.
   (h) *the blowup of* $Q^4$ *along a line.*
   (i) $\mathbb{P}_{Q^3}(\mathscr{O}_{Q^3}(-1) \oplus \mathscr{O}_{Q^3})$.
   (j) $\mathbb{P}_{\mathbb{P}^3}(\mathscr{O}_{\mathbb{P}^3}(-1) \oplus \mathscr{O}_{\mathbb{P}^3}(1))$.

**39** (Nef values). Let $X$ be a $\mathbf{Q}$-factorial terminal $n$-dimensional projective variety, and let $L$ be an ample $\mathbf{Q}$-divisor on $X$. The *nef value* of $L$ is defined as

$$\tau(L) := \min\{t \geqslant 0 : K_X + tL \text{ is nef}\}.$$

It is a rational number by the Rationality Theorem ([35, Theorem 3.5]). By the Basepoint Free Theorem ([35, Theorem 3.7.3]), for $m$ sufficiently large and divisible, the linear system $|m(K_X + \tau(L)L)|$ defines a morphism $\varphi_L : X \to X'$ with connected fibers onto a normal variety. We refer to $\varphi_L$ as the *nef value morphism* of the polarized variety $(X, L)$.

The next theorem summarizes the classification of smooth polarized varieties $(X, L)$ such that $K_X + (n-3)L$ is not nef, i.e., $\tau(L) > n - 3$.

**Theorem 40.** *Let $(X, L)$ be an $n$-dimensional smooth polarized variety, with $\rho(X) > 1$ and $n \geqslant 4$. Set $\mathscr{L} := \mathscr{O}_X(L)$. Suppose that $\tau(L) > n-3$. Then $\tau(L) \in \{n-2, n-1, n\}$, unless $(n, \tau(L)) \in \{(5, \frac{5}{2}), (4, \frac{3}{2}), (4, \frac{4}{3})\}$.*

(1) *Suppose that $\tau(L) = n$. Then $\varphi_L$ makes $X$ a $\mathbb{P}^{n-1}$-bundle over a smooth curve $C$, and for a general fiber $F \cong \mathbb{P}^{n-1}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(1)$.*

(2) *Suppose that $\tau(L) = n-1$. Then $(X, \mathscr{L})$ satisfies one of the following conditions:*
   (a) $(X, \mathscr{L}) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1) \boxtimes \mathscr{O}_{\mathbb{P}^2}(1))$.
   (b) *$\varphi_L$ makes $X$ a quadric bundle over a smooth curve $C$, and for a general fiber $F \cong Q^{n-1}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{Q^{n-1}}(1)$.*
   (c) *$\varphi_L$ makes $X$ a $\mathbb{P}^{n-2}$-bundle over a smooth surface $S$, and for a general fiber $F \cong \mathbb{P}^{n-2}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-2}}(1)$.*
   (d) *$\varphi_L$ is the blowup of a smooth projective variety at finitely many points, and for any component $E \cong \mathbb{P}^{n-1}$ of the exceptional locus of $\varphi_L$, $\mathscr{L}_{|E} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(1)$.*

(3) *Suppose that $\tau(L) = n-2$. Then $(X, \mathscr{L})$ satisfies one of the following conditions.*
   (a) *$-K_X \sim (n-2)L$, and $(X, \mathscr{L})$ is as in Theorem* 38.
   (b) *$\varphi_L$ makes $X$ a generic del Pezzo fibration over a smooth curve $C$, and for a general fiber $F$ of $\varphi_L$, either* $\operatorname{Pic}(F) = \mathbb{Z}[\mathscr{L}_{|F}]$ *or* $(F, \mathscr{L}_{|F}) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1) \boxtimes \mathscr{O}_{\mathbb{P}^2}(1))$.

(c) *$\varphi_L$ makes $X$ a generic quadric bundle over a normal surface $S$, and for a general fiber $F \cong Q^{n-2}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{Q^{n-2}}(1)$.*

(d) *$\varphi_L$ makes $X$ a generic $\mathbb{P}^{n-3}$-bundle over a normal 3-fold $Y$, and for a general fiber $F \cong \mathbb{P}^{n-3}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-3}}(1)$.*

(e) *$\varphi_L : X \to X'$ is the composition of finitely many disjoint divisorial contractions. In particular, $X'$ is $\mathbf{Q}$-factorial and terminal (see Proposition 41 below for the description of the possible divisorial contractions).*

(4) *Suppose that $n = 5$ and $\tau(L) = \frac{5}{2}$. Then $\varphi_L$ makes $X$ a $\mathbb{P}^4$-bundle over a smooth curve $C$ and for a general fiber $F \cong \mathbb{P}^4$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^4}(2)$.*

(5) *Suppose that $n = 4$ and $\tau(L) = \frac{3}{2}$. Then $(X, \mathscr{L})$ satisfies one of the following conditions:*

(a) *$(X, \mathscr{L}) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2) \boxtimes \mathscr{O}_{\mathbb{P}^2}(2))$.*

(b) *$\varphi_L$ makes $X$ a generic quadric bundle over a smooth curve $C$, and for a general fiber $F \cong Q^3$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{Q^3}(2)$.*

(c) *$\varphi_L$ makes $X$ a generic $\mathbb{P}^2$-bundle over a normal surface $S$, and for a general fiber $F \cong \mathbb{P}^2$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^2}(2)$.*

(6) *Suppose that $n = 4$ and $\tau(L) = \frac{4}{3}$. Then $\varphi_L$ makes $X$ a $\mathbb{P}^3$-bundle over a smooth curve $C$, and for a general fiber $F \cong \mathbb{P}^3$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^3}(3)$.*

*Proof.* The main references for the proof of Theorem 40 are [7, Chapter 7] and [1].

By [7, Proposition 7.2.2, Theorems 7.2.3 and 7.2.4], either $\tau(L) = n$ and $(X, L)$ is as in (1) above, or $\tau(L) \leqslant n - 1$.

Suppose that $\tau(L) \leqslant n - 1$. By [7, Theorems 7.3.2 and 7.3.4], one of the following holds:

(2a) $-K_X \sim (n-1)L$, and hence $(X, \mathscr{L}) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1) \boxtimes \mathscr{O}_{\mathbb{P}^2}(1))$ by Theorem 38.

(2b) $\varphi_L$ makes $X$ a generic quadric bundle over a smooth curve $C$, and for a general fiber $F \cong Q^{n-1}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{Q^{n-1}}(1)$.

(2c) $\varphi_L$ makes $X$ a generic $\mathbb{P}^{n-2}$-bundle over a normal surface $S$, and for a general fiber $F \cong \mathbb{P}^{n-2}$ of $\varphi_L$, $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-2}}(1)$.

(2d) $\varphi_L$ is the blowup of a smooth projective variety at finitely many points, and for any component $E \cong \mathbb{P}^{n-1}$ of the exceptional locus of $\varphi_L$, $\mathscr{L}_{|E} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(1)$.

(e) $\tau(L) \leqslant n - 2$.

In case (2b), it follows from [1, Theorem 5.1] that $X$ is in fact a quadric bundle over $C$. In case (2c), it follows from [1, Theorem 5.1] that $S$ is smooth and $X$ is in fact a $\mathbb{P}^{n-2}$-bundle over $S$.

If $\tau(L) = n - 2$, the classification under (3) follows from [7, Theorem 7.5.3].

If $\tau(L) < n - 2$, then, by [7, Theorems 7.7.2, 7.7.3, 7.7.5 and 7.7.8]

$$(n, \tau(L)) \in \left\{ \left(5, \frac{5}{2}\right), \left(4, \frac{3}{2}\right), \left(4, \frac{4}{3}\right) \right\}$$

and $(X, L)$ is as in (4)–(6) above. □

We will also need the following result.

**Proposition 41** ([7, Theorems 7.5.3, 7.5.6, 7.7.2, 7.7.3, 7.7.5 and 7.7.8]). *Let $(X, L)$ be an $n$-dimensional smooth polarized variety, with $n \geqslant 4$. Suppose that $\tau(L) = n-2$, and the nef value morphism $\varphi_L : X \to X'$ is birational. Then $\varphi_L$ is the composition of finitely many disjoint divisorial contractions $\varphi_i : X \to X_i$, with exceptional divisor $E_i$, of the following types:*

- *$\varphi_i : X \to X_i$ is the blowup of a smooth curve $C_i \subset X_i$. In this case $X_i$ is smooth and the restriction of $\mathscr{L}$ to a fiber $F \cong \mathbb{P}^{n-2}$ of $(\varphi_i)_{|E_i} : E_i \to C_i$ satisfies $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-2}}(1)$.*
- *$(E_i, \mathscr{N}_{E_i/X}, \mathscr{L}_{|E_i}) \cong (\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(-2), \mathscr{O}_{\mathbb{P}^{n-1}}(1))$. In this case $X_i$ is 2-factorial. In even dimension it is Gorenstein.*
- *$(E_i, \mathscr{N}_{E_i/X}, \mathscr{L}_{|E_i}) \cong (Q^{n-1}, \mathscr{O}_{Q^{n-1}}(-1), \mathscr{O}_{Q^{n-1}}(1))$. In this case $X_i$ is singular and factorial.*

*Set $L' := (\varphi_L)_*(L)$. Then $K_{X'} + (n-3)L'$ is nef except in the following cases:*

(1) *$n = 6$ and $(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(2))$.*

(2) *$n = 5$ and one of the following holds:*

 (a) *$(X', \mathscr{O}_{X'}(L')) \cong (Q^5, \mathscr{O}_{Q^5}(2))$.*

 (b) *$X'$ is a $\mathbb{P}^4$-bundle over a smooth curve, and the restriction of $\mathscr{O}_{X'}(L')$ to a general fiber is $\mathscr{O}_{\mathbb{P}^4}(2)$.*

 (c) *$(X, \mathscr{O}_X(L)) \cong (\mathbb{P}_{\mathbb{P}^4}(\mathscr{O}_{\mathbb{P}^4}(3) \oplus \mathscr{O}_{\mathbb{P}^4}(1)), \mathscr{O}_{\mathbb{P}}(1))$.*

(3) *$n = 4$ and one of the following holds:*

 (a) *$(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(3))$.*

 (b) *$X'$ is a Gorenstein del Pezzo 4-fold and $3L' \sim_{\mathbf{Q}} -2K_{X'}$.*

 (c) *$\varphi_{L'}$ makes $X'$ a generic quadric bundle over a smooth curve $C$, and for a general fiber $F \cong Q^3$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{Q^3}(2)$.*

 (d) *$\varphi_{L'}$ makes $X'$ a generic $\mathbb{P}^2$-bundle over a normal surface $S$, and for a general fiber $F \cong \mathbb{P}^2$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{\mathbb{P}^2}(2)$.*

 (e) *$(X', \mathscr{O}_{X'}(L')) \cong (Q^4, \mathscr{O}_{Q^4}(3))$.*

 (f) *$\varphi_L : X \to X'$ factors through $\tilde{X}$, the blowup of $\mathbb{P}^4$ along a cubic surface contained in a hyperplane $E \subset \mathbb{P}^4$. Denote by $\tilde{E} \subset \tilde{X}$ the strict transform of $E$, and by $\tilde{L}$ the push-forward of $L$ to $\tilde{X}$. Then $N_{\tilde{E}/\tilde{X}} \cong \mathscr{O}_{\mathbb{P}^3}(-2)$, $\mathscr{O}_{\tilde{E}}(\tilde{L}_{|\tilde{E}}) \cong \mathscr{O}_{\mathbb{P}^3}(1)$, and only $\tilde{E}$ is contracted by $\tilde{X} \to X'$.*

 (g) *$\varphi_L : X \to X'$ factors through $\tilde{X}$, a conic bundle over $\mathbb{P}^3$. Denote by $\tilde{L}$ the push-forward of $L$ to $\tilde{X}$. The morphism $\tilde{X} \to X'$ only contracts a subvariety $\tilde{E} \cong \mathbb{P}^3$ such that $N_{\tilde{E}/\tilde{X}} \cong \mathscr{O}_{\mathbb{P}^3}(-2)$ and $\mathscr{O}_{\tilde{E}}(\tilde{L}_{|\tilde{E}}) \cong \mathscr{O}_{\mathbb{P}^3}(1)$.*

 (h) *$\varphi_{L'}$ makes $X'$ a $\mathbb{P}^3$-bundle over a smooth curve $C$, and for a general fiber $F \cong \mathbb{P}^3$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{\mathbb{P}^3}(3)$.*

 (i) *$(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(4))$.*

 (j) *$X' \subset \mathbb{P}^{10}$ is a cone over $(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2))$ and $L' \sim_{\mathbf{Q}} 2H$, where $H$ denotes a hyperplane section in $\mathbb{P}^{10}$.*

**Remark 42.** In [7, Theorems 7.5.3], the description of the first type of divisorial contraction is as follows: $X_i$ is smooth, and $\varphi_i : X \to X_i$ contracts a smooth divisor $E_i \subset X$ onto a smooth curve $C_i \subset X_i$. By [39, Theorem 2], $\varphi_i$ is a smooth blowup.

**4.2. Codimension 1 Mukai foliations on projective space bundles over curves.** In this subsection, we work under Assumptions 37, supposing moreover that $\tau(L) = n$, and thus $\varphi_L$ makes $X$ a $\mathbb{P}^{n-1}$-bundle over a smooth curve $C$. We start with the following observation, which is a special case of Proposition 23.

**Proposition 43.** *Let $\mathscr{F}$ be a codimension 1 Fano foliation on a smooth projective variety $X$. Then $\mathscr{F}$ is not the relative tangent sheaf of any surjective morphism $\pi : X \to C$ onto a smooth curve.*

As a consequence of Proposition 43, $\mathscr{H} := T_{X/C} \cap \mathscr{F}$ is a codimension 2 foliation on $X$. The restriction of $\mathscr{H}$ to the general fiber $F \cong \mathbb{P}^{n-1}$ of $\varphi_L$ inherits positivity of $\mathscr{F}$, which allows us to describe it explicitly. In order to do so, we recall the description of families of degree 0 foliations on projective spaces from [3, 7.8].

**44** (Families of degree 0 foliations on $\mathbb{P}^m$). Let $Y$ be a positive-dimensional smooth projective variety, and let $\mathscr{E}$ be a locally free sheaf of rank $m+1 \geqslant 2$ on $Y$. Set $X := \mathbb{P}_Y(\mathscr{E})$, denote by $\mathscr{O}_X(1)$ the tautological line bundle on $X$, by $\pi : X \to Y$ the natural projection, and by $F \cong \mathbb{P}^m$ a general fiber of $\pi$. Let $\mathscr{H} \subsetneq T_{X/Y}$ be a foliation of rank $r \leqslant m-1$ on $X$, and suppose that $\mathscr{H}|_F \cong \mathscr{O}_{\mathbb{P}^m}(1)^{\oplus r} \subset T_{\mathbb{P}^m}$.

Let $\mathscr{V}^*$ be the saturation of $\pi_*(\mathscr{H} \otimes \mathscr{O}_X(-1))$ in $\mathscr{E}^* \cong \pi_*(T_{X/Y} \otimes \mathscr{O}_X(-1))$, and set $\mathscr{V} := (\mathscr{V}^*)^*$. Then

$$\mathscr{H} \cong (\pi^*\mathscr{V}^*) \otimes \mathscr{O}_X(1).$$

In particular, $\det(\mathscr{H}) \cong \pi^* \det(\mathscr{V}^*) \otimes \mathscr{O}_X(r)$.

The description of $\mathscr{H}|_F$ as the relative tangent sheaf of a linear projection $\mathbb{P}^m \dashrightarrow \mathbb{P}^{m-r}$ globalizes as follows. Let $\mathscr{K}$ be the (rank $m+1-r$) kernel of the dual map $\mathscr{E} \to \mathscr{V}$. Then there exists an open subset $Y^\circ \subset Y$, with $\operatorname{codim}_Y(Y \setminus Y^\circ) \geqslant 2$, over which we have an exact sequence of vector bundles

$$0 \to \mathscr{K}|_{Y^\circ} \to \mathscr{E}|_{Y^\circ} \to \mathscr{V}|_{Y^\circ} \to 0.$$

This induces a relative linear projection $\varphi : \mathbb{P}_{Y^\circ}(\mathscr{E}|_{Y^\circ}) \dashrightarrow \mathbb{P}_{Y^\circ}(\mathscr{K}|_{Y^\circ}) =: Z$, which restricts to a smooth morphism $\varphi^\circ : X^\circ \to Z$, where $X^\circ \subset X$ is an open subset with the property that $\operatorname{codim}_X(X \setminus X^\circ) \geqslant 2$. The restriction of $\mathscr{H}$ to $X^\circ$ is precisely $T_{X^\circ/Z}$.

**Proposition 45.** *Let $X$, $\mathscr{F}$, $L$ and $\mathscr{L}$ be as in Assumptions 37. Suppose that $\tau(L) = n$, and thus $\varphi_L$ makes $X$ a $\mathbb{P}^{n-1}$-bundle over a smooth curve $C$. Set $\mathscr{E} := (\varphi_L)_*\mathscr{L}$, so that $X \cong \mathbb{P}_C(\mathscr{E})$. Then one of the following holds:*

(1) *$C \cong \mathbb{P}^1$, $\mathscr{F}$ is algebraically integrable, and its restriction to a general fiber is induced by a pencil of hyperquadrics in $\mathbb{P}^{n-1}$ containing a double hyperplane.*

(2) *There exist*

- *an exact sequence*
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0$$
*of vector bundles on $C$, with* $\operatorname{rank}(\mathscr{K}) = 3$,
- *a rank 2 foliation $\mathscr{G}$ on $\mathbb{P}_C(\mathscr{K})$, generically transverse to the natural projection $p : \mathbb{P}_C(\mathscr{K}) \to C$, satisfying* $\det(\mathscr{G}) \cong p^*(\det(\mathscr{V}))$ *and* $r^a_{\mathscr{G}} \geqslant 1$,

*such that $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the induced relative linear projection*

$$\mathbb{P}_C(\mathscr{E}) \dashrightarrow \mathbb{P}_C(\mathscr{K}).$$

*In this case,* $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$.

(3) *There exist*

- *an exact sequence*
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0$$
*of vector bundles on $C$, with* $\operatorname{rank}(\mathscr{K}) = 2$,
- *a foliation by curves $\mathscr{G}$ on $\mathbb{P}_C(\mathscr{K})$, generically transverse to the natural projection $p : \mathbb{P}_C(\mathscr{K}) \to C$, and satisfying $\mathscr{G} \cong p^*(\det(\mathscr{V})) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{K})}(-1)$,*

*such that $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the induced relative linear projection*

$$\mathbb{P}_C(\mathscr{E}) \dashrightarrow \mathbb{P}_C(\mathscr{K}).$$

*In this case,* $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$.

*Proof.* By Proposition 43, $\mathscr{F} \neq T_{X/C}$. So $\mathscr{H} := \mathscr{F} \cap T_{X/C}$ is a codimension 2 foliation on $X$. Set $\mathscr{Q} := (\mathscr{F}/\mathscr{H})^{**}$. It is an invertible subsheaf of $(\varphi_L)^*T_C$, and we have

$$\det(\mathscr{H}) \cong \det(\mathscr{F}) \otimes \mathscr{Q}^*. \tag{4.1}$$

We want to describe the codimension 1 foliation $\mathscr{H}_F$ obtained by restricting $\mathscr{H}$ to a general fiber $F \cong \mathbb{P}^{n-1}$ of $\varphi_L$. By Paragraph 14, there exists a non-negative integer $b$ such that

$$-K_{\mathscr{H}_F} = (n-3+b)H,$$

where $H$ denotes a hyperplane in $F \cong \mathbb{P}^{n-1}$. By Theorem 3, we must have $b \in \{0, 1\}$.

First we suppose that $b = 0$, i.e., $\mathscr{H}_F$ is a degree 1 foliation on $\mathbb{P}^{n-1}$. Then $\mathscr{Q}|_F \cong \mathscr{O}_{\mathbb{P}^{n-1}}$ and thus $\mathscr{Q} \cong (\varphi_L)^*\mathscr{C}$ for some line bundle $\mathscr{C} \subset T_C$ on $C$. Recall that there are two types of codimension 1 degree 1 foliations on $\mathbb{P}^{n-1}$:

(i) $\mathscr{H}_F$ is induced by pencil of hyperquadrics containing a double hyperplane,

(ii) $\mathscr{H}_F$ is the linear pullback of a foliation on $\mathbb{P}^2$ induced by a global holomorphic vector field.

Suppose that we are in case (i). Then $\mathscr{H}$ is algebraically integrable, and its general log leaf is $(Q^{n-2}, H)$, where $Q^{n-2} \subset F \cong \mathbb{P}^{n-1}$ is an irreducible (possibly singular) hyperquadric, and $H$ is a hyperplane section. Note that $(Q^{n-2}, H)$ is log canonical, unless $Q^{n-2}$ is a cone over a conic curve and $H$ is a tangent hyperplane through the $(n-4)$-dimensional vertex. The latter situation falls under case (ii), treated below. So we may assume that the general log leaf of $\mathscr{H}$ is log canonical. By Proposition 23, $\det(\mathscr{H})$ cannot be ample. By (4.1), we must have $\deg(\mathscr{C}) > 0$, and hence $C \cong \mathbb{P}^1$.

Next we show that $\mathscr{F}$ is algebraically integrable. It then follows that we are in case (1) in the statement of Proposition 45. Since $\mathscr{H}$ is algebraically integrable, there is a smooth surface $S$ with a generic $\mathbb{P}^1$-bundle structure $p : S \to \mathbb{P}^1$, and a rational map $\psi : X \dashrightarrow S$ over $\mathbb{P}^1$ inducing $\mathscr{H}$. By Paragraph 12, $\mathscr{F}$ is the pullback via $\psi$ of a rank 1 foliation $\mathscr{G}$ on $S$. Moreover, there is an inclusion $p^*\mathscr{C} \subset \mathscr{G}$. It follows from Theorem 17 that the leaves of $\mathscr{G}$ are algebraic, and so are the leaves of $\mathscr{F}$.

Suppose that we are in case (ii). Then there exists a codimension 3 foliation $\mathscr{W} \subset \mathscr{H}$ whose restriction to $F \cong \mathbb{P}^{n-1}$ is a degree 0 foliation on $\mathbb{P}^{n-1}$. By Paragraph 44, there exists an exact sequence of vector bundles on $C$

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

with $\operatorname{rank}(\mathscr{K}) = 3$, such that $\mathscr{W} \cong ((\varphi_L)^*\mathscr{V}^*) \otimes \mathscr{L}$ is the tangent sheaf to the relative linear projection $\varphi : X \cong \mathbb{P}_C(\mathscr{E}) \dashrightarrow \mathbb{P}_C(\mathscr{K})$. Denote by $p : \mathbb{P}_C(\mathscr{K}) \to C$ the natural projection. By (2.2), there is a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}_C(\mathscr{K})$ such that $\mathscr{F}$ is the pullback of $\mathscr{G}$ via $\varphi$, and $\det(\mathscr{G}) \cong p^*(\det(\mathscr{V}))$. Note that $\det(\mathscr{V})$ is an ample line bundle on $C$. Thus, applying Theorem 17 to a suitable destabilizing subsheaf of $\mathscr{G}$, we conclude that $r^a_{\mathscr{G}} \geqslant 1$. We are in case (2) in the statement of Proposition 45.

From now on we assume that $b = 1$, i.e., $\mathscr{H}_F$ is a degree 0 foliation on $\mathbb{P}^{n-1}$. By Paragraph 44, there exists an exact sequence of vector bundles on $C$

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

with $\operatorname{rank}(\mathscr{K}) = 2$, such that $\mathscr{H} \cong ((\varphi_L)^*\mathscr{V}^*) \otimes \mathscr{L}$ is the tangent sheaf to the relative linear projection $\varphi : X \cong \mathbb{P}_C(\mathscr{E}) \dashrightarrow \mathbb{P}_C(\mathscr{K})$. Denote by $p : \mathbb{P}_C(\mathscr{K}) \to C$ the natural projection. By (2.2), there is a foliation by curves on $\mathbb{P}_C(\mathscr{K})$

$$\mathscr{G} \cong p^*(\det(\mathscr{V})) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{K})}(-1) \hookrightarrow T_{\mathbb{P}(\mathscr{K})}$$

such that $\mathscr{F}$ is the pullback of $\mathscr{G}$ via $\varphi$. We are in case (3) in the statement of Proposition 45. □

We describe the codimension 1 foliations on $\mathbb{P}_C(\mathscr{K})$ that appear in Proposition 45 (2).

**Proposition 46.** *Let $\mathscr{K}$ be a rank 3 vector bundle on a smooth complete curve $C$, and set $Y := \mathbb{P}_C(\mathscr{K})$, with natural projection $p : Y \to C$. Let $\mathscr{G}$ be a rank 2 foliation on $Y$, generically transverse to $p : Y \to C$, and satisfying $\det(\mathscr{G}) \cong p^*\mathscr{A}$ for some ample line bundle $\mathscr{A}$ on $C$. Then one of the following holds:*

(1) *There exist*

- *an exact sequence*
$$0 \to \mathscr{K}_1 \to \mathscr{K} \to \mathscr{B} \to 0$$
*of vector bundles on $C$, with $\operatorname{rank}(\mathscr{K}_1) = 2$,*
- *a rank 1 foliation $\mathscr{N}$ on $\mathbb{P}_C(\mathscr{K}_1)$, generically transverse to the natural projection $p_1 : \mathbb{P}_C(\mathscr{K}_1) \to C$, and satisfying $\mathscr{N} \cong p_1^*(\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{K}_1)}(-1)$,*

*such that $\mathscr{G}$ is the pullback of $\mathscr{N}$ via the induced relative linear projection*
$$Y = \mathbb{P}_C(\mathscr{K}) \dashrightarrow \mathbb{P}_C(\mathscr{K}_1).$$

(2) *There exist*

- *a $\mathbb{P}^1$-bundle $q : S \to C$,*
- *a rational map $\psi : Y \dashrightarrow S$ over $C$ whose restriction to a general fiber $F \cong \mathbb{P}^2$ of $p$ is given by a pencil of conics containing a double line,*
- *a rank 1 foliation $\mathscr{N}$ on $S$, generically transverse to $q : S \to C$, and satisfying $\mathscr{N} \cong q^*(T_C(-B))$ for some effective divisor $B$ on $C$,*

*such that $\mathscr{G}$ is the pullback of $\mathscr{N}$ via $\psi$. Moreover, the set of critical values of $\psi$ is invariant by $\mathscr{N}$.*

(3) $C \cong \mathbb{P}^1$, *and there exist*

- *a rational map* $\psi : Y \dashrightarrow \mathbb{P}^2$, *which restricts to an isomorphism on all but possibly one fiber of* $p : Y \to C$,
- *a rank* 1 *foliation* $\mathscr{N}$ *on* $\mathbb{P}^2$ *induced by a global vector field,*

*such that* $\mathscr{G}$ *is the pullback of* $\mathscr{N}$ *via* $\psi$.

*Proof.* Consider the rank 1 foliation $\mathscr{C} := \mathscr{G} \cap T_{Y/C} \subsetneq T_{\mathbb{P}_C(\mathscr{K})}$. It induces a rank 1 foliation $\mathscr{C}_F$ on a general fiber $F \cong \mathbb{P}^2$ of $p$. By Paragraph 14, there exists a non-negative integer $b$ such that

$$-K_{\mathscr{C}_F} = bH,$$

where $H$ denotes a hyperplane in $F \cong \mathbb{P}^2$. By Theorem 3, we must have $b \in \{0, 1\}$.

First we suppose that $b = 1$, i.e., $\mathscr{C}_F$ is a degree 0 foliation on $\mathbb{P}^2$. The same argument used in the last paragraph of the proof of Proposition 45 shows that $(\mathscr{K}, \mathscr{G})$ satisfies condition (1) in the statement of Proposition 46.

From now on we assume that $b = 0$, i.e., $\mathscr{C}_F$ is a degree 1 foliation on $\mathbb{P}^2$. It follows that $\mathscr{C}_{|F} \cong \mathscr{O}_{\mathbb{P}^2}$ and there exists an effective divisor $B$ on $C$ such that

$$\mathscr{C} \cong p^*(\mathscr{A} \otimes \mathscr{O}_C(K_C + B)).$$

We will distinguish two cases, depending on whether or not $\mathscr{C}$ is algebraically integrable.

Suppose first that $\mathscr{C}$ is algebraically integrable. Then $\mathscr{C}_F$ is induced by a pencil of conics containing a double line. We will show that $\mathscr{G}$ satisfies condition (2) in the statement of Proposition 46.

Let $S$ be the space of leaves of $\mathscr{C}$. Then $S$ comes with a natural morphism onto $C$, whose general fiber parametrizes a pencil of conics in $F \cong \mathbb{P}^2$. We conclude that $S \to C$ is a $\mathbb{P}^1$-bundle.

So $\mathscr{C}$ is induced by a rational map $\psi : Y \dashrightarrow S$ over $C$, and the restriction of $\psi$ to a general fiber $F \cong \mathbb{P}^2$ of $p$, $\psi_{|F} : F \cong \mathbb{P}^2 \to \mathbb{P}^1$, is given by a pencil of conics containing a double line $2\ell_F$. Let $R \subset Y$ be the closure of the union of the lines $\ell_F$ when $F$ runs through general fibers of $p : Y \to C$. Then $p_{|R} : R \to C$ is a $\mathbb{P}^1$-bundle.

Next we show that $R$ is the singular locus of $\psi$. Let $F' \cong \mathbb{P}^2$ be a special fiber of $p$ such that $\mathscr{C}_{|F'} \hookrightarrow T_{F'}$ vanishes in codimension 1. Then the foliation $\mathscr{C}_{F'}$ on $F'$ induced by $\mathscr{C}$ is a degree 0 foliation on $\mathbb{P}^2$, and the cycle in $S$ corresponding to the leaf $\ell$ of $\mathscr{C}_{F'}$ is $R \cap F' + \ell$. We conclude that $R$ is the singular locus of $\psi$.

By Paragraph 12, there is a foliation by curves $\mathscr{N}$ on $S$ such that $\mathscr{G}$ is the pullback of $\mathscr{N}$ via $\psi$. If $\psi(R)$ is not invariant by $\mathscr{N}$, then

$$\psi^*\mathscr{N} \cong \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(R) \otimes p^*\mathscr{O}_C(-K_C - B) = \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(R) \otimes \psi^*(q^*\mathscr{O}_C(-K_C - B))$$

by (2.1). Thus

$$\mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(R) \cong \psi^*(\mathscr{N} \otimes q^*\mathscr{O}_C(-K_C - B)),$$

yielding a contradiction. Therefore $\psi(R)$ is invariant by $\mathscr{N}$. Moreover,

$$\psi^*\mathscr{N} \cong p^*\mathscr{O}_C(-K_C - B).$$

Suppose from now on that $\mathscr{C}$ is not algebraically integrable, and hence neither is $\mathscr{G}$. We will show that $\mathscr{G}$ satisfies condition (3) in the statement of Proposition 46.

Let $\mathscr{L}$ be a very ample line bundle on $Y$. By [3, Proposition 7.5], there exists an algebraically integrable subfoliation by curves $\mathscr{M} \subset \mathscr{G}$, $\mathscr{M} \not\subset T_{Y/C}$, such that

$$\mathscr{M} \cdot \mathscr{L}^2 \geqslant \det(\mathscr{G}) \cdot \mathscr{L}^2 \geqslant 1.$$

Moreover, the general leaf of $\mathscr{M}$ is a rational curve. Since $\mathscr{M} \not\subset T_{Y/C}$, the general leaf of $\mathscr{M}$ dominates $C$, and we conclude that $C \cong \mathbb{P}^1$.

Next we show that $\mathscr{M} \cong p^*\mathscr{O}_{\mathbb{P}^1}(c)$, with $c \in \{1, 2\}$. Write

$$\mathscr{M} \cong \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(a) \otimes p^*\mathscr{O}_{\mathbb{P}^1}(c)$$

for some integers $a$ and $c$. First note that, for a general line $\ell \subset F$, $\mathscr{C}_{|\ell} \subset \mathscr{G}_{|\ell}$ is a subbundle. Since $\mathscr{C}_{|\ell} \cong \det(\mathscr{G})_{|\ell} \cong \mathscr{O}_\ell$, we must have $\mathscr{G}_\ell \cong \mathscr{O}_\ell \oplus \mathscr{O}_\ell$. This implies that $a \leqslant 0$. Now observe that, since $\mathscr{G}$ is not algebraically integrable, $\mathscr{M}$ does not depend on the choice of $\mathscr{L}$. Therefore

$$\mathscr{M} \cdot (\mathscr{O}_Y(kF) \otimes \mathscr{L})^2 \geqslant \det(\mathscr{G}) \cdot (\mathscr{O}_Y(kF) \otimes \mathscr{L})^2 > 0 \quad \text{for all } k \geqslant 1.$$

Thus $\mathscr{M} \cdot F \cdot \mathscr{L} \geqslant 0$, and hence $a \geqslant 0$. We conclude that $a = 0$ and $\mathscr{M} \cong p^*\mathscr{O}_{\mathbb{P}^1}(c)$. Since $\mathscr{M} \cdot \mathscr{L}^2 \geqslant 1$, we have $c \geqslant 1$. Since $\mathscr{M} \subset p^*T_{\mathbb{P}^1}$, we conclude that $c \in \{1, 2\}$.

If $c = 2$, then $\mathscr{M}$ yields a flat connection on $p$. Hence,

$$\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1}(d) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$$

for some integer $d$, and $\mathscr{M}$ is induced by the projection $\psi : Y \cong \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$.

Now suppose that $c = 1$. We may assume that $\mathscr{K}$ is of the form

$$\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-a_1) \oplus \mathscr{O}_{\mathbb{P}^1}(-a_2)$$

for some integers $a_2 \geqslant a_1 \geqslant 0$. Let $\tilde{C} \subset Y$ be the closure of a general leaf of $\mathscr{M}$. We will show that $\tilde{C}$ is a section of $p$. Suppose to the contrary that $\tilde{C}$ has degree $\geqslant 2$ over $\mathbb{P}^1$. By [16, Lemme 1.2 and Corollaire 1.3], $\tilde{C}$ has degree 2 over $\mathbb{P}^1$, and $\mathscr{M}$ is regular in a neighborhood of $\tilde{C}$. In particular, we have $\mathscr{N}_{\tilde{C}/Y} \cong \mathscr{O}_{\tilde{C}} \oplus \mathscr{O}_{\tilde{C}}$. Write $\tilde{C} \sim 2\sigma + kf$, where $\sigma$ is the section of $p$ corresponding to the surjection $\mathscr{K} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^1}(-a_2)$, and $f$ is a line on a fiber of $p$. Let $E \subset Y$ be the divisor corresponding to the surjection $\mathscr{K} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^1}(-a_1) \oplus \mathscr{O}_{\mathbb{P}^1}(-a_2)$, so that $\mathscr{O}_Y(E) \cong \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(1)$. Since the deformations of $\tilde{C}$ sweep out a dense open subset of $Y$, we must have

$$E \cdot \tilde{C} = -2a_2 + k \geqslant 0. \tag{4.2}$$

On the other hand, since $\mathscr{N}_{\tilde{C}/Y} \cong \mathscr{O}_{\tilde{C}} \oplus \mathscr{O}_{\tilde{C}}$, we have

$$0 = \deg(\mathscr{N}_{\tilde{C}/Y}) = \deg((T_Y)_{|\tilde{C}}) - 2 = 2 + 2a_1 - 4a_2 + 3k. \tag{4.3}$$

Equations (4.2) and (4.3) together yield a contradiction, proving that $\tilde{C}$ is a section of $p$. The map $\mathscr{M} \cong p^*\mathscr{O}_{\mathbb{P}^1}(1) \to p^*T_{\mathbb{P}^1}$ vanishes exactly along one fiber $F_0$ of $p$. This implies that $\mathscr{M} \subset T_Y$ restricts to a regular foliation (with algebraic leaves) over $Y \setminus F_0$. This foliation is induced by a smooth morphism $\psi : Y \setminus F_0 \to \mathbb{P}^2$, which restricts to an isomorphism on all fibers $F \neq F_0$ of $p : Y \to \mathbb{P}^1$.

In either case, by Paragraph 12, $\mathscr{G}$ is the pullback via $\psi$ of a rank 1 foliation $\mathscr{N}$ on $\mathbb{P}^2$. A straightforward computation shows that $\mathscr{N} \cong \mathscr{O}_{\mathbb{P}^2}$. This completes the proof of the proposition. □

We construct examples of foliations described in Proposition 46 (2).

**Example 47.** Let $C$ be a smooth complete curve, let $\mathscr{A}$ be an ample line bundle on $C$, and let $P \in C$. Set

$$\begin{aligned} \mathscr{L} &:= \mathscr{A} \otimes \mathscr{O}_C(K_C + P), \\ \mathscr{K} &:= \mathscr{L}^{\otimes 2} \oplus \mathscr{L} \oplus \mathscr{O}_C, \\ \mathscr{W} &:= \mathscr{L}^{\otimes 4} \oplus \mathscr{L}^{\otimes 2}. \end{aligned}$$

Suppose that $\deg(\mathscr{L}) \neq 0$.

Let $s$ be a local frame for $\mathscr{L}$. It induces local frames $(k_1, k_2, k_3)$ and $(w_1, w_2)$ for $\mathscr{K}$ and $\mathscr{W}$, respectively. We view $\mathscr{W}$ as a subbundle of $\mathrm{Sym}^2 \mathscr{K}$ by mapping $w_1$ to $k_1 \otimes k_1$, and $w_2$ to $k_2 \otimes k_2 - k_1 \otimes k_3$. This gives rise to a rational map $\psi : \mathbb{P}_C(\mathscr{K}) \dashrightarrow \mathbb{P}_C(\mathscr{W})$ such that $\psi^* \mathscr{O}_{\mathbb{P}_C(\mathscr{W})}(1) \cong \mathscr{O}_{\mathbb{P}_C(\mathscr{K})}(2)$. Note that the set $\sigma$ of critical values of $\psi$ is the section of $q$ corresponding to $\mathscr{W} = \mathscr{L}^{\otimes 4} \oplus \mathscr{L}^{\otimes 2} \twoheadrightarrow \mathscr{L}^{\otimes 2}$.

Denote by $q : \mathbb{P}_C(\mathscr{W}) \to C$ the natural morphism. By [3, Lemma 9.5], the inclusion $\mathscr{N} := q^*(T_C(-P)) \hookrightarrow p^* T_C$ lifts to an inclusion $\iota : \mathscr{N} \hookrightarrow T_{\mathbb{P}_C(\mathscr{W})}$. We claim that the cokernel of $\iota$ is torsion-free, and thus it defines a foliation on $\mathbb{P}_C(\mathscr{W})$. Indeed, if $T_{\mathbb{P}_C(\mathscr{W})}/\mathscr{N}$ is not torsion-free, then we get an inclusion $\mathscr{N} \subset T_{\mathbb{P}_C(\mathscr{W})} \otimes q^* \mathscr{O}_C(-P)$ (see [3, Lemma 9.7]). Thus $p^* T_C \cong \mathscr{N} \otimes q^* \mathscr{O}_C(P) \subset T_{\mathbb{P}_C(\mathscr{W})}$, and the natural exact sequence

$$0 \to T_{\mathbb{P}_C(\mathscr{W})/C} \to T_{\mathbb{P}_C(\mathscr{W})} \to p^* T_C \to 0$$

splits. This implies that $\mathscr{K}$ admits a flat projective connection, which is absurd. This proves the claim. An easy computation shows that $\deg(\omega_\sigma \otimes \mathscr{N}_{|\sigma}) = -\deg(\mathscr{A}) - \deg(\mathscr{L}) < 0$, and thus $\sigma$ is invariant under $\mathscr{N}$.

Now set $\mathscr{G} := \psi^{-1}(\mathscr{N})$. Then $\mathscr{G}$ is a rank 2 foliation on $\mathbb{P}_C(\mathscr{K})$, generically transverse to the natural projection $p : \mathbb{P}_C(\mathscr{K}) \to C$, and satisfies $\det(\mathscr{G}) \cong p^* \mathscr{A}$.

Next we construct examples of foliations described in Proposition 45 (3).

**Example 48.** Let $C$ be a smooth complete curve and let $\mathscr{V}$ be an ample vector bundle of rank $n-2$ on $C$. Let $\mathscr{K}_0$ be a vector bundle of rank 2 on $C$, and suppose that $\mathscr{K}_0$ does not admit a flat projective connection. Choose a sufficiently ample line bundle $\mathscr{A}$ on $C$ such that the following conditions hold:

(1) $\mathscr{K} := \mathscr{K}_0 \otimes \mathscr{A}$ is an ample vector bundle,

(2) there is a nowhere vanishing section $\alpha \in H^0(C, T_C \otimes \det(\mathscr{V}^*) \otimes \mathscr{K}_0 \otimes \mathscr{A})$,

(3) $h^1(C, \det(\mathscr{V}^*) \otimes \mathrm{Sym}^3(\mathscr{K}_0) \otimes \det(\mathscr{K}_0^*) \otimes \mathscr{A}) = 0$.

Set $S := \mathbb{P}_C(\mathscr{K})$, denote by $p : S \to C$ the natural projection, and by $\mathscr{O}_S(1)$ the tautological line bundle. The section $\alpha$ from condition (2) yields an inclusion

$$\mathscr{G} := p^*(\det(\mathscr{V})) \otimes \mathscr{O}_S(-1) \hookrightarrow p^* T_C,$$

which does not vanish identically on any fiber of $p$. Notice that $\mathscr{G} \otimes \mathscr{O}_S(B) \cong p^* T_C$ for some section $B$ of $p$. By Lemma 50 below, condition (3) implies that the inclusion $\mathscr{G} \hookrightarrow p^* T_C$ can be lifted to an inclusion

$$\iota : \mathscr{G} \hookrightarrow T_S.$$

We claim that the cokernel of $\iota$ is torsion-free, and thus it defines a foliation on $S$. Indeed, if $T_S/\mathscr{G}$ is not torsion-free, then we get an inclusion $\mathscr{G} \subset T_S(-B)$ (see [3, Lemma 9.7]). Thus

$$p^*T_C \cong \mathscr{G} \otimes \mathscr{O}_S(B) \subset T_S,$$

and the natural exact sequence

$$0 \to T_{S/C} \to T_S \to p^*T_C \to 0$$

splits. This implies that $\mathscr{K}$ admits a flat projective connection, contradicting our assumption. This proves the claim.

Now set $\mathscr{E} := \mathscr{V} \oplus \mathscr{K}$, $X := \mathbb{P}_C(\mathscr{E})$, denote by $\pi : X \to C$ the natural projection, and by $\mathscr{O}_X(1)$ the tautological line bundle. Condition (1) above implies that $\mathscr{O}_X(1)$ is an ample line bundle on $X$. The natural quotient $\mathscr{E} \to \mathscr{V}$ defines a relative linear projection $\varphi : X \dashrightarrow S$. Let $\mathscr{F}$ be the codimension 1 foliation on $X$ obtained as pullback of $\mathscr{G}$ via $\varphi$. Recall that $T_{X/S} \cong \pi^* \det(\mathscr{V}^*) \otimes \mathscr{O}_X(1)$, and thus, by (2.2),

$$\det(\mathscr{F}) \cong \mathscr{O}_X(n-3),$$

i.e., $\mathscr{F}$ is a codimension 1 Mukai foliation on $X$.

**Remark 49.** In Example 48, one can choose the ample line bundle $\mathscr{A}$ so that $K_{\mathscr{G}}$ is ample. In this case, the resulting Mukai foliation $\mathscr{F}$ presents a behavior very different than all other cases described in Theorem 8: it is the pullback of a foliation of general type on a surface.

**Lemma 50.** *Let $\mathscr{K}$ be a vector bundle of rank 2 on a smooth projective curve $C$, let $p : S = \mathbb{P}_C(\mathscr{K}) \to C$ be the corresponding ruled surface, and let $\mathscr{O}_S(1)$ be the tautological line bundle. Let $\mathscr{B}$ be a line bundle on $C$ such that there is an inclusion*

$$j : p^*\mathscr{B} \otimes \mathscr{O}_S(-1) \hookrightarrow p^*T_C.$$

*If $h^1(C, \mathscr{B} \otimes \operatorname{Sym}^3(\mathscr{K}) \otimes \det(\mathscr{K}^*)) = 0$, then $j$ can be lifted to an inclusion*

$$p^*\mathscr{B} \otimes \mathscr{O}_S(-1) \hookrightarrow T_S.$$

*Proof.* Let $e$ be the class in $H^1(S, T_{S/C} \otimes p^*\omega_C)$ corresponding to the exact sequence

$$0 \to T_{S/C} \to T_S \to p^*T_C \to 0.$$

An inclusion of line bundles $j : \mathscr{G} \hookrightarrow p^*T_C$ extends to an inclusion $\mathscr{G} \hookrightarrow T_S$ if and only if the induced section $j^*e \in H^1(S, T_{S/C} \otimes \mathscr{G}^*)$ vanishes identically.

Setting $\mathscr{G} := p^*\mathscr{B} \otimes \mathscr{O}_S(-1)$, we get

$$H^1(S, T_{S/C} \otimes \mathscr{G}^*) = H^1(C, p_*(T_{S/C} \otimes \mathscr{O}_S(1)) \otimes \mathscr{B}).$$

Since $T_{S/C} \cong p^*(\det(\mathscr{K}^*)) \otimes \mathscr{O}_S(2)$, this gives

$$\begin{aligned} H^1(S, T_{S/C} \otimes \mathscr{G}^*) &= H^1(C, p_*\mathscr{O}_S(3) \otimes \det(\mathscr{K}^*) \otimes \mathscr{B}) \\ &= H^1(C, \operatorname{Sym}^3(\mathscr{K}) \otimes \det(\mathscr{K}^*) \otimes \mathscr{B}). \end{aligned}$$

The latter vanishes by assumption, and thus $j : p^*\mathscr{B} \otimes \mathscr{O}_S(-1) \hookrightarrow p^*T_C$ extends to an inclusion $p^*\mathscr{B} \otimes \mathscr{O}_S(-1) \hookrightarrow T_S$. □

**4.3. Codimension 1 Mukai foliations on quadric bundles over curves.** In this subsection, we work under Assumptions 37, supposing moreover that $\tau(L) = n-1$ and $\varphi_L$ makes $X$ a quadric bundle over a smooth curve $C$. This is case (2b) of Theorem 40.

We start with two useful observations.

**Remark 51.** Let $\varphi : X \to C$ be a quadric bundle over a smooth curve, with $X$ smooth. An easy computation shows that the (finitely many) singular fibers of $\pi$ have only isolated singularities.

**Lemma 52.** *Let $T$ be a complex variety, and let $\varphi : X \to T$ be a flat projective morphism whose fibers are all irreducible and reduced. Let $\mathscr{Q}$ be a line bundle on $X$ such that $\mathscr{Q}_{|F} \cong \mathscr{O}_F$ for a general fiber $F$ of $\varphi$. Then there exists a line bundle $\mathscr{M}$ on $T$ such that $\mathscr{Q} \cong \varphi^*\mathscr{M}$.*

*Proof.* Let $t \in T$ be any point, and denote by $X_t$ the corresponding fiber of $\varphi$. By the Semicontinuity Theorem, $h^0(X_t, \mathscr{Q}_{|X_t}) \geqslant 1$, and $h^0(X_t, \mathscr{Q}^*_{|X_t}) \geqslant 1$. It follows that

$$\mathscr{Q}_{|X_t} \cong \mathscr{O}_{X_t},$$

since $X_t$ is irreducible and reduced. By [26, Corollary III.12.9], $\mathscr{M} := \varphi_*\mathscr{Q}$ is a line bundle on $T$, and the evaluation map $\varphi^*\mathscr{M} = \varphi^*\varphi_*\mathscr{Q} \to \mathscr{Q}$ is an isomorphism. □

**Proposition 53.** *Let $X$, $\mathscr{F}$ and $L$ be as in Assumptions* 37. *Suppose that $\tau(L) = n-1$, and $\varphi_L$ makes $X$ a quadric bundle over a smooth curve $C$. Then $C \cong \mathbb{P}^1$, and there exist*

- *an exact sequence of vector bundles on $\mathbb{P}^1$*

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

  *with* $\mathrm{rank}(\mathscr{E}) = n+1$, $\mathrm{rank}(\mathscr{K}) = 2$, *and natural projections $\pi : \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}^1$ and $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$,*
- *an integer $b$ and a foliation by rational curves $\mathscr{G} \cong q^*(\det(\mathscr{V}) \otimes \mathscr{O}_{\mathbb{P}^1}(b))$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$,*

*such that $X \in |\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2) \otimes \pi^*\mathscr{O}(b)|$, and $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the restriction to $X$ of the relative linear projection $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. Moreover, one of the following holds:*

(1) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 3}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$ and $b = 2$.*

(2) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1), \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$ and $b = 1$.*

(3) $(\mathscr{E}, \mathscr{K}) \cong (\mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}, \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ *for some integer $a \geqslant 1$ and $b = 0$.*

(4) $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}$ *for some integer $a$, and $\mathscr{E}$ is an ample vector bundle of rank* 5 *or* 6 *with* $\deg(\mathscr{E}) = 2 + 2a - b$.

(5) $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(c)$ *for distinct integers $a$ and $c$, and $\mathscr{E}$ is an ample vector bundle of rank* 5 *or* 6 *with* $\deg(\mathscr{E}) = 1 + a + c - b$.

*In particular, $n \in \{4, 5\}$ and $\mathscr{F}$ is algebraically integrable.*

*Conversely, given $\mathscr{K}$, $\mathscr{E}$ and $b$ satisfying any of the conditions* (1)–(5), *and a smooth member $X \in |\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^1}(b)|$, there exists a codimension* 1 *Mukai foliation on $X$ as described above.*

*Proof.* Denote by $F \cong Q^{n-1} \subset \mathbb{P}^n$ a general (smooth) fiber of $\varphi_L$, and recall from Theorem 40 (2b) that $\mathscr{L}_{|F} \cong \mathscr{O}_{Q^{n-1}}(1)$. Set $\mathscr{E} := (\varphi_L)_*\mathscr{L}$, and denote by $\pi : \mathbb{P}_C(\mathscr{E}) \to C$ the natural projection. Then $X$ is a divisor of relative degree 2 on $\mathbb{P}_C(\mathscr{E})$, that is, we have $X \in |\mathscr{O}_{\mathbb{P}_C(\mathscr{E})}(2) \otimes \pi^*\mathscr{B}|$ for some line bundle $\mathscr{B}$ on $C$.

By Proposition 43, $\mathscr{F} \neq T_{X/C}$. So $\mathscr{H} := \mathscr{F} \cap T_{X/C}$ is a codimension 2 foliation on $X$. Set $\mathscr{Q} := (\mathscr{F}/\mathscr{H})^{**}$. It is an invertible subsheaf of $(\varphi_L)^*T_C$, and $\det(\mathscr{H}) \cong \det(\mathscr{F}) \otimes \mathscr{Q}^*$. Denote by $\mathscr{H}_F$ the codimension 1 foliation on $F \cong Q^{n-1}$ obtained by restriction of $\mathscr{H}$. By Paragraph 14, there exists a non-negative integer $d$ such that $-K_{\mathscr{H}_F} = (n-3+d)H$, where $H$ denotes a hyperplane section of $Q^{n-1} \subset \mathbb{P}^n$. By Theorem 3, we must have $d = 0$. Hence:

- $\mathscr{H}_F$ is induced by a pencil of hyperplane sections on $Q^{n-1} \subset \mathbb{P}^n$ by Theorem 4. In particular the general log leaf of $\mathscr{H}$ is log canonical.
- $\det(\mathscr{H})_{|F} \cong \det(\mathscr{F})_{|F}$, and thus $\mathscr{Q} \cong (\varphi_L)^*\mathscr{M}$ for some line bundle $\mathscr{M} \subset T_C$ by Lemma 52.

By Proposition 23, $\det(\mathscr{H})$ is not ample. Since $\det(\mathscr{H}) \cong \det(\mathscr{F}) \otimes (\varphi_L)^*(\mathscr{M}^*)$ and $\det(\mathscr{F})$ is ample, the line bundle $\mathscr{M}$ has positive degree. Hence $C \cong \mathbb{P}^1$, $\mathscr{B} \cong \mathscr{O}_{\mathbb{P}^1}(b)$ for some $b \in \mathbb{Z}$, and $\deg(\mathscr{M}) \in \{1, 2\}$.

The linear span of $\mathrm{Sing}(\mathscr{H}_F)$ in $\mathbb{P}^n$ is the base locus of the pencil of hyperplanes in $\mathbb{P}^n$ inducing $\mathscr{H}_F$ on $F \cong Q^{n-1} \subset \mathbb{P}^n$. So $\mathscr{H}$ is the restriction to $X$ of a foliation $\tilde{\mathscr{H}}$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$ whose restriction to a general fiber of $\pi$ is a degree 0 foliation on $\mathbb{P}^n$. By Paragraph 44, there is a sequence of vector bundles on $\mathbb{P}^1$,

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0, \tag{4.4}$$

with $\mathrm{rank}(\mathscr{K}) = 2$ and natural projection $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$, such that $\tilde{\mathscr{H}}$ is induced by the relative linear projection $\tilde{\psi} : \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. So $\mathscr{H}$ is induced by the restriction

$$\psi = \tilde{\psi}_{|X} : X \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}).$$

By Remark 51, there is an open subset $X^\circ \subset X$ with $\mathrm{codim}_X(X \setminus X^\circ) \geqslant 2$ such that

$$\psi^\circ = \psi_{|X^\circ} : X^\circ \to \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$$

is a smooth morphism with connected fibers. In particular, we have $\mathscr{H} \cong T_{X/\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})}$, where $T_{X/\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})}$ denotes the saturation of $T_{X^\circ/\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})}$ in $T_X$. By Paragraph 12, $\mathscr{F}$ is the pullback via $\psi$ of a rank 1 foliation $\mathscr{G}$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. By (2.2), $\mathscr{G} \cong q^*\mathscr{M}$ and

$$\mathscr{L}^{\otimes n-3} \cong \det(\mathscr{F}) \cong \det(T_{X/\mathbb{P}(\mathscr{K})}) \otimes (\varphi_L)^*\mathscr{M}.$$

Since $\deg(\mathscr{M}) > 0$, the leaves of $\mathscr{G}$ are rational curves by Theorem 17. A straightforward computation gives $\mathscr{M} \cong \det(\mathscr{V}) \otimes \mathscr{O}_{\mathbb{P}^1}(b)$, and so

$$\deg(\mathscr{M}) = \deg(\mathscr{V}) + b \in \{1, 2\}. \tag{4.5}$$

If $\deg(\mathscr{M}) = 2$, i.e., if $\mathscr{M} \cong T_C$, then $q^*\mathscr{M} \subset T_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})}$ yields a flat connection on the natural projection $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$. Hence,

$$\mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

and $\mathscr{G}$ is induced by the projection to $\mathbb{P}^1$ transversal to $q$. In this case, $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}$ for some integer $a$. If $\deg(\mathscr{M}) = 1$, then one has $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(c)$ for distinct integers $a$ and $c$. This can be seen from the explicit description of the Atiyah classes in the proof of [3, Theorem 9.6].

By [10, Theorem 4.13], the vector bundle

$$(\varphi_L)_*\mathscr{O}_X(K_{X/\mathbb{P}^1}+mL) \cong S^{m-n+1}\mathscr{E}\otimes\det(\mathscr{E})\otimes\mathscr{O}_{\mathbb{P}^1}(b)$$

is nef for all $m \geqslant n-1$. Therefore $\mathscr{E}$ is a nef vector bundle on $\mathbb{P}^1$, and we write

$$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^1}(a_{n+1}),$$

with $0 \leqslant a_1 \leqslant \cdots \leqslant a_{n+1}$.

First suppose that $\mathscr{E}$ is not ample. Let $r \in \{1,\dots,n+1\}$ be the largest positive integer such that $a_1 = \cdots = a_r = 0$. Let

$$\varpi : \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \to \mathbb{P}(H^0(\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}),\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1))^*)$$

be the morphism induced by the complete linear system $|\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)|$. If $r = n+1$, then

$$\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \cong \mathbb{P}^1\times\mathbb{P}^n,$$

and $\varpi$ is induced by the projection morphism $\mathbb{P}^1\times\mathbb{P}^n \to \mathbb{P}^n$. If $r \leqslant n$, then $\varpi$ is a birational morphism, and its restriction to the exceptional locus

$$\operatorname{Exc}(\varpi) = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^1}(a_r)) \cong \mathbb{P}^1\times\mathbb{P}^{r-1}$$

corresponds to the projection $\mathbb{P}^1\times\mathbb{P}^{r-1}\to\mathbb{P}^{r-1}$. Thus, if $Z \subset \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$ is any closed subset, then $\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)_{|Z}$ is ample if and only if $Z$ does not contain any fiber of

$$\operatorname{Exc}(\varpi) \cong \mathbb{P}^1\times\mathbb{P}^{r-1}\to\mathbb{P}^{r-1}.$$

Since $\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)_{|X} \cong \mathscr{L}$ is ample, $X$ does not contain any fiber of

$$\operatorname{Exc}(\varpi) \cong \mathbb{P}^1\times\mathbb{P}^{r-1}\to\mathbb{P}^{r-1}.$$

Since

$$(\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2)\otimes\pi^*\mathscr{O}_{\mathbb{P}^1}(b))_{|\operatorname{Exc}(\varpi)} \cong \mathscr{O}_{\mathbb{P}^1}(b)\boxtimes\mathscr{O}_{\mathbb{P}^{r-1}}(2)$$

and $X \in |\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2)\otimes\pi^*\mathscr{O}_{\mathbb{P}^1}(b)|$, we must have $b \geqslant 0$ and

$$h^0(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(b)) = b+1 \geqslant r. \tag{4.6}$$

If $\deg(\mathscr{V}) = 0$, then

$$\mathscr{V} \cong \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-1)}$$

and the exact sequence (4.4) splits. This implies that $r \geqslant n-1$. On the other hand, by (4.5) and (4.6), $r \leqslant b+1 \leqslant 3$. Therefore $n = 4$, $r = 3$, $b = 2$, and $\deg(\mathscr{M}) = 2$. This is case (1) described in the statement of Proposition 53.

If $\deg(\mathscr{V}) \geqslant 1$, then $b \in \{0,1\}$ by (4.5).

Suppose that $b = 1$. Then $\deg(\mathscr{V}) = 1$ by (4.5). Thus

$$\mathscr{V} \cong \mathscr{O}_{\mathbb{P}^1}^{\oplus(n-2)}\oplus\mathscr{O}_{\mathbb{P}^1}(1),$$

and the exact sequence (4.4) splits. By (4.6), we have $n-2 \leqslant 2$. This implies $n = 4$, and $\deg(\mathscr{M}) = 2$. This is case (2) described in the statement of Proposition 53.

Suppose that $b = 0$. Then we must have $r = 1$ by (4.6). By (4.5), we have $\deg(\mathscr{V}) \leqslant 2$. On the other hand, $\deg(\mathscr{V}) \geqslant a_1+\cdots+a_{n-1} \geqslant n-2$. Thus $n = 4$, $\mathscr{V} \cong \mathscr{O}_{\mathbb{P}^1}\oplus\mathscr{O}_{\mathbb{P}^1}(1)^2$, the exact sequence (4.4) splits, and $\deg(\mathscr{M}) = 2$. This is case (3) described in the statement of Proposition 53.

Suppose from now on that $\mathscr{E}$ is an ample vector bundle on $\mathbb{P}^1$. Since

$$\deg(\mathscr{V}) \geqslant a_1 + \cdots + a_{n-1},$$

we have $b \leqslant 2 - (a_1 + \cdots + a_{n-1})$ by (4.5).

We claim that $n \in \{4, 5\}$. Suppose to the contrary that $n \geqslant 6$. Then

$$h^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(a_i + a_j + b)) = 0$$

if either $1 \leqslant i, j \leqslant n-2$, or $1 \leqslant i \leqslant n-2$ and $j = n-1$. This implies that in suitable homogeneous coordinates $(x_1 : \cdots : x_{n+1})$, $F \cong Q^{n-1} \subset \mathbb{P}^n$ is given by equation

$$c_{n-1}x_{n-1}^2 + x_n l_n(x_1, \ldots, x_{n+1}) + x_{n+1} l_{n+1}(x_1, \ldots, x_{n+1}) = 0,$$

where $c_{n-1} \in \mathbb{C}$, and $l_n$ and $l_{n+1}$ are linear forms. This contradicts the fact that $F$ is smooth, and proves the claim. So we are in one of cases (4) and (5) of Proposition 53, depending on whether $\deg(\mathscr{M})$ is 2 or 1, respectively.

Now we proceed to prove the converse statement. Let $\mathscr{K}$, $\mathscr{E}$ and $b$ satisfy one of the conditions (1)–(5) in the statement of Proposition 53, and let $X \in |\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2) \otimes \pi^*\mathscr{O}_{\mathbb{P}^1}(b)|$ be a smooth member. Then, one easily checks that $\mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)_{|X}$ is an ample line bundle on $X$. We shall construct a codimension 1 Mukai foliation $\mathscr{F}$ on $X$ such that

$$\mathscr{O}_X(-K_{\mathscr{F}}) \cong \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)_{|X}.$$

First, let $\mathscr{V}$ be a vector bundle of rank $n-1$ on $\mathbb{P}^1$ fitting into an exact sequence of vector bundles

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{V} \to 0,$$

and consider the induced rational map

$$\tilde{\psi} : \mathbb{P}_{\mathbb{P}^1}(\mathscr{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}).$$

Let $\psi := \tilde{\psi}_{|X} : X \dashrightarrow \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$ be the restriction of $\tilde{\psi}$ to $X$, and let $q : \mathbb{P}_{\mathbb{P}^1}(\mathscr{K}) \to \mathbb{P}^1$ be the natural projection. By Remark 51, there is an open subset $X^\circ \subset X$ with $\operatorname{codim}_X(X \setminus X^\circ) \geqslant 2$ such that $\psi^\circ = \psi_{|X^\circ} : X^\circ \to \mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$ is a smooth morphism with connected fibers. Set

$$\mathscr{M} := \det(\mathscr{V}) \otimes \mathscr{O}_{\mathbb{P}^1}(b) \subset T_{\mathbb{P}^1}.$$

This inclusion lifts to an inclusion of vector bundles $q^*\mathscr{M} \subset T_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})}$. Let $\mathscr{F}$ be the pullback via $\psi$ of the foliation defined by $q^*\mathscr{M}$ on $\mathbb{P}_{\mathbb{P}^1}(\mathscr{K})$. One computes that

$$\mathscr{O}_X(-K_{\mathscr{F}}) \cong \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(1)_{|X}. \qquad \square$$

**Example 54.** Set $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_6)$, with

$$1 = a_1 = a_2 \leqslant a_3 \leqslant a_4 \leqslant a_5 = a_6 = a \quad \text{and} \quad a_3 + a_4 \leqslant a + 1.$$

Set $\mathscr{K} = \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus 2}$ and $b = -(a_3 + a_4)$. Then we have that $\mathscr{E}$ and $b$ satisfy condition (4) in Proposition 53. Let $\lambda_{4,4} \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2a_4 + b))$, $\lambda_{2,5} \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(a_2 + a_5 + b))$ and $\lambda_{1,6} \in H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(a_1 + a_6 + b))$ be general sections. Then

$$\lambda_{4,4} + \lambda_{2,5} + \lambda_{1,6} \in H^0(\mathbb{P}^1, S^2\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}^1}(b)) \cong H^0(\mathbb{P}_{\mathbb{P}^1}(\mathscr{E}), \mathscr{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathscr{E})}(2) \otimes \pi^*\mathscr{O}(b))$$

defines a smooth hypersurface $X \subset \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$.

**4.4. Codimension 1 Mukai foliations on projective space bundles over surfaces.** In this subsection, we work under Assumptions 37, supposing moreover that $\tau(L) = n - 1$ and $\varphi_L$ makes $X$ a $\mathbb{P}^{n-2}$-bundle over a smooth surface $S$. This is case (2c) of Theorem 40.

We start with some easy observations.

**Lemma 55.** *Let $C$ be a smooth proper curve of genus $g \geqslant 0$, and let $p : S \to C$ be a ruled surface. Suppose that $-K_S \sim A + B$ where $A$ is an ample divisor, and $B$ is effective. Then $g = 0$. If moreover $B$ is nonzero and supported on fibers of $p$, then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

*Proof.* Let $\mathscr{E}$ be a normalized vector bundle on the curve $C$ such that $S \cong \mathbb{P}_C(\mathscr{E})$, and set $e := -\deg(\mathscr{E})$, as in [26, Notation V.2.8.1]. Denote by $f$ a general fiber, and by $C_0$ a minimal section of $p : S \to C$. Let $a$ and $b$ be integers such that $A \sim aC_0 + bf$. We have $-K_S \sim 2C_0 + (2-2g+e)f$, and hence $a \in \{1, 2\}$, and $B \sim (2-a)C_0 + (e-2g+2-b)f$. Since $B$ is effective, $e - 2g + 2 - b \geqslant 0$.

We claim that $e \geqslant 0$. Suppose to the contrary that $e < 0$. By [26, Proposition V.2.21], we have $b > \frac{1}{2}ae$. Thus

$$2g - 2 \leqslant e - b < e - \frac{1}{2}ae < 0,$$

and $g = 0$. But this contradicts [26, Theorem V.2.12], proving the claim.

By [26, Proposition V.2.20], we must have $b \geqslant ae + 1$, and thus

$$-2 \leqslant 2g - 2 \leqslant e - b \leqslant (1 - a)e - 1 < 0.$$

This implies $g = 0$, and $b - e \in \{1, 2\}$. If moreover $a = 2$ and $B \neq 0$, then $e = 0$, completing the proof of the lemma. □

**Lemma 56.** *Let $S$ be a smooth projective surface such that $-K_S \sim A + B$ where $A$ is ample, and $B \neq 0$ is effective. Then either $S \cong \mathbb{P}^2$, or $S$ is a Hirzebruch surface.*

*Proof.* It is enough to show that either $S$ is minimal, or $S \cong \mathbb{F}_1$. Suppose otherwise, and let $c : S \to T$ be a proper birational morphism onto a ruled surface $q : T \to C$. Set $A_T := c_*A$ and $B_T := c_*B$. Then $A_T$ is ample, $B_T$ is effective, and $-K_T \sim A_T + B_T$. By Lemma 55, $C \cong \mathbb{P}^1$ and $T \cong \mathbb{F}_e$ for some $e \geqslant 0$.

Let $p : S \to C$ be the induced morphism, and denote by $f$ a fiber of $p$ or $q$. Since $c$ is not an isomorphism by assumption, $A \cdot f \geqslant 2$. On the other hand, $-K_S \cdot f = 2$, and thus $A \cdot f = -K_S \cdot f - B \cdot f \leqslant 2$. Hence $A \cdot f = A_T \cdot f = 2$, and $B \cdot f = B_T \cdot f = 0$. These equalities, together with the fact that $A_T$ is ample and $-K_T \sim A_T + B_T$, imply that one of the following holds:

(1) $T \cong \mathbb{F}_1$ and $B_T = 0$,

(2) $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ and either $B_T = 0$ or $B = f$.

Suppose that $B_T = 0$. Intersecting $-K_S$ with the (disjoint) curves $E_i$ contracted by $c$ gives that

$$A = c^*A_T - \sum_i E_i = -K_S.$$

But this forces $B = 0$, contrary to our assumptions.

So we must have $T \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $B = f$. Intersecting $-K_S$ with the (disjoint) curves $E_i$ contracted by $c$ gives that $A = c^* A_T - \sum_i E_i$ and $B = f$. Let $\ell \subset T$ be a fiber of the projection $T \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ transversal to $q : T \to \mathbb{P}^1$. Then $A_T \cdot \ell = 1$. We choose $\ell$ to contain the image of some $E_i$, and let $\tilde{\ell} \subset S$ be its strict transform. Then $A \cdot \tilde{\ell} \leqslant 0$, contrary to our assumptions.

We conclude that either $S$ is minimal, or $S \cong \mathbb{F}_1$. □

**Proposition 57.** *Let $X$, $\mathscr{F}$ and $L$ be as in Assumptions* 37. *Suppose that $\tau(L) = n-1$, and $\varphi_L$ makes $X$ a $\mathbb{P}^{n-2}$-bundle over a smooth surface $S$. Then there exist*

- *an exact sequence of sheaves of $\mathscr{O}_S$-modules*
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0,$$
*where $\mathscr{K}$, $\mathscr{E}$, and $\mathscr{V} := \mathscr{Q}^{**}$ are vector bundles on $S$, $\mathscr{E}$ is ample of rank $n-1$, and* $\operatorname{rank}(\mathscr{K}) = 2$,
- *a codimension* 1 *foliation $\mathscr{G}$ on $\mathbb{P}_S(\mathscr{K})$, generically transverse to the natural projection $q : \mathbb{P}_S(\mathscr{K}) \to S$, satisfying $\det(\mathscr{G}) \cong q^* \det(\mathscr{V})$ and $r^a_{\mathscr{G}} \geqslant 1$,*

*such that $X \cong \mathbb{P}_S(\mathscr{E})$, and $\mathscr{F}$ is the pullback of $\mathscr{G}$ via the induced relative linear projection $\mathbb{P}_S(\mathscr{E}) \dashrightarrow \mathbb{P}_S(\mathscr{K})$. In this case, $r^a_{\mathscr{F}} \geqslant r_{\mathscr{F}} - 1$. Moreover, one of the following holds:*

(1) *$S \cong \mathbb{P}^2$, $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{P}^2}(i)$ for some $i \in \{1, 2, 3\}$, and $4 \leqslant n \leqslant 3 + i$.*

(2) *$S$ is a del Pezzo surface $\not\cong \mathbb{P}^2$, $\det(\mathscr{V}) \cong \mathscr{O}_S(-K_S)$, and $4 \leqslant n \leqslant 5$.*

(3) *$S \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\det(\mathscr{V})$ is a line bundle of type $(1, 1)$, $(2, 1)$ or $(1, 2)$, and $n = 4$.*

(4) *$S \cong \mathbb{F}_e$ for some integer $e \geqslant 1$, $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{F}_e}(C_0 + (e+i)f)$, where $i \in \{1, 2\}$, $C_0$ is the minimal section of the natural morphism $\mathbb{F}_e \to \mathbb{P}^1$, $f$ is a general fiber, and $n = 4$.*

*Conversely, given $S$, $\mathscr{K}$, $\mathscr{E}$, $\mathscr{V}$ as above, and a codimension* 1 *foliation $\mathscr{G} \subset T_{\mathbb{P}_S(\mathscr{K})}$ satisfying $\det(\mathscr{G}) \cong q^* \det(\mathscr{V})$, the pullback of $\mathscr{G}$ via the relative linear projection*
$$X \cong \mathbb{P}_S(\mathscr{E}) \dashrightarrow \mathbb{P}_S(\mathscr{K})$$
*is a codimension* 1 *Mukai foliation on $X$.*

*Proof.* Denote by $F \cong \mathbb{P}^{n-2}$ a general fiber of $\varphi_L$, and recall from Theorem 40 (2c) that $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^{n-2}}(1)$. Set $\mathscr{E} := (\varphi_L)_* \mathscr{L}$. Then $\mathscr{E}$ is an ample vector bundle of rank $n-1$, and $X \cong \mathbb{P}_S(\mathscr{E})$.

We claim that $T_{X/S} \not\subseteq \mathscr{F}$. Indeed, if $T_{X/S} \subseteq \mathscr{F}$, then $\mathscr{F}$ would be the pullback via $\varphi_L$ of a foliation on $S$, and so $\mathscr{L}^{\otimes n-3}{}_{|F} \cong \det(\mathscr{F})_{|F} \cong \det(T_{X/S})_{|F} \cong \mathscr{L}^{\otimes n-1}{}_{|F}$ by Paragraph 12, which is absurd. Hence $\mathscr{H} := \mathscr{F} \cap T_{X/S}$ is a codimension 3 foliation on $X$. Denote by $\mathscr{H}_F$ the codimension 1 foliation on $F \cong \mathbb{P}^{n-2}$ obtained by restriction of $\mathscr{H}$. By Paragraph 14, there exists a non-negative integer $c$ such that $-K_{\mathscr{H}_F} = (n-3+c)H$, where $H$ denotes a hyperplane in $\mathbb{P}^{n-2}$. By Theorem 3, $c = 0$, and $\mathscr{H}_F$ is a degree 0 foliation on $F \cong \mathbb{P}^{n-2}$.

Let $\mathscr{K}$ and $\mathscr{V}$ be as defined in Paragraph 44. By [27, Corollary 1.4] and [3, Remark 2.3], $\mathscr{K}$ and $\mathscr{V}$ are vector bundles on $S$, and there is an exact sequence
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$
with $\mathscr{Q}^{**} \cong \mathscr{V}$. The foliation $\mathscr{H}$ is induced by the relative linear projection
$$\psi : X \cong \mathbb{P}_S(\mathscr{E}) \dashrightarrow \mathbb{P}_S(\mathscr{K}).$$

So, by Paragraph 12, $\mathscr{F}$ is the pullback via $\psi$ of a rank 2 foliation $\mathscr{G}$ on $\mathbb{P}_S(\mathscr{K})$. There is an open subset $X^\circ \subset X$ with $\operatorname{codim}_X(X \setminus X^\circ) \geqslant 2$ such that $\psi^\circ = \psi_{|X^\circ} : X^\circ \to \mathbb{P}_S(\mathscr{K})$ is a smooth morphism with connected fibers. So, by (2.2),

$$\mathscr{L}^{\otimes n-3} \cong \det(\mathscr{F}) \cong \det(T_{X/\mathbb{P}_S(\mathscr{K})}) \otimes \varphi^* \det(\mathscr{G}),$$

where $T_{X/\mathbb{P}_S(\mathscr{K})}$ denotes the saturation of $T_{X^\circ/\mathbb{P}_S(\mathscr{K})}$ in $T_X$. A straightforward computation gives $\det(\mathscr{G}) \cong q^* \det(\mathscr{V})$, where $q : \mathbb{P}_S(\mathscr{K}) \to S$ denotes the natural projection. By Lemma 58, $\det(\mathscr{V})$ is an ample line bundle on $S$. Thus, applying Theorem 17 to a suitable destabilizing subsheaf of $\mathscr{G}$, we conclude that $r^a_{\mathscr{G}} \geqslant 1$.

The natural morphism $\mathscr{G} \to q^*T_S$ is injective since $T_{\mathbb{P}_S(\mathscr{K})/S} \not\subseteq \mathscr{G}$. Let $q^*B$ be the divisor of zeroes of the induced map $q^* \det(\mathscr{V}) \cong \det(\mathscr{G}) \to q^* \det(T_S)$.

Suppose first that $B = 0$. Then $\det(\mathscr{V}) \cong \mathscr{O}_S(-K_S)$, and hence $S$ is a del Pezzo surface. If $S \cong \mathbb{P}^2$, then $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{P}^2}(3)$. Since the restriction of $\mathscr{V}$ to a general line on $\mathbb{P}^2$ is an ample vector bundle, it follows that $\operatorname{rank}(\mathscr{V}) \leqslant 3$, and hence $4 \leqslant n \leqslant 6$. Suppose that $S \not\cong \mathbb{P}^2$, and let $\ell \subset S$ be a general free rational curve of minimal anticanonical degree. Then we have $\det(\mathscr{V}) \cdot \ell = -K_S \cdot \ell = 2$. Since $\mathscr{V}_{|\ell}$ is an ample vector bundle, it follows that $\operatorname{rank}(\mathscr{V}) \leqslant 2$, and hence $4 \leqslant n \leqslant 5$.

Suppose now that $B \neq 0$. By Lemma 56, either $S \cong \mathbb{P}^2$, or $S$ is a Hirzebruch surface. If $S \cong \mathbb{P}^2$, then $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{P}^2}(i)$, with $i \in \{1, 2\}$. As above, we see that $\operatorname{rank}(\mathscr{V}) \leqslant i$, and hence $4 \leqslant n \leqslant 3 + i$. If $S \cong \mathbb{F}_e$ for some $e \geqslant 0$, then a straightforward computation gives that either $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{F}_e}(C_0 + (e+i)f)$, with $i \in \{1, 2\}$, or $e = 0$ and $\det(\mathscr{V}) \cong \mathscr{O}_{\mathbb{F}_e}(2C_0 + f)$. In any case, $\det(\mathscr{V}) \cdot \ell = 1$ for a suitable free rational curve $\ell \subset \mathbb{F}_e$. Since $\mathscr{V}_{|\ell}$ is an ample vector bundle, it follows that $\operatorname{rank}(\mathscr{V}) = 1$, and hence $n = 4$.

Conversely, given $S$, $\mathscr{K}$, $\mathscr{E}$, and $\mathscr{V}$ satisfying one the conditions in the statement of Proposition 57, and a codimension 1 foliation $\mathscr{G} \subset T_{\mathbb{P}_S(\mathscr{K})}$ satisfying $\det(\mathscr{G}) \cong q^* \det(\mathscr{V})$, a straightforward computation shows that the pullback of $\mathscr{G}$ via the relative linear projection

$$X \cong \mathbb{P}_S(\mathscr{E}) \dashrightarrow \mathbb{P}_S(\mathscr{K})$$

is a codimension 1 foliation on $X$ with determinant $\mathscr{O}_{\mathbb{P}_S(\mathscr{E})}(n-3)$. □

**Lemma 58.** *Let $S$ be a smooth projective surface, let $W \subset S$ be a closed subscheme with* $\operatorname{codim}_S W \geqslant 2$*, let $\mathscr{E}$ be an ample vector bundle on $S$, and let $\mathscr{V}$ be a vector bundle on $S$ such that there exists a surjective morphism of sheaves of $\mathscr{O}_S$-modules $\mathscr{E} \twoheadrightarrow \mathscr{I}_W\mathscr{V}$. Then* $\det(\mathscr{V})$ *is an ample line bundle.*

*Proof.* Let $r$ be the rank of $\mathscr{V}$. The $r$-th wedge product of the morphism $\mathscr{E} \twoheadrightarrow \mathscr{I}_W\mathscr{V}$ gives rise to a surjective morphism $\wedge^r \mathscr{E}_{|S\setminus \operatorname{Supp}(W)} \twoheadrightarrow \det(\mathscr{V})_{|S\setminus \operatorname{Supp}(W)}$. It follows that

$$\det(\mathscr{V}) \cdot C \geqslant 1$$

for any curve $C \subset S$. To conclude that $\det(\mathscr{V})$ is ample, it is enough to show that

$$h^0(S, \det(\mathscr{V})^{\otimes m}) \geqslant 1$$

for some integer $m \geqslant 1$ by the Nakai–Moishezon Criterion.

Set $Y := \mathbb{P}_S(\wedge^r \mathscr{E})$. Denote by $\mathscr{O}_Y(1)$ the tautological line bundle on $Y$ and by $q : Y \to S$ the natural projection. Let $T \subset Y$ be the closure of the section of $q_{|S\setminus \operatorname{Supp}(W)}$ corresponding to $\wedge^r \mathscr{E}_{|S\setminus \mathrm{Supp}(W)} \twoheadrightarrow \det(\mathscr{V})_{|S\setminus \mathrm{Supp}(W)}$. Then

$$\mathscr{O}_Y(1)_{|T} \cong (q_{|T})^* \det(\mathscr{V}) \otimes \mathscr{O}_T(E)$$

for some divisor $E \subset T$ with $\mathrm{Supp}(E) \subset T \cap q^{-1}(\mathrm{Supp}(W))$. Since $\mathscr{O}_Y(1)_{|T}$ is an ample line bundle, we must have

$$\begin{aligned} h^0(S \setminus \mathrm{Supp}(W), \det(\mathscr{V})^{\otimes m}{}_{|S\setminus \mathrm{Supp}(W)}) &\geqslant h^0(T \setminus \mathrm{Supp}(E), \mathscr{O}_Y(m)_{|T\setminus \mathrm{Supp}(E)}) \\ &\geqslant h^0(T, \mathscr{O}_Y(m)_{|T}) \\ &\geqslant 1 \end{aligned}$$

for some $m \geqslant 1$, and hence $h^0(S, \det(\mathscr{V})^{\otimes m}) \geqslant 1$. □

Our next goal is to classify pairs $(\mathscr{K}, \mathscr{G})$ that appear in Proposition 57. When

$$\det(\mathscr{G}) \cong q^*\mathscr{O}_S(-K_S),$$

the situation is easily described as follows. This includes the cases described in Proposition 57 (1, $i = 3$) and (2).

**Remark 59.** Let $Z$ be a simply connected smooth projective variety, and let $\mathscr{K}$ be a rank 2 vector bundle on $Z$. Set $Y := \mathbb{P}_Z(\mathscr{K})$, with natural projection $q : Y \to Z$. Denote by $\mathscr{O}_Y(1)$ the tautological line bundle on $Y$. Let $\mathscr{G} \subset T_Y$ be a codimension 1 foliation on $Y$ such that $\mathscr{G} \cong q^*\mathscr{O}_Z(-K_Z)$. Then $\mathscr{G} \subset T_Y$ induces a flat connection on $q$. Thus

$$\mathscr{K} \cong \mathscr{M} \oplus \mathscr{M}$$

for some line bundle $\mathscr{M}$ on $Z$, and $\mathscr{G}$ is induced by the natural morphism

$$\mathbb{P}_Z(\mathscr{K}) \cong Z \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Suppose now that $S \cong \mathbb{P}^2$ or $\mathbb{F}_e$, and $\det(\mathscr{G}) \not\cong q^*\mathscr{O}_S(-K_S)$. We will describe $\mathscr{K}$ and $\mathscr{G}$ that appear in Proposition 57 by restricting them to special rational curves on $S$. Our analysis will rely on the following result.

**Lemma 60.** *Let $m \geqslant 0$ be an integer, and consider the ruled surface $q : \mathbb{F}_m \to \mathbb{P}^1$. Let $\mathscr{C} \cong q^*\mathscr{O}_{\mathbb{P}^1}(a)$ be a foliation by curves on $\mathbb{F}_m$ with $a > 0$. Then $a \in \{1, 2\}$, and one of the following holds:*

(1) *If $a = 2$, then $m = 0$, and $\mathscr{C}$ is induced by the projection $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ transversal to $q$.*

(2) *If $a = 1$, then $m \geqslant 1$, and $\mathscr{C}$ is induced by a pencil containing $C_0 + mf_0$, where $C_0$ denotes the minimal section and $f_0$ a fiber of $q : \mathbb{F}_m \to \mathbb{P}^1$.*

*Proof.* Notice that $\mathscr{C} \neq T_{\mathbb{F}_m/\mathbb{P}^1}$, thus the natural map

$$q^*\mathscr{O}_{\mathbb{P}^1}(a) \cong \mathscr{C} \to q^*T_{\mathbb{P}^1} \cong q^*\mathscr{O}_{\mathbb{P}^1}(2)$$

is nonzero, and hence $a \in \{1, 2\}$. If $a = 2$, then, as in Remark 59, $\mathscr{C}$ yields a flat connection on $q$, $m = 0$, and $\mathscr{C}$ is induced by the projection $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ transversal to $q$, proving (1).

From now on we assume that $a = 1$. Then we must have $m \geqslant 1$, since the map $\mathscr{C} \to T_{\mathbb{F}_m}$ does not vanish in codimension 1. Denote by $C_0$ the minimal section, and by $f$ a fiber of $q : \mathbb{F}_m \to \mathbb{P}^1$.

By Theorem 17, $\mathscr{C}$ is algebraically integrable and its leaves are rational curves. So it is induced by a rational map with irreducible general fibers $\pi : \mathbb{F}_m \dashrightarrow \mathbb{P}^1$. Let $C$ be the closure of a general leaf of $\mathscr{C}$. As in the proof of Proposition 46, one shows that $C$ is a section of $q$, and $\mathscr{C}$ is not regular along $C$. Write $C \sim C_0 + bf$ with $b \geqslant m$ (see [26, Proposition V.2.20]).

The foliation $\mathscr{C}$ is induced by a pencil $\Pi$ of members of $|\mathscr{O}_{\mathbb{F}_m}(C)|$. Observe that the space of reducible members of $|\mathscr{O}_{\mathbb{F}_m}(C)|$ is a codimension 1 linear subspace. Therefore, $\Pi$ has a unique reducible member.

Let $f_0$ be the divisor of zeroes of $q^*\mathscr{O}_{\mathbb{P}^1}(1) \cong \mathscr{C} \to q^*T_{\mathbb{P}^1} \cong q^*\mathscr{O}_{\mathbb{P}^1}(2)$. It is a fiber of $q$. Note that $\mathscr{C}$ induces a flat connection on $q$ over $\mathbb{F}_m \setminus f_0$. In particular, $\mathscr{C}$ is regular over $\mathbb{F}_m \setminus f_0$. Let $R(\pi)$ be the ramification divisor of $\pi$, and notice that $R(\pi)$ is supported on $f_0$. A straightforward computation gives

$$R(\pi) \equiv (2b - (m+1))f.$$

Let $C_1 + kf_0$ be the reducible member of $\Pi$ (with $k \geqslant 1$), where $C_1$ is irreducible. Write $C_1 \sim C_0 + b_1 f$. Then

$$k = 2b - (m+1) + 1 \quad \text{and} \quad b_1 + k = b.$$

Thus

$$b - k = m - b \leqslant 0 \quad \text{and} \quad b - k = b_1 \geqslant 0.$$

Hence $b_1 = 0$, and $k = b = m$. This proves (2). □

**Proposition 61.** *Let $\mathscr{K}$ be a rank 2 vector bundle on a ruled surface $p : \mathbb{F}_e \to \mathbb{P}^1$, with $e \geqslant 0$. Set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{F}_e$, and tautological line bundle $\mathscr{O}_Y(1)$. Let $\mathscr{G} \subset T_Y$ be a codimension 1 foliation on $Y$ with $\det(\mathscr{G}) \cong q^*\mathscr{A}$ for some ample line bundle $\mathscr{A}$ on $\mathbb{F}_e$. Then one of the following holds:*

(1) *$e \in \{0, 1\}$ and there exists a line bundle $\mathscr{B}$ on $\mathbb{F}_e$ such that*
- *$\mathscr{K} \cong \mathscr{B} \oplus \mathscr{B}$,*
- *$\mathscr{G}$ is induced by the natural morphism $Y \cong \mathbb{F}_e \times \mathbb{P}^1 \to \mathbb{P}^1$ and thus*

$$\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(-K_{\mathbb{F}_e}).$$

(2) *There exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, integers $s \geqslant 1$ and $t \geqslant 0$, a minimal section $C_0$ and a fiber $f$ of $p : \mathbb{F}_e \to \mathbb{P}^1$ such that*
- *$\mathscr{K} \cong \mathscr{B} \otimes (\mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0 + tf))$,*
- *$\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ containing $\Sigma + q^*(sC_0 + tf)$, where $\Sigma$ is the section of $q : Y \to \mathbb{F}_e$ corresponding to the surjection*

$$\mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0 + tf) \twoheadrightarrow \mathscr{O}_{\mathbb{F}_e},$$

- *one has*

$$\det(\mathscr{G}) \cong \begin{cases} q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+2)f) & \text{if } t = 0, \\ q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f) & \text{if } t > 0. \end{cases}$$

(3) *$e \in \{0,1\}$, there exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, an integer $s \geqslant 1$, and an irreducible divisor $B \sim C_0 + f$ on $\mathbb{F}_e$, where $C_0$ is a minimal section and $f$ a fiber of $p : \mathbb{F}_e \to \mathbb{P}^1$, such that*

- $\mathscr{K} \cong \mathscr{B} \otimes (\mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(s(C_0 + f)))$,
- *$\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ containing $\Sigma + sq^*B$, where $\Sigma$ is the section of $q : Y \to \mathbb{F}_e$ corresponding to the surjection*
$$\mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(s(C_0 + f)) \twoheadrightarrow \mathscr{O}_{\mathbb{F}_e},$$
- $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f)$.

(4) *There exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, integers $s, t \geqslant 1$, a minimal section $C_0$ and a fiber $f$ of $p : \mathbb{F}_e \to \mathbb{P}^1$ such that*

- $\mathscr{K} \cong \mathscr{B} \otimes (\mathscr{O}_{\mathbb{F}_e}(sC_0) \oplus \mathscr{O}_{\mathbb{F}_e}(tf))$,
- *$\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ generated by $\Sigma + sq^*C_0$ and $\Sigma' + tq^*f$, where $\Sigma$ and $\Sigma'$ are sections of $q : Y \to \mathbb{F}_e$ corresponding to the surjections*
$$\mathscr{O}_{\mathbb{F}_e}(sC_0) \oplus \mathscr{O}_{\mathbb{F}_e}(tf) \twoheadrightarrow \mathscr{O}_{\mathbb{F}_e}(tf)$$
*and*
$$\mathscr{O}_{\mathbb{F}_e}(sC_0) \oplus \mathscr{O}_{\mathbb{F}_e}(tf) \twoheadrightarrow \mathscr{O}_{\mathbb{F}_e}(sC_0),$$
*respectively,*
- $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f)$.

(5) *There exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, integers $s \geqslant 1$ and $\lambda \geqslant 0$, a minimal section $C_0$ and a fiber $f$ of $p : \mathbb{F}_e \to \mathbb{P}^1$ such that*

- *$\mathscr{K}$ fits into an exact sequence*
$$0 \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K} \otimes \mathscr{B}^* \to \mathscr{O}_f(-\lambda) \to 0,$$
- *$\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ generated by $\Sigma + sq^*C_0$ and $\Sigma'$, where $\Sigma$ is the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*(\mathscr{B}^* \otimes \mathscr{O}_{\mathbb{F}_e}(-sC_0))$ corresponding to*
$$\mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K} \otimes \mathscr{B}^*,$$
*and $\Sigma'$ corresponds to*
$$\mathscr{O}_{\mathbb{F}_e} \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K} \otimes \mathscr{B}^*,$$
- $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f)$.

(6) *There exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, an integer $t \geqslant 0$, a minimal section $C_0$ and a fiber $f$ of $p : \mathbb{F}_e \to \mathbb{P}^1$, and a local complete intersection subscheme $\Lambda \subset \mathbb{F}_e$ of codimension 2, with $h^0(\mathbb{F}_e, \mathscr{I}_\Lambda\mathscr{O}_{\mathbb{F}_e}(C_0)) \geqslant 1$, such that*

- *$\mathscr{K}$ fits into an exact sequence*
$$0 \to \mathscr{O}_{\mathbb{F}_e}(tf) \to \mathscr{K} \otimes \mathscr{B}^* \to \mathscr{I}_\Lambda\mathscr{O}_{\mathbb{F}_e}(C_0) \to 0,$$
- *$\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ containing $\Sigma + tq^*f$, where $\Sigma$ is the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*(\mathscr{B}^* \otimes \mathscr{O}_{\mathbb{F}_e}(-tf))$ corresponding to*
$$\mathscr{O}_{\mathbb{F}_e}(tf) \to \mathscr{K} \otimes \mathscr{B}^*,$$
- $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f)$.

(7) *There exist a line bundle $\mathscr{B}$ on $\mathbb{F}_e$, a minimal section $C_0$ and a fiber $f$ of $p : \mathbb{F}_e \to \mathbb{P}^1$, and a local complete intersection subscheme $\Lambda \subset \mathbb{F}_e$ of codimension 2 such that*

- *there exists a curve $C \sim C_0 + f$ with $\Lambda \subset C$ and $\Lambda \not\subset C'$ for any proper subcurve $C' \subsetneq C$,*
- *$\mathscr{K}$ fits into an exact sequence*
$$0 \to \mathscr{O}_{\mathbb{F}_e} \to \mathscr{K} \otimes \mathscr{B}^* \to \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f) \to 0,$$
- *$\mathscr{G}$ is induced by a pencil of irreducible members of $|\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*|$ containing $\Sigma$, the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*\mathscr{B}^*$ corresponding to $\mathscr{O}_{\mathbb{F}_e} \to \mathscr{K} \otimes \mathscr{B}^*$,*
- $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{F}_e}(C_0 + (e+1)f)$.

*Proof.* To ease notation, set $S := \mathbb{F}_e$. Denote by $C_0$ a minimal section, and by $f$ a general fiber of $p : S \to \mathbb{P}^1$. Denote by $F \cong \mathbb{P}^1$ a general fiber of $q : Y \to S$. Given any curve $C \subset S$, we set $Y_C := q^{-1}(C)$, and denote by $q_C : Y_C \to C$ the restriction of $q$ to $Y_C$.

We claim that $T_{Y/S} \not\subseteq \mathscr{G}$. Indeed, if $T_{Y/S} \subseteq \mathscr{G}$, then $\mathscr{G}$ would be the pullback via $q$ of a foliation on $S$, and so $\mathscr{O}_{\mathbb{P}^1} \cong \det(\mathscr{G})_{|F} \cong (T_{Y/S})_{|F} \cong \mathscr{O}_{\mathbb{P}^1}(2)$ by Paragraph 12, which is absurd. Thus, the natural map $T_Y \to q^*T_S$ induces an injective morphism of sheaves $\mathscr{G} \to q^*T_S$. Let $q^*B$ be the divisor of zeroes of the induced map $q^*\det(\mathscr{A}) \cong \det(\mathscr{G}) \to q^*\det(T_S)$.

Suppose that $B = 0$. Then $\mathscr{A} \cong \mathscr{O}_S(-K_S)$ is ample, and hence $e \in \{0, 1\}$. Moreover, $\mathscr{G} \subset T_Y$ induces a flat connection on $q$. Thus $\mathscr{K} \cong \mathscr{B} \oplus \mathscr{B}$ for some line bundle $\mathscr{B}$ on $S$, and $\mathscr{G}$ is induced by the natural morphism $Y \cong S \times \mathbb{P}^1 \to \mathbb{P}^1$. This is case (1) in the statement of Proposition 61.

Suppose from now on that $B \neq 0$. A straightforward computation shows that one of the following holds (up to possibly exchanging $C_0$ and $f$ when $e = 0$):

- $\mathscr{A} \cong \mathscr{O}_S(C_0 + (e+1)f)$ and $B \sim C_0 + f$,
- $\mathscr{A} \cong \mathscr{O}_S(C_0 + (e+2)f)$ and $B \sim C_0$.

In either case, $B$ contains a unique irreducible component dominating $\mathbb{P}^1$. We denote this irreducible component by $B_1$, and set $B_2 := B - B_1$.

Let $C \cong \mathbb{P}^1$ be a general member of $|\mathscr{O}_S(C_0 + ef)|$. Since $T_{Y/S} \not\subseteq \mathscr{G}$, $\mathscr{G}$ induces foliations by curves $\mathscr{C}_f \subset T_{Y_f}$ and $\mathscr{C}_C \subset T_{Y_C}$ on $Y_f$ and $Y_C$, respectively. By Paragraph 14, there exist effective divisors $D_f$ on $Y_f$ and $D_C$ on $Y_C$ such that

$$\begin{aligned} \mathscr{O}_{Y_f}(-K_{\mathscr{C}_f}) &\cong (q^*\mathscr{A})_{|Y_f} \otimes \mathscr{O}_{Y_f}(D_f), \\ \mathscr{O}_{Y_C}(-K_{\mathscr{C}_C}) &\cong q^*(\mathscr{A} \otimes \mathscr{O}_S(-C_0 - ef))_{|Y_C} \otimes \mathscr{O}_{Y_C}(D_C). \end{aligned} \tag{4.7}$$

**Claim.** *The following hold:*

(a) *One has $\mathscr{C}_f \cong q_f^*\mathscr{O}_{\mathbb{P}^1}(1)$, $Y_f \cong \mathbb{F}_m$ with $m \geqslant 1$, and $\mathscr{C}_f$ is induced by a pencil containing $\sigma_0 + m\ell_0$, where $\sigma_0$ denotes the minimal section and $\ell_0$ a fiber of $q_f : Y_f \to \mathbb{P}^1$.*

(b) *If $B \sim C_0 + f$ (and so $C \cap \mathrm{Supp}(B) \neq \emptyset$), then $\mathscr{C}_C \cong q_C^*\mathscr{O}_{\mathbb{P}^1}(1)$, $Y_C \cong \mathbb{F}_m$ with $m \geqslant 1$, and $\mathscr{C}_C$ is induced by a pencil containing $\sigma_0 + m\ell_0$, where $\sigma_0$ denotes the minimal section and $\ell_0$ a fiber of $q_C : Y_C \to \mathbb{P}^1$.*

(c) *If $B \sim C_0$ (and so $C \cap \mathrm{Supp}(B) = \emptyset$), then $\mathscr{C}_C \cong q_C^*\mathscr{O}_{\mathbb{P}^1}(2)$, $Y_C \cong C \times \mathbb{P}^1$, and $\mathscr{C}_C$ is induced by the projection morphism $C \times \mathbb{P}^1 \to \mathbb{P}^1$.*

On the open subset $Y \setminus q^{-1}(\mathrm{Supp}(B))$, $\mathscr{G}$ is regular and induces a flat connection. Therefore $\mathscr{G}_{|Y_f}$ intersects $T_{Y_f/f}$ transversely over $f \setminus \{f \cap \mathrm{Supp}(B)\}$, and thus the support of $D_f$ is contained in $q^{-1}(f \cap \mathrm{Supp}(B))$. It follows from (4.7) that $\mathscr{C}_f \cong q_f^*\mathscr{O}_{\mathbb{P}^1}(k)$ for some positive integer $k$. Since the natural map $\mathscr{C}_f \to q_f^*T_f$ is injective, we must have $k \in \{1,2\}$. The same argument shows that $\mathscr{C}_C \cong q_C^*\mathscr{O}_{\mathbb{P}^1}(l)$ with $l \in \{1,2\}$. If $\mathscr{C}_f \cong q_f^*\mathscr{O}_{\mathbb{P}^1}(2)$, then $\mathscr{C}_f$ is a regular foliation, $Y_f \cong f \times \mathbb{P}^1$, and $\mathscr{C}_f$ is induced by the projection morphism $f \times \mathbb{P}^1 \to \mathbb{P}^1$. On the other hand, if $b$ is a general point in $\mathrm{Supp}(B)$, then $q^{-1}(b)$ is tangent to $\mathscr{G}$, while $\mathscr{G}$ is regular at a general point of $q^{-1}(b)$. Since $f$ is assumed to be general, $f \cap \mathrm{Supp}(B)$ is a general point $b \in \mathrm{Supp}(B)$, and we conclude from this observation that $Y_f$ must be a leaf of $\mathscr{G}$, which is absurd. Therefore we must have $\mathscr{C}_f \cong q_f^*\mathscr{O}_{\mathbb{P}^1}(1)$ and $D_f = 0$. Analogously, we prove that if $C \cap \mathrm{Supp}(B) \neq \emptyset$, then $\mathscr{C}_C \cong q_C^*\mathscr{O}_{\mathbb{P}^1}(1)$ and $D_C = 0$. The description of $(Y_f, \mathscr{C}_f)$ and $(Y_C, \mathscr{C}_C)$ in this case follow from Lemma 60. Finally, if $B \sim C_0$, then $D_C = 0$, and $\mathscr{C}_C$ induces a flat connection on $q_C$. Therefore $\mathscr{C}_C \cong q_C^*\mathscr{O}_{\mathbb{P}^1}(2)$, $Y_C \cong C \times \mathbb{P}^1$, and $\mathscr{C}_C$ is induced by the projection morphism $C \times \mathbb{P}^1 \to \mathbb{P}^1$. This proves the claim.

Next we show that $\mathscr{G}$ has algebraic leaves, and that a general leaf has relative degree 1 over $S$. From the claim, we know that the general leaves of $\mathscr{C}_f$ and $\mathscr{C}_C$ are sections of $q_f : Y_f \to f$ and $q_C : Y_C \to C$, respectively. Let $F_C$ be a general leaf of $\mathscr{C}_C$ mapping onto $C$. For a general fiber $f$ of $p : S \to \mathbb{P}^1$, $Y_f$ meets $F_C$ in a single point, and there is a unique leaf $F_f$ of $\mathscr{C}_f$ through this point. We let $\Sigma$ be the closure of the union of the leaves $F_f$ obtained in this way, as $f$ varies through general fibers of $p : S \to \mathbb{P}^1$. It is a general leaf of $\mathscr{G}$, and has relative degree 1 over $S$.

Since $\mathscr{G}$ is algebraically integrable, we can consider the rational first integral for $\mathscr{G}$, $\pi : Y \dashrightarrow \tilde{W}$, as described in Paragraph 16. Since $Y$ is a rational variety, $\tilde{W} \cong \mathbb{P}^1$. So $\mathscr{G}$ is induced by a pencil $\Pi$ in the linear system $|\mathscr{O}_Y(1) \otimes q^*\mathscr{M}|$ for some line bundle $\mathscr{M}$ on $S$. Notice that $\pi_f := \pi_{|Y_f} : Y_f \dashrightarrow \mathbb{P}^1$ and $\pi_C := \pi_{|Y_C} : Y_C \dashrightarrow \mathbb{P}^1$ are rational first integrals for $\mathscr{C}_f$ and $\mathscr{C}_C$, respectively, and $\mathscr{C}_f$ and $\mathscr{C}_C$ are induced by the restricted pencils $\Pi_{|Y_f}$ and $\Pi_{|Y_C}$, respectively.

Our next task is to determine the line bundle $\mathscr{M}$. From Claim (a)–(c), there are integers $a, b, s, t$, with $s \geqslant 1$ and $t \geqslant 0$ such that

$$\mathscr{K}_{|f} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(a+s) \quad \text{and} \quad \mathscr{K}_{|C} \cong \mathscr{O}_{\mathbb{P}^1}(b) \oplus \mathscr{O}_{\mathbb{P}^1}(b+t).$$

Moreover, $\mathscr{M}_{|f} \cong \mathscr{O}_{\mathbb{P}^1}(-a)$ and $\mathscr{M}_{|C} \cong \mathscr{O}_{\mathbb{P}^1}(-b)$. This implies that $\mathscr{M} \cong \mathscr{O}_S(-aC_0 - bf)$.

Any member of $\Pi$ can be written as $\Sigma + uq^*B_1 + vq^*B_2$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $u, v \geqslant 0$ are integers. In particular, the ramification divisor $R(\pi)$ of $\pi$ must be of the form $R(\pi) = cq^*B_1 + dq^*B_2$, with $c, d \geqslant 0$ integers. We have

$$N_{\mathscr{G}} \cong (\pi^*\Omega^1_{\mathbb{P}^1} \otimes \mathscr{O}_Y(R(\pi)))^* \cong \mathscr{O}_Y(2) \otimes q^*\mathscr{M}^{\otimes 2} \otimes \mathscr{O}_Y(-R(\pi)).$$

On the other hand,

$$\begin{aligned} N_{\mathscr{G}} &\cong \mathscr{O}_Y(-K_Y) \otimes \mathscr{O}_Y(K_{\mathscr{G}}) \\ &\cong \mathscr{O}_Y(2) \otimes q^*(\det(\mathscr{K}^*) \otimes \mathscr{O}_S(-K_S) \otimes \mathscr{A}^*) \\ &\cong \mathscr{O}_Y(2) \otimes q^*(\det(\mathscr{K}^*) \otimes \mathscr{O}_S(B)), \end{aligned}$$

and hence

$$\begin{aligned} \mathscr{O}_Y(R(\pi)) &\cong q^*(\mathscr{M}^{\otimes 2} \otimes \det(\mathscr{K}) \otimes \mathscr{O}_S(-B)) \\ &\cong q^*(\mathscr{O}_S(sC_0 + tf) \otimes \mathscr{O}_S(-B)). \end{aligned}$$

It follows from Claim (b)–(c) that, if $B \sim C_0$, then $t = d = 0$, and if $B \not\sim C_0$, then $c = s - 1$ and $d = t - 1$. Notice also that, if $s \geqslant 2$, then the pencil $\Pi$ contains a member of the form $\Sigma + sq^*B_1 + vq^*B_2$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $v \geqslant 0$. Similarly, if $t \geqslant 2$, then the pencil $\Pi$ contains a member of the form $\Sigma + uq^*B_1 + tq^*B_2$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $u \geqslant 0$.

**Case 1.** Suppose that $\Pi$ contains a member of the form $\Sigma + uq^*B_1 + vq^*B_2$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, $B_2 \neq 0$, and $u, v > 0$.

Up to replacing $C_0$ and $f$ with linearly equivalent curves on $S$, we may write

$$B = C_0 + f.$$

It follows from Claim (a)–(b) that $u = s$ and $v = t$. Moreover $\Sigma \cap Y_f$ and $\Sigma \cap Y_C$ are the minimal sections of $q_f : Y_f \to f$ and $q_C : Y_C \to C$, respectively. If $\Sigma'$ is the closure of a general leaf of $\mathscr{G}$, then $\Sigma \cap \Sigma' \cap Y_f = \emptyset = \Sigma \cap \Sigma' \cap Y_C$. This implies that $\Sigma \cap \Sigma' \cap q^{-1}(b) = \emptyset$ for a general point $b \in \operatorname{Supp}(B)$. One can find an open subset $V \subset S$, with $\operatorname{codim}_S(S \setminus V) \geqslant 2$, such that $\Sigma \cap q^{-1}(V)$ and $\Sigma' \cap q^{-1}(V)$ are sections of $q_{|q^{-1}(V)}$, and $\Sigma \cap \Sigma' \cap q^{-1}(V) = \emptyset$. Therefore, there are line bundles $\mathscr{B}_1$ and $\mathscr{B}_2$ on $S$ such that $\mathscr{K} \cong \mathscr{B}_1 \oplus \mathscr{B}_2$, and $\Sigma$ corresponds to the surjection $\mathscr{B}_1 \oplus \mathscr{B}_2 \twoheadrightarrow \mathscr{B}_1$. From the description of $\mathscr{K}_{|f}$ and $\mathscr{K}_{|C}$ above, we see that $\mathscr{B}_{1|f} \cong \mathscr{O}_{\mathbb{P}^1}(a)$, $\mathscr{B}_{1|C} \cong \mathscr{O}_{\mathbb{P}^1}(b)$, $\mathscr{B}_{2|f} \cong \mathscr{O}_{\mathbb{P}^1}(a+s)$, and $\mathscr{B}_{2|C} \cong \mathscr{O}_{\mathbb{P}^1}(b+t)$. Thus $\mathscr{B}_1 \cong \mathscr{O}_S(aC_0 + bf)$, and $\mathscr{B}_2 \cong \mathscr{O}_S((a+s)C_0 + (b+t)f)$. We are in case (2) in the statement of Proposition 61, with $t \geqslant 1$.

**Case 2.** Suppose that $\Pi$ contains a member of the form $\Sigma + uq^*B_1$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $u > 0$.

It follows from Claim (a) that $u = s$. Next we prove that any other reducible divisor of $\Pi$ must be of the form $\Sigma' + tq^*B_2$, where $\Sigma'$ is irreducible and has relative degree 1 over $S$. In particular, if there exists such a divisor in $\Pi$, we must have $B_2 \neq 0$ and $t > 0$. Indeed, let $D = \Sigma' + iq^*B_1 + jq^*B_2$ be a reducible member of $\Pi$, where $\Sigma'$ is irreducible and has relative degree 1 over $S$. If $i > 0$, then it follows from the claim that $i = s$ and $\Sigma = \Sigma'$. Since $D \sim \Sigma + sq^*B_1$, we must have $D = \Sigma + sq^*B_1$. If $i = 0$, then we must have $B_2 \neq 0$ and $j > 0$. It follows from Claim (b) that $j = t$, and so $D = \Sigma' + tq^*B_2$.

We consider three cases.

**Case 2.1.** Suppose that $B_2 = 0$, and $B = B_1 \sim C_0 + f$. Then we must have $e \in \{0, 1\}$.

Let $\Sigma'$ be the closure of a general leaf of $\mathscr{G}$. It follows from Claim (a) that $\Sigma \cap Y_f$ is the minimal section of $q_f$, and $\Sigma \cap \Sigma' \cap Y_f = \emptyset$. Proceeding as in case 1, we show that we must be in case (3) in the statement of Proposition 61.

**Case 2.2.** Suppose that $B_2 = 0$, and $B = B_1 \sim C_0$.

Up to replacing $C_0$ with a linearly equivalent curve on $S$, we may write $B = C_0$. Since $\Sigma$ is irreducible and has relative degree 1 over $S$, it contains only finitely many fibers of $q$, and corresponds to a surjective morphism of sheaves $\mathscr{K} \twoheadrightarrow \mathscr{I}_\Lambda \mathscr{S}$, where $\mathscr{S}$ is a line bundle on $S$, and $\Lambda \subset S$ is a closed subscheme with $\operatorname{codim}_S \Lambda \geqslant 2$. Denote by $\mathscr{T}$ the kernel of this morphism. Then $\mathscr{T}$ is a line bundle on $S$, and

$$\mathscr{T} \otimes \mathscr{S} \cong \det(\mathscr{K}) \cong \mathscr{O}_S((2a+s)C_0 + 2bf).$$

Since $\Sigma + sq^*C_0 \in \Pi \subset |\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf)|$, we must have

$$\mathscr{T} \otimes \mathscr{O}_S(-aC_0 - bf) \cong \mathscr{O}_S(sC_0) \quad \text{and} \quad \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf) \cong \mathscr{O}_S.$$

Let $\Sigma'$ be the closure of a general leaf of $\mathscr{G}$, and let $\sigma' \in H^0(Y, \mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf))$ be a nonzero section vanishing along $\Sigma'$. We claim that $\sigma'$ is mapped to a nonzero element in $H^0(S, \mathscr{I}_\Lambda \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf)) \cong H^0(S, \mathscr{I}_\Lambda)$ under the natural morphism

$$H^0(S, \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)) \to H^0(S, \mathscr{I}_\Lambda \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf)).$$

Indeed, if $0 \neq \sigma \in H^0(Y, \mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf)))$ comes from

$$\begin{aligned} H^0(S, \mathscr{T} \otimes \mathscr{O}_S(-aC_0 - bf)) &\subset H^0(S, \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)) \\ &\cong H^0(Y, \mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf)), \end{aligned}$$

then its zero locus on $Y$ must be reducible, yielding a contradiction. We conclude that $\Lambda = \emptyset$, and the exact sequence $0 \to \mathscr{T} \to \mathscr{K} \to \mathscr{S} \to 0$ splits. So we are in case (2) in the statement of Proposition 61, with $s > 0$ and $t = 0$.

**Case 2.3.** Suppose that $B_2 \neq 0$.

Up to replacing $C_0$ and $f$ with linearly equivalent curves on $S$, we may write

$$B = C_0 + f.$$

As in Case 2.2, $\Sigma$ corresponds to a surjective morphism of sheaves $\mathscr{K} \twoheadrightarrow \mathscr{I}_\Lambda \mathscr{S}$, where $\mathscr{S}$ is a line bundle on $S$, and $\Lambda \subset S$ is a closed subscheme with $\operatorname{codim}_S \Lambda \geqslant 2$. Denote by $\mathscr{T}$ the kernel of this morphism. Then $\mathscr{T}$ is a line bundle on $S$, and

$$\begin{aligned} \mathscr{T} \otimes \mathscr{S} \cong \det(\mathscr{K}) &\cong \mathscr{O}_S((2a+s)C_0 + (2b+t)f) \\ &\cong \mathscr{O}_S(sC_0 + tf) \otimes \mathscr{O}_S(2aC_0 + 2bf). \end{aligned}$$

Since $\Sigma + sq^*C_0 \in |\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf)|$, we must have

$$\mathscr{T} \otimes \mathscr{O}_S(-aC_0 - bf) \cong \mathscr{O}_S(sC_0) \quad \text{and} \quad \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf) \cong \mathscr{O}_S(tf).$$

Thus, we have an exact sequence

$$0 \to \mathscr{O}_S(sC_0) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf) \to \mathscr{I}_\Lambda \mathscr{O}_S(tf) \to 0,$$

where $\mathscr{O}_S(sC_0) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)$ is the map corresponding to $\Sigma + sq^*C_0 \in \Pi$.

Suppose that $\Sigma + sq^*C_0$ is the only reducible member of $\Pi$. Then we must have $t = 1$. Let $\Sigma'$ be the closure of a general leaf of $\mathscr{G}$, and let $\sigma' \in H^0(Y, \mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf))$ be a nonzero section vanishing along $\Sigma'$. As in Case 2.2, we see that $\sigma'$ is mapped to a nonzero element $\bar{\sigma}' \in H^0(S, \mathscr{I}_\Lambda \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf)) \cong H^0(S, \mathscr{I}_\Lambda \mathscr{O}_S(f))$ under the natural morphism

$$H^0(S, \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)) \to H^0(S, \mathscr{I}_\Lambda \mathscr{S} \otimes \mathscr{O}_S(-aC_0 - bf)).$$

Let $f' \sim f$ be the divisor of zeroes of $\bar{\sigma}'$. Then $\Lambda \subset f'$. Since $\Sigma \cap \Sigma' \cap q^{-1}(b) = \emptyset$ for any point $b \in C_0 \setminus f$, we must have $f = f'$. We obtain an exact sequence

$$0 \to \mathscr{O}_S \oplus \mathscr{O}_S(sC_0) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf) \to \mathscr{O}_f(-\Lambda) \to 0,$$

where $\mathscr{O}_S \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)$ is the map given by $\sigma'$. We are in case (5) in the statement of Proposition 61.

Now suppose that $\Pi$ contains a second reducible divisor. We have seen above that it must be of the form $\Sigma' + tq^*f$, where $\Sigma'$ is irreducible and has relative degree 1 over $S$. As before, it gives rise to an exact sequence

$$0 \to \mathscr{O}_S(tf) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf) \to \mathscr{I}_{\Lambda'}\mathscr{O}_S(sC_0) \to 0,$$

where $\Lambda' \subset S$ is a closed subscheme with $\operatorname{codim}_S \Lambda' \geqslant 2$. Notice that $\Sigma \neq \Sigma'$, and so the induced morphism

$$\mathscr{O}_S(sC_0) \oplus \mathscr{O}_S(tf) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf)$$

is injective. Since $\det(\mathscr{O}_S(sC_0) \oplus \mathscr{O}_S(tf)) \cong \det(\mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf))$, it is in fact an isomorphism. We are in case (4) in the statement of Proposition 61.

**Case 3.** Suppose that $\Pi$ contains a member of the form $\Sigma + vq^*B_2$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, $B_2 \neq 0$, and $v > 0$.

Up to replacing $C_0$ and $f$ with linearly equivalent curves on $S$, we may write

$$B = C_0 + f.$$

It follows from Claim (b) that $v = t$.

As in Case 2, we see that any other reducible divisor of $\Pi$ must be of the form $\Sigma' + sq^*C_0$, where $\Sigma'$ is irreducible and has relative degree 1 over $S$. If there exists such a divisor, we are in Case 2.3 above. So we may assume that $\Sigma + tq^*f$ is the only reducible member of $\Pi$. This implies that $s = 1$, and $\Sigma$ gives rise to an exact sequence

$$0 \to \mathscr{O}_S(tf) \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf) \to \mathscr{I}_{\Lambda}\mathscr{O}_S(C_0) \to 0,$$

where $\Lambda \subset S$ is a closed subscheme with $\operatorname{codim}_S \Lambda \geqslant 2$. If $\Lambda = \emptyset$, then the sequence splits since $h^1(S, \mathscr{O}_S(-C_0 + tf)) = 0$, and we are in case (4) in the statement of Proposition 61, with $s = 1$. If $\Lambda \neq \emptyset$, then $\Lambda$ is a local complete intersection subscheme, and we are in case (6) in the statement of Proposition 61, with $t \geqslant 1$.

**Case 4.** Suppose that all members of $\Pi$ are irreducible. Then $s = 1$ and $t \leqslant 1$.

Let $\Sigma'$ be the closure of a general leaf of $\mathscr{G}$. It gives rise to an exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{K} \otimes \mathscr{O}_S(-aC_0 - bf) \to \mathscr{I}_{\Lambda}\mathscr{O}_S(C_0 + tf) \to 0,$$

where $\Lambda \subset S$ is a closed subscheme with $\operatorname{codim}_S \Lambda \geqslant 2$. We claim that $\Lambda \neq \emptyset$. Indeed, if $\Lambda = \emptyset$, then the sequence splits since $h^1(S, \mathscr{O}_S(-C_0 - f)) = 0$. But this implies that $\Pi$ contains a reducible member, contrary to our assumptions. Hence $\operatorname{codim}_S \Lambda = 2$, and $\Lambda$ is a local complete intersection subscheme.

If $t = 0$, then we are in case (6) in the statement of Proposition 61.

Suppose from now on that $t = 1$. Then we must have $h^0(S, \mathscr{I}_{\Lambda}\mathscr{O}_S(C_0 + f)) \geqslant 1$. We will show that, there exists a curve $C \sim C_0 + f$ with $\Lambda \subset C$ such that, for any proper subcurve $C' \subsetneq C$, $\Lambda \not\subset C'$. Suppose to the contrary that any curve $C \sim C_0 + f$ with $\Lambda \subset C$ can be written as $C = C_1 \cup f_1$ with $C_1 \sim C_0$, $f_1 \sim f$, and either $\Lambda \subset C_1$, or $\Lambda \subset f_1$. This implies that the set of reducible members of $|\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-aC_0 - bf)|$ has codimension 1 since $h^0(S, \mathscr{O}_S(-f)) = 0 = h^0(S, \mathscr{O}_S(-C_0))$. This contradicts the fact that all members of $\Pi$ are irreducible. We are in case (7) in the statement of Proposition 61. □

Next we investigate whether all the seven cases described in Proposition 61 in fact occur.

**62** (Proposition 61 (1)–(4)). Let $\mathscr{K}$ be a rank 2 vector bundle on a ruled surface

$$p : \mathbb{F}_e \to \mathbb{P}^1,$$

with $e \geqslant 0$. Set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{F}_e$, and denote by $\mathscr{O}_Y(1)$ the tautological line bundle on $Y$. Suppose that $\mathscr{K}$ satisfies one of the conditions (1)–(4) in the statement of Proposition 61. Then the pencil $\Pi$ described in the statement yields a codimension 1 foliation $\mathscr{G}$ on $Y$ with $\det(\mathscr{G}) \cong q^*\mathscr{A}$ where $\mathscr{A}$ is an ample line bundle.

**63** (Proposition 61 (5)). Consider the ruled surface $p : \mathbb{F}_e \to \mathbb{P}^1$, with $e \geqslant 0$, denote by $C_0$ a minimal section, and by $f$ a fiber of $p$. Let $s, \lambda \geqslant 0$ be integers, and suppose that $\mathscr{K}$ is a coherent sheaf on $\mathbb{F}_e$ fitting into an exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}_{\mathbb{F}_e}(f) \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0 + f) \to \mathscr{O}_{f_0}(s + \lambda + 1) \to 0. \tag{4.8}$$

By [28, Proposition 5.2.2], $\mathscr{K}$ is a rank 2 vector bundle on $\mathbb{F}_e$. Since

$$\det(\mathscr{K}) \cong \mathscr{O}_{\mathbb{F}_e}(sC_0 + f),$$

we have

$$\mathscr{K}^* \cong \mathscr{K} \otimes \mathscr{O}_{\mathbb{F}_e}(-sC_0 - f).$$

Dualizing sequence (4.8), and twisting it with $\mathscr{O}_{\mathbb{F}_e}(sC_0 + f)$ yields

$$0 \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K} \to \mathscr{O}_{f_0}(-\lambda) \to 0. \tag{4.9}$$

Conversely, dualizing sequence (4.9), and twisting it with $\mathscr{O}_{\mathbb{F}_e}(sC_0+f)$ yields sequence (4.8).

Now set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{F}_e$ and tautological line bundle $\mathscr{O}_Y(1)$. Let $\Sigma$ be the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{F}_e}(-sC_0)$ corresponding to the map $\mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K}$ induced by (4.9). Similarly, let $\Sigma'$ be the zero locus of the section of $\mathscr{O}_Y(1)$ corresponding to the map $\mathscr{O}_{\mathbb{F}_e} \to \mathscr{O}_{\mathbb{F}_e} \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0) \to \mathscr{K}$ induced by (4.9). Let $\Pi$ be the pencil in $|\mathscr{O}_Y(1)|$ generated by $\Sigma + sq^*C_0$ and $\Sigma'$.

If $\Sigma + sq^*C_0$ is the only reducible member of $\Pi$, then this pencil induces a foliation $\mathscr{G}$ on $Y$ as in Proposition 61 (5). So we investigate this condition. Suppose that there exists another reducible divisor in $\Pi$, and write it as $\Sigma'' + q^*D \neq \Sigma + sq^*C_0$, where $D \sim uC_0 + vf$ is a nonzero effective divisor on $\mathbb{F}_e$. By restricting $\Pi$ to $Y_f = q^{-1}(f)$, we see that $u = 0$ and $v > 0$. Thus

$$h^0(\mathbb{F}_e, \mathscr{K} \otimes \mathscr{O}_{\mathbb{F}_e}(-vf_0)) \geqslant 1.$$

On the other hand,

$$h^0(\mathbb{F}_e, \mathscr{O}_{\mathbb{F}_e}(-vf_0) \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0 - vf_0)) = 0.$$

This implies $\lambda = 0$, and thus $\mathscr{K} \cong \mathscr{O}_{\mathbb{F}_e}(f) \oplus \mathscr{O}_{\mathbb{F}_e}(sC_0)$.

**64** (Proposition 61 (6)). Consider the ruled surface $p : \mathbb{F}_e \to \mathbb{P}^1$, with $e \geqslant 0$, denote by $C_0$ a minimal section, and by $f$ a fiber of $p$. Let $\Lambda \subset \mathbb{F}_e$ be a local complete intersection subscheme of codimension 2, and let $t \geqslant 0$ be an integer. By [28, Theorem 5.1.1], there exists a vector bundle $\mathscr{K}$ on $\mathbb{F}_e$ fitting into an exact sequence

$$0 \to \mathscr{O}_{\mathbb{F}_e}(tf) \to \mathscr{K} \to \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0) \to 0.$$

Set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{F}_e$ and tautological line bundle $\mathscr{O}_Y(1)$.

Note that the map

$$H^0(\mathbb{F}_e, \mathscr{K}) \to H^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0))$$

is surjective since $h^1(\mathbb{F}_e, \mathscr{O}_{\mathbb{F}_e}(tf)) = 0$. Suppose moreover that $h^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0)) \geqslant 1$. Let $s' \in H^0(Y, \mathscr{O}_Y(1))$ be a section mapping to a nonzero section in $H^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0))$, and denote by $\Sigma'$ its zero locus. We claim that $\Sigma'$ is irreducible. Indeed, if $\Sigma'$ is reducible, then, up to replacing $C_0$ by a linearly equivalent curve, we see that $s'$ must vanish along $q^{-1}(C_0)$. Therefore, $h^0(\mathbb{F}_e, \mathscr{K} \otimes \mathscr{O}_{\mathbb{F}_e}(-C_0)) \geqslant 1$, and hence $h^0(\mathbb{F}_e, \mathscr{O}_{\mathbb{F}_e}(tf - C_0)) \geqslant 1$, which is absurd. This shows that $\Sigma'$ is irreducible. Let $\Sigma$ be the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{F}_e}(-tf)$ corresponding to $\mathscr{O}_{\mathbb{F}_e}(tf) \to \mathscr{K}$. Then the pencil $\Pi \subset |\mathscr{O}_Y(1)|$ generated by $\Sigma + tq^* f_0$ and $\Sigma'$ induces a foliation $\mathscr{G}$ on $Y$ as in Proposition 61 (6).

**65** (Proposition 61 (7)). Consider the ruled surface $p : \mathbb{F}_e \to \mathbb{P}^1$, with $e \geqslant 0$, denote by $C_0$ a minimal section, and by $f$ a fiber of $p$. Let $\Lambda \subset \mathbb{F}_e$ be a local complete intersection subscheme of codimension 2. By [28, Theorem 5.1.1], there exists a vector bundle $\mathscr{K}$ on $\mathbb{F}_e$ fitting into an exact sequence

$$0 \to \mathscr{O}_{\mathbb{F}_e} \to \mathscr{K} \to \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f) \to 0.$$

Set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{F}_e$ and tautological line bundle $\mathscr{O}_Y(1)$.

Note that the map

$$H^0(\mathbb{F}_e, \mathscr{K}) \to H^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f))$$

is surjective since $h^1(\mathbb{F}_e, \mathscr{O}_{\mathbb{F}_e}) = 0$. We assume moreover that $h^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f)) \geqslant 1$, and that there exists a curve $C \sim C_0 + f$ with $\Lambda \subset C$ such that $\Lambda \not\subset C'$ for any proper subcurve $C' \subsetneq C$.

Let $\Sigma$ be the zero locus of the section of $\mathscr{O}_Y(1)$ corresponding to the map $\mathscr{O}_{\mathbb{F}_e} \to \mathscr{K}$. Then $\Sigma$ is a section of $q$ over $q^{-1}(\mathrm{Supp}(\Lambda))$, and hence it is irreducible. Let $\Sigma'$ be the zero locus of the section $s' \in H^0(\mathbb{F}_e, \mathscr{O}_Y(1))$ that lifts the section of $H^0(\mathbb{F}_e, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f))$ whose divisor of zeroes is $C$. We claim that $\Sigma'$ is irreducible. Suppose otherwise. Then $s'$ must vanish along a subcurve $C'$ of $C$. This implies that $h^0(S, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{F}_e}(C_0 + f - C')) \neq 0$. Therefore, $C'$ is a proper subcurve of $C$, and $\Lambda \subset C''$ where $C''$ is such that $C = C' \cup C''$, contrary to our assumptions. This shows that $\Sigma'$ is irreducible. The pencil $\Pi \subset |\mathscr{O}_Y(1)|$ generated by $\Sigma$ and $\Sigma'$ has only irreducible members, and induces a foliation $\mathscr{G}$ on $Y$ as in Proposition 61 (7).

Now we go back to the problem of describing $\mathscr{K}$ and $\mathscr{G}$ that appear in Proposition 57. It remains to consider the case when $S \cong \mathbb{P}^2$, and $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{P}^2}(a)$ with $a \in \{1, 2\}$. Proposition 66 below addresses the case $a = 2$, while Proposition 69 addresses the case $a = 1$.

**Proposition 66.** *Let $\mathscr{K}$ be a rank 2 vector bundle on $\mathbb{P}^2$. Set $Y := \mathbb{P}_{\mathbb{P}^2}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{P}^2$ and tautological line bundle $\mathscr{O}_Y(1)$. Let $\mathscr{G} \subset T_Y$ be a codimension 1 foliation on $Y$ with $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{P}^2}(2)$. Then one of the following holds:*

(1) *There exist integers $a$ and $s$, with $s \geqslant 1$, such that $\mathscr{K} \cong \mathscr{O}_{\mathbb{P}^2}(a) \oplus \mathscr{O}_{\mathbb{P}^2}(a+s)$, and $\mathscr{G}$ is induced by a pencil in $|\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{P}^2}(-a)|$ containing a divisor of the form $\Sigma + sq^*\ell_0$, where $\Sigma$ is the section of $q$ corresponding to the map $\mathscr{O}_{\mathbb{P}^2}(a) \oplus \mathscr{O}_{\mathbb{P}^2}(a+s) \twoheadrightarrow \mathscr{O}_{\mathbb{P}^2}(a)$, and $\ell_0 \subset \mathbb{P}^2$ is a line.*

(2) *There exist an integer $a$ and a local complete intersection subscheme $\Lambda \subset \mathbb{P}^2$ of codimension 2 such that $h^0(\mathbb{P}^2, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1)) \geqslant 1$, $\mathscr{K}$ fits into an exact sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{K} \otimes \mathscr{O}_{\mathbb{P}^2}(-a) \to \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1) \to 0,$$

*and $\mathscr{G}$ is induced by a pencil of irreducible members of $|\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{P}^2}(-a)|$ containing the zero locus of the section of $\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{P}^2}(-a)$ corresponding to the map $\mathscr{O}_{\mathbb{P}^2} \to \mathscr{K} \otimes \mathscr{O}_{\mathbb{P}^2}(-a)$.*

*Proof.* The proof is very similar to that of Proposition 61, and so we leave some easy details to the reader. To ease notation, set $S := \mathbb{P}^2$, and write $\mathscr{O}_S(1)$ for $\mathscr{O}_{\mathbb{P}^2}(1)$.

Since $T_{Y/S} \not\subseteq \mathscr{G}$, the natural map $T_Y \to q^*T_S$ induces an injective morphism of sheaves $\mathscr{G} \to q^*T_S$. There is a line $\ell_0 \subset \mathbb{P}^2$ such that the divisor of zeroes of the induced map

$$q^*\mathscr{O}_{\mathbb{P}^2}(2) \cong \det(\mathscr{G}) \to q^* \det(T_S)$$

is $q^*\ell_0$. Note that $\mathscr{G}$ induces a flat connection on $q : Y \to S$ over $q^{-1}(\mathrm{Supp}(\ell_0))$.

Let $\ell \subset \mathbb{P}^2$ be a general line, set $Y_\ell := q^{-1}(\ell)$ and $q_\ell := q_{|Y_\ell} : Y_\ell \to \ell$. Since $T_{Y/S} \not\subseteq \mathscr{G}$, $\mathscr{G}$ induces a foliation by curves $\mathscr{C}_\ell \subset T_{Y_\ell}$. As in the claim in the proof of Proposition 61, one checks that $\mathscr{C}_\ell \cong q_\ell^*\mathscr{O}_{\mathbb{P}^1}(1)$, $Y_\ell \cong \mathbb{F}_s$ with $s \geqslant 1$, and $\mathscr{C}_\ell$ is induced by a pencil containing $\sigma_0 + sf$, where $\sigma_0$ denotes the minimal section, and $f$ a fiber of $q_\ell : Y_\ell \to \ell$.

One then shows that $\mathscr{G}$ has algebraic leaves, and it is induced by a pencil

$$\Pi \subset |\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_S(-a)|,$$

where $a$ is such that $\mathscr{K}_{|\ell} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(a+s)$.

Any member of $\Pi$ is of the form $\Sigma + uq^*\ell_0$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $u \geqslant 0$ is an integer. In particular, the ramification divisor of the rational first integral for $\mathscr{G}$, $\pi : Y \dashrightarrow \mathbb{P}^1$, must be of the form $R(\pi) = cq^*\ell_0$, with $c \geqslant 0$. An easy computation shows that $c = s - 1$. In particular, if $s \geqslant 2$, then $\Pi$ contains a member of the form $\Sigma + sq^*\ell_0$, where $\Sigma$ is irreducible and has relative degree 1 over $S$.

**Case 1.** Suppose that $\Pi$ contains a member of the form $\Sigma + uq^*\ell_0$, where $\Sigma$ is irreducible and has relative degree 1 over $S$, and $u \geqslant 1$ is an integer. It follows from the description of $\mathscr{C}_\ell$ above that $u = s$, and $\Sigma \cap Y_\ell$ is the minimal section of $q_\ell : Y_\ell \cong \mathbb{F}_s \to \ell$. If $\Sigma'$ is the closure of a general leaf of $\mathscr{G}$, then $\Sigma \cap \Sigma' \cap Y_\ell = \emptyset$. This implies that $\Sigma \cap \Sigma' \cap q^{-1}(b) = \emptyset$ for a general point $b \in \ell_0$. One can find an open subset $V \subset S$, with $\mathrm{codim}_S(S \setminus V) \geqslant 2$, such that $\Sigma \cap q^{-1}(V)$ and $\Sigma' \cap q^{-1}(V)$ are sections of $q_{|q^{-1}(V)}$, and $\Sigma \cap \Sigma' \cap q^{-1}(V) = \emptyset$. Therefore, there are line bundles $\mathscr{B}_1$ and $\mathscr{B}_2$ on $S$ such that $\mathscr{K} \cong \mathscr{B}_1 \oplus \mathscr{B}_2$, and $\Sigma$ corresponds to the surjection $\mathscr{B}_1 \oplus \mathscr{B}_2 \twoheadrightarrow \mathscr{B}_1$. From the description of $\mathscr{K}_{|\ell}$, we see that $\mathscr{B}_1 \cong \mathscr{O}_S(a)$ and $\mathscr{B}_2 \cong \mathscr{O}_S(a+s)$. This is case (1) in the statement of Proposition 66.

**Case 2.** Suppose then that all members of $\Pi$ are irreducible. In particular, we must have $s = 1$. Then the section $\Sigma$ gives rise to an exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{K} \otimes \mathscr{M} \to \mathscr{I}_\Lambda \mathscr{O}_S(1) \to 0,$$

where $\Lambda \subset S$ is a closed subscheme with $\mathrm{codim}_S \Lambda \geqslant 2$. If $\Lambda = \emptyset$, then the sequence splits since $h^1(S, \mathscr{O}_S(-1)) = 0$. But then $\Pi$ contains a reducible member, a contradiction. Thus $\Lambda \neq \emptyset$, and $\Lambda$ is a local complete intersection subscheme. This is case (2) in the statement of Proposition 66. □

Next we give examples of foliations of the type described in Proposition 66 (2).

**67.** Let $\Lambda \subset \mathbb{P}^2$ be a local complete intersection subscheme of codimension 2. Then, by [28, Theorem 5.1.1], there exists a vector bundle $\mathscr{K}$ on $\mathbb{P}^2$ fitting into an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{K} \to \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1) \to 0.$$

Set $Y := \mathbb{P}_{\mathbb{F}_e}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{P}^2$ and tautological line bundle $\mathscr{O}_Y(1)$.

Note that the map

$$H^0(\mathbb{P}^2, \mathscr{K}) \to H^0(\mathbb{P}^2, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1))$$

is surjective since $H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2})$ vanishes. We assume moreover that $h^0(\mathbb{P}^2, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1)) \geqslant 1$, and let $\ell_0$ be a line in $\mathbb{P}^2$ such that $\Lambda \subset \ell_0$. Let $\Sigma$ be the zero locus of the section of $\mathscr{O}_Y(1)$ corresponding to $\mathscr{O}_{\mathbb{P}^2} \to \mathscr{K}$. Then $\Sigma$ is a section of $q$ over $q^{-1}(\mathrm{Supp}(\Lambda))$, and hence it is irreducible. Let $s' \in H^0(\mathbb{P}^2, \mathscr{O}_Y(1))$ be a section lifting the section $H^0(\mathbb{P}^2, \mathscr{I}_\Lambda \mathscr{O}_{\mathbb{P}^2}(1))$ corresponding to $\ell_0$, and denote by $\Sigma'$ its zero locus. We claim that $\Sigma'$ is irreducible. Indeed, if $\Sigma'$ is reducible, then $s'$ must vanish along $\ell_0$. On the other hand,

$$h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-1)) = h^0(\mathbb{P}^2, \mathscr{I}_\Lambda) = 0,$$

and hence $h^0(\mathbb{P}^2, \mathscr{K} \otimes \mathscr{O}_{\mathbb{P}^2}(-1)) = 0$, yielding a contradiction.

Therefore, the pencil $\Pi \subset |\mathscr{O}_Y(1)|$ generated by $\Sigma$ and $\Sigma'$ induces a foliation $\mathscr{G}$ on $Y$ as in Proposition 66 (2).

**Example 68.** Set $Y := \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$, denote by $q : Y \to \mathbb{P}^2$ the natural projection, and by $\mathscr{O}_Y(1)$ the tautological line bundle. Let $p : Y \to \mathbb{P}^2$ be the morphism induced by the linear system $|\mathscr{O}_Y(1) \otimes q^*\mathscr{O}_{\mathbb{P}^2}(-1)|$. If $\mathscr{C} \subset T_{\mathbb{P}^2}$ is a degree 0 foliation, then $\mathscr{G} := p^{-1}(\mathscr{C})$ is a codimension 1 foliation on $Y$ with $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{P}^2}(2)$ as in Proposition 66 (2). Let $F$ be the closure of a leaf of $\mathscr{G}$. Then $q_{|F} : F \to \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at a point on the line $q(p^{-1}(\mathrm{Sing}(\mathscr{C})))$.

**Proposition 69.** *Let $\mathscr{K}$ be a rank 2 vector bundle on $\mathbb{P}^2$. Set $Y := \mathbb{P}_{\mathbb{P}^2}(\mathscr{K})$, with natural projection $q : Y \to \mathbb{P}^2$ and tautological line bundle $\mathscr{O}_Y(1)$. Let $\mathscr{G}$ be a codimension 1 foliation on $Y$ with $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{P}^2}(1)$, and suppose that $\mathscr{G}$ is not algebraically integrable. Then there exist*

- *a rational map $\psi : Y \dashrightarrow S = \mathbb{P}_\ell(\mathscr{K}_{|\ell})$, where $\ell \subset \mathbb{P}^2$ is a general line, giving rise to a foliation by rational curves $\mathscr{M} \cong q^*\mathscr{O}_{\mathbb{P}^2}(1)$ on $Y$, which lifts a degree 0 foliation on $\mathbb{P}^2$,*
- *a rank 1 foliation $\mathscr{N}$ on $S$ induced by a global vector field,*

*such that $\mathscr{G}$ is the pullback of $\mathscr{N}$ via $\psi$.*

*Proof.* First note that $T_{Y/\mathbb{P}^2} \not\subseteq \mathscr{G}$. Indeed, if $T_{Y/\mathbb{P}^2} \subseteq \mathscr{G}$, then $\mathscr{G}$ would be the pullback via $q$ of a foliation on $\mathbb{P}^2$. Denote by $f \cong \mathbb{P}^1$ a general fiber of $q : Y \to \mathbb{P}^2$. Then, by Paragraph 12, $\mathscr{O}_{\mathbb{P}^1} \cong \det(\mathscr{G})_{|f} \cong (T_{Y/S})_{|f} \cong \mathscr{O}_{\mathbb{P}^1}(2)$ which is absurd.

Let $\mathscr{A}$ be a very ample line bundle on $Y$. Since $\mathscr{G}$ is not algebraically integrable, by [3, Proposition 7.5], there exists an algebraically integrable subfoliation by curves $\mathscr{M} \subset \mathscr{G}$ such that $\mathscr{M} \cdot \mathscr{A}^2 \geqslant \det(\mathscr{G}) \cdot \mathscr{A}^2 \geqslant 1$. Moreover, $\mathscr{M}$ does not depend on the choice of the very ample line bundle $\mathscr{A}$ on $Y$. Therefore

$$\mathscr{M} \cdot (q^*\mathscr{O}_{\mathbb{P}^2}(k) \otimes \mathscr{A})^2 \geqslant \det(\mathscr{G}) \cdot (q^*\mathscr{O}_{\mathbb{P}^2}(k) \otimes \mathscr{A})^2 > 0 \quad \text{for all } k \geqslant 1,$$

and hence $\mathscr{M} \cdot f \geqslant 0$.

So we can write $\mathscr{M} \cong \mathscr{O}_Y(a) \otimes q^*\mathscr{O}_{\mathbb{P}^2}(b)$ for some integers $a$ and $b$, with $a \geqslant 0$. Since $T_{Y/\mathbb{P}^2} \not\subseteq \mathscr{G}$, there exists an injection of sheaves $\mathscr{M} \to q^*T_{\mathbb{P}^2}$, and hence we must have $a = 0$. On the other hand, since $\mathscr{M} \cdot \mathscr{A}^2 \geqslant 1$, we must have $b \geqslant 1$. Since the map $\mathscr{M} \to q^*T_{\mathbb{P}^2}$ induces a nonzero map $\mathscr{O}_{\mathbb{P}^2}(b) \to T_{\mathbb{P}^2}$, we conclude that $b = 1$ by Bott's formulas. So

$$\mathscr{M} \cong q^*\mathscr{O}_{\mathbb{P}^2}(1).$$

Let $p \in \mathbb{P}^2$ be the singular locus of the degree 0 foliation $\mathscr{O}_{\mathbb{P}^2}(1) \subset T_{\mathbb{P}^2}$ induced by the map $\mathscr{M} \to q^*T_{\mathbb{P}^2}$. It follows that $\mathscr{M} \subset T_Y$ is a regular foliation (with algebraic leaves) over $q^{-1}(\mathbb{P}^2 \setminus \{p\})$. By Lemma 60, a general leaf of $\mathscr{M}$ maps isomorphically to a line in $\mathbb{P}^2$ through the point $p$. This implies that the space of leaves of $\mathscr{M}$ can be naturally identified with $S = \mathbb{P}_\ell(\mathscr{K}_{|\ell})$, where $\ell \subset \mathbb{P}^2$ is a general line. Moreover, the natural morphism $q^{-1}(\mathbb{P}^2 \setminus \{p\}) \to S$ is smooth. Hence, by Paragraph 12, $\mathscr{G}$ is the pullback of a rank 1 foliation $\mathscr{N} \cong \mathscr{O}_S$ on $S$. □

**Remark 70.** Let $\mathscr{E}$ be a vector bundle on a smooth complex variety $Z$. Set $Y := \mathbb{P}_Z(\mathscr{E})$, with natural projection $q : Y \to Z$. Let $\mathscr{W}$ be a line bundle on $Z$, and let $V \in H^0(Z, T_Z \otimes \mathscr{W})$ be a twisted vector field on $Z$. By [11, Proposition 1.1] and [3, Lemma 9.5], the map $\mathscr{W}^* \subset T_Z$ induced by $V$ lifts to a map $q^*\mathscr{W}^* \to T_Y$ if and only if $\mathscr{E}$ is $V$-equivariant, i.e., if there exists a $\mathbb{C}$-linear map $\tilde{V} : \mathscr{E} \to \mathscr{W} \otimes \mathscr{E}$ lifting the derivation $V : \mathscr{O}_T \to \mathscr{W}$.

**Example 71.** Set $Y := \mathbb{P}_{\mathbb{P}^2}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$, with natural projection $q : Y \to \mathbb{P}^2$ and tautological line bundle $\mathscr{O}_Y(1)$. Let $p : Y \to \mathbb{P}^3$ be the morphism induced by the linear system $|\mathscr{O}_Y(1)|$. Note that it is the blowup of $\mathbb{P}^3$ at a point $x$. Let $y \in \mathbb{P}^3 \setminus \{x\}$, and denote by $\varpi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ the linear projection from $y$. Let $\mathscr{C} \cong \mathscr{O}_{\mathbb{P}^2} \subset T_{\mathbb{P}^2}$ be a degree 1 foliation on $\mathbb{P}^2$, singular at the point $\varpi(x)$, and let $\mathscr{G} \subset T_Y$ be the pullback of $\mathscr{C}$ via $\varpi \circ p$. It is a codimension 1 foliation on $Y$. An easy computation shows that $\det(\mathscr{G}) \cong q^*\mathscr{O}_{\mathbb{P}^2}(1)$. The rational map $\varpi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ induces a foliation by curves $\mathscr{M} \cong q^*\mathscr{O}_{\mathbb{P}^2}(1)$ on $Y$, which lifts the degree 0 foliation on $\mathbb{P}^2$ given by the linear projection from the point $q(p^{-1}(y)) \in \mathbb{P}^2$. The space $T$ of leaves of $\mathscr{M}$ can be naturally identified with $S = \mathbb{F}_1$, and $\mathscr{G}$ is the pullback via the induced rational map $Y \dashrightarrow S$ of a foliation induced by a global vector field on $S$.

**4.5. Proof of Theorem 8.** Let $X$, $\mathscr{F}$ and $L$ be as in Assumptions 37. By Theorem 19, $K_X + (n-3)L$ is not nef, i.e., $\tau(L) > n-3$. By Theorem 40, one has $\tau(L) \in \{n-2, n-1, n\}$, unless $(n, \tau(L)) \in \{(5, \frac{5}{2}), (4, \frac{3}{2}), (4, \frac{4}{3})\}$.

**Step 1.** We show that $\tau(L) \geqslant n-2$.

Suppose to the contrary that $(n, \tau(L)) \in \{(5, \frac{5}{2}), (4, \frac{3}{2}), (4, \frac{4}{3})\}$ (Theorem 40 (4)–(6)).

In cases (4), (5b) and (6) described in Theorem 40, $\varphi_L$ makes $X$ a fibration over a smooth curve $C$. Denote by $F$ the general fiber of $\varphi_L$, which is either a projective space or a quadric. By Proposition 43, $\mathscr{F} \neq T_{X/C}$. Therefore, if $\ell \subset X$ is a general line on $F$, then $\ell$ is not tangent to $\mathscr{F}$. By Lemma 21, $(n-3)L \cdot \ell = -K_{\mathscr{F}} \cdot \ell \leqslant -K_F \cdot \ell - 2$. One easily checks that this inequality is violated for those $(X, L)$ in Theorem 40 (4), (5b) and (6), yielding a contradiction.

It remains to consider the two cases (5a) and (5c) described in Theorem 40. In both cases, $X$ admits a morphism $\pi : X \to S$ onto a normal surface with general fiber $F \cong \mathbb{P}^2$, and $\mathscr{L}_{|F} \cong \mathscr{O}_{\mathbb{P}^2}(2)$. Let $\ell \subset X$ be a general line on $F \cong \mathbb{P}^2$. Then $L \cdot \ell = 2$. It follows from Lemma 21 that $\ell$ is tangent to $\mathscr{F}$. By Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a foliation by curves $\mathscr{G}$ on $S$. Thus

$$L_{|F} = (-K_{\mathscr{F}})_{|F} = (-K_{X/S})_{|F},$$

which is a contradiction.

We conclude that $\tau(L) \geqslant n-2$.

**Step 2.** We show that $\tau(L) \geqslant n-1$.

Suppose to the contrary that $\tau(L) = n-2$. Then one of the following holds:

- $(X, L)$ is as in Theorem 40 (3a–d),
- $\varphi_L : X \to X'$ is birational.

Suppose that $(X, L)$ is one of the pairs described in Theorem 40 (3a–d) and Theorem 38. Then $X$ admits a morphism $\pi : X \to Y$ onto a normal variety of dimension $d$, $1 \leqslant d \leqslant 3$, with general fiber $F$ a Fano manifold of dimension $n-d$, index $\iota_F = n-2$, and

$$-K_F = (n-2)L_{|F}. \tag{4.10}$$

Since $\iota_F \geqslant \dim(F) - 1$, $F$ is covered by rational curves of $L$-degree 1. So we can apply Lemma 21, and conclude that $F$ is tangent to $\mathscr{F}$. By Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a codimension 1 foliation $\mathscr{G}$ on $Y$. So

$$(n-3)L_{|F} = (-K_{\mathscr{F}})_{|F} = (-K_{X/Y})_{|F},$$

which contradicts (4.10).

Suppose now that $\varphi_L : X \to X'$ is birational. By Proposition 41, $\varphi_L$ is the composition of finitely many disjoint divisorial contractions $\varphi_i : X \to X_i$ of the following types:

(E) $\varphi_i : X \to X_i$ is the blowup of a smooth curve $C_i \subset X_i$, with exceptional divisor $E_i$. In this case $X_i$ is smooth, and the restriction of $\mathscr{L}$ to a fiber of $(\varphi_i)_{|E_i} : E_i \to C_i$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{n-2}}(1)$.

(F) $\varphi_i : X \to X_i$ contracts a divisor $F_i \cong \mathbb{P}^{n-1}$ to a singular point, and

$$(F_i, \mathscr{N}_{F_i/X}, \mathscr{L}_{|F_i}) \cong (\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(-2), \mathscr{O}_{\mathbb{P}^{n-1}}(1)).$$

In this case $X_i$ is 2-factorial. In even dimension it is Gorenstein.

(G) $\varphi_i : X \to X_i$ contracts a divisor $G_i \cong Q^{n-1}$ to a singular point, and

$$(G_i, \mathscr{N}_{G_i/X}, \mathscr{L}_{|G_i}) \cong (Q^{n-1}, \mathscr{O}_{Q^{n-1}}(-1), \mathscr{O}_{Q^{n-1}}(1)).$$

In this case $X_i$ factorial.

In particular $X'$ is $\mathbf{Q}$-factorial and terminal.

Set $L' := (\varphi_L)_*(L)$. The Mukai foliation $\mathscr{F}$ induces a foliation $\mathscr{F}'$ on $X'$ such that

$$-K_{\mathscr{F}'} \sim (n-3)L'.$$

We claim that $K_{X'} + (n-3)L'$ is not pseudo-effective. To prove this, let $\Delta \sim_{\mathbb{Q}} (n-3)L$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, \Delta)$ is klt, and set $\Delta' := (\varphi_L)_*(\Delta) \sim_{\mathbb{Q}} (n-3)L'$. Since $-(K_X + \Delta)$ is $\varphi_L$-ample, $(X', \Delta')$ is also klt. Suppose that $K_{X'} + (n-3)L' \sim_{\mathbb{Q}} K_{X'} + \Delta'$ is pseudo-effective. Under these assumptions, [4, Theorem 2.11] states that for any integral divisor $D$ on $X'$ such that $D \sim_{\mathbb{Q}} K_{X'} + \Delta'$, $h^0(X, \Omega^1_{X'}[\otimes]\mathscr{O}_{X'}(-D)) = 0$. On the other hand, by Paragraph 11, $\mathscr{F}'$ gives rise to a nonzero global section

$$\omega \in H^0(X, \Omega^1_{X'}\,[\otimes]\,\mathscr{O}_{X'}(-(K_{X'} - (n-3)L'))),$$

yielding a contradiction and proving the claim. In particular, $K_{X'} + (n-3)L'$ is not nef, and Proposition 41 implies that one of the following holds:

(1) $n = 6$ and $(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(2))$.

(2) $n = 5$ and one of the following holds:
   - (a) $(X', \mathscr{O}_{X'}(L')) \cong (Q^5, \mathscr{O}_{Q^5}(2))$.
   - (b) $X'$ is a $\mathbb{P}^4$-bundle over a smooth curve, and the restriction of $\mathscr{O}_{X'}(L')$ to a general fiber is $\mathscr{O}_{\mathbb{P}^4}(2)$.
   - (c) $(X, \mathscr{O}_X(L)) \cong (\mathbb{P}_{\mathbb{P}^4}(\mathscr{O}_{\mathbb{P}^4}(3) \oplus \mathscr{O}_{\mathbb{P}^4}(1)), \mathscr{O}_{\mathbb{P}}(1))$.

(3) $n = 4$ and one of the following holds:
   - (a) $(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(3))$.
   - (b) $X'$ is a Gorenstein del Pezzo 4-fold and $3L' \sim_{\mathbf{Q}} -2K_{X'}$.
   - (c) $\varphi_{L'}$ makes $X'$ a generic quadric bundle over a smooth curve $C$, and for a general fiber $F \cong Q^3$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{Q^3}(2)$.
   - (d) $\varphi_{L'}$ makes $X'$ a generic $\mathbb{P}^2$-bundle over a normal surface $S$, and for a general fiber $F \cong \mathbb{P}^2$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{\mathbb{P}^2}(2)$.
   - (e) $(X', \mathscr{O}_{X'}(L')) \cong (Q^4, \mathscr{O}_{Q^4}(3))$.
   - (f) $\varphi_L : X \to X'$ factors through $\tilde{X}$, the blowup of $\mathbb{P}^4$ along a cubic surface $S$ contained in a hyperplane. The exceptional locus of the contraction $\tilde{X} \to X'$ is the strict transform of the hyperplane of $\mathbb{P}^4$ containing $S$, and it is of type (F) above.
   - (g) $\varphi_L : X \to X'$ factors through $\tilde{X}$, a conic bundle over $\mathbb{P}^3$. The exceptional locus of the contraction $\tilde{X} \to X'$ consists of a single prime divisor of type (F) above.
   - (h) $\varphi_{L'}$ makes $X'$ a $\mathbb{P}^3$-bundle over a smooth curve $C$, and for a general fiber $F \cong \mathbb{P}^3$ of $\varphi_{L'}$, $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{\mathbb{P}^3}(3)$.
   - (i) $(X', \mathscr{O}_{X'}(L')) \cong (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(4))$.
   - (j) $X' \subset \mathbb{P}^{10}$ is a cone over $(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2))$ and $L' \sim_{\mathbf{Q}} 2H$, where $H$ denotes a hyperplane section in $\mathbb{P}^{10}$.

If $X'$ is a Fano manifold with $\rho(X') = 1$, then $\mathscr{F}'$ is a Fano foliation with

$$-K_{\mathscr{F}'} \sim (n-3)L'.$$

By Theorem 3, $\iota_{\mathscr{F}'} \leqslant n-1$, and equality holds only if $X' \cong \mathbb{P}^n$, in which case $\mathscr{F}'$ is induced by a pencil of hyperplanes. As a consequence, $(X', L')$ cannot be as in (1), (2a), (3e) and (3i).

Suppose that $(X', \mathscr{O}(L')) \cong (\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(3))$, i.e., $(X, L)$ is as in (3a). Then $\mathscr{F}'$ is induced by a pencil of hyperplanes in $\mathbb{P}^4$. Denote by $H \cong \mathbb{P}^2$ the base locus of this pencil. Since $X' \cong \mathbb{P}^4$ is smooth, by Proposition 41, $\varphi_L : X \to \mathbb{P}^4$ is the blowup of finitely many disjoint smooth curves $C_i \subset \mathbb{P}^4$, $1 \leqslant i \leqslant k$. Denote by $E_i \subset X$ the exceptional divisor over $C_i$, and by $F_i \cong \mathbb{P}^{n-2}$ a fiber of $(\varphi_L)_{|E_i}$. Let

$$\omega \in H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes \mathscr{O}_{\mathbb{P}^4}(2))$$

be the 1-form defining $\mathscr{F}'$. An easy computation shows that $(\varphi_L)^*\omega$ vanishes along $E_i$ (with multiplicity exactly 2) if and only if $C_i \subset H$. So $(\varphi_L)^*\omega$ induces a section that does not vanish in codimension 1

$$\omega_{\mathscr{F}} \in H^0\left(X, \Omega^1_X \otimes (\varphi_L)^*(\mathscr{O}_{\mathbb{P}^4}(2)) \otimes \mathscr{O}_X\left(-\sum_{i=1}^{k} \epsilon_i E_i\right)\right),$$

where $\epsilon_i = 2$ if $C_i \subset H$, and $\epsilon_i = 0$ otherwise. This is precisely the 1-form defining $\mathscr{F}$. Hence,

$$N_{\mathscr{F}} \cong \mathscr{O}_X(-K_X + K_{\mathscr{F}}) \cong (\varphi_L)^*(\mathscr{O}_{\mathbb{P}^4}(2)) \otimes \mathscr{O}_X\left(-\sum_{i=1}^{k} \epsilon_i E_i\right),$$

and thus

$$\mathscr{O}_{\mathbb{P}^{n-2}}(1) \cong \mathscr{O}_X(-K_X + K_{\mathscr{F}})_{|F_i} \cong \mathscr{O}_{\mathbb{P}^{n-2}}(\epsilon_i),$$

yielding a contradiction. We conclude that $(X', L')$ cannot be as in (3a).

Next we consider the cases in which $X'$ admits a morphism $\pi : X' \to C$ onto a smooth curve, with general fiber $F$ isomorphic to either $\mathbb{P}^{n-1}$ or $Q^{n-1}$ (these are the three cases (2b), (3c) and (3h)). By Proposition 43, $\mathscr{F}$ is not the relative tangent to the composed morphism $\varphi_L \circ \pi : X \to C$. Hence, $\mathscr{F}' \neq T_{X'/C}$, and a general line $\ell \subset F$ is not tangent to $\mathscr{F}'$. By Lemma 20,

$$(n-3)L' \cdot \ell = -K_{\mathscr{F}'} \cdot \ell \leqslant -K_F \cdot \ell - 2.$$

One easily checks that this inequality is violated for those $(X', L')$ in cases (2b), (3c) and (3h).

Next we show that $(X, L)$ cannot be as in (2c). Suppose to the contrary that

$$(X, \mathscr{O}_X(L)) \cong (\mathbb{P}_{\mathbb{P}^4}(\mathscr{O}_{\mathbb{P}^4}(3) \oplus \mathscr{O}_{\mathbb{P}^4}(1)), \mathscr{O}_{\mathbb{P}}(1)).$$

Let $\ell \subset X$ be a general fiber of the natural projection $\pi : X = \mathbb{P}_{\mathbb{P}^4}(\mathscr{O}_{\mathbb{P}^4}(3) \oplus \mathscr{O}_{\mathbb{P}^4}(1)) \to \mathbb{P}^4$. Since $-K_{\mathscr{F}} \cdot \ell = 2$, $\ell$ is tangent to $\mathscr{F}$ by Lemma 21. By Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}^4$. By (2.2), $\det(\mathscr{G}) \cong \mathscr{O}_{\mathbb{P}^4}(4)$, which is impossible by Theorem 3.

We show that $(X', L')$ cannot be as in (3b). Suppose to the contrary that $X'$ is a Gorenstein del Pezzo 4-fold and $3L' \sim_{\mathbf{Q}} -2K_{X'}$. Then there is an ample Cartier divisor $H'$ on $X'$ such that $L' \sim_{\mathbf{Q}} 2H'$ and $-K_{X'} \sim 3H'$. Notice that $X'$ has isolated singularities. Let $Y \in |H'|$ be a general member.

We claim that $Y$ is a smooth 3-fold. Suppose first that $(H')^4 \geqslant 2$. Then $|H'|$ is basepoint free by [23, Corollary 1.5], and hence $Y$ is smooth by Bertini's Theorem. Suppose now that $(H')^4 = 1$. By [22, Theorem 4.2] (see also [22, 6.3 and 6.4]), one has $\dim(\mathrm{Bs}(H')) \leqslant 0$. Thus, if $H'_1, \dots, H'_4$ are general members of $|H'|$, then $H'_1, \dots, H'_4$ meets properly in a (possibly empty) finite set of points, and $(H')^4 = \deg(H'_1 \cap \cdots \cap H'_4)$ (see [24, Example 2.4.8]). Since $(H')^4 = 1$, it follows that $H'_1 \cap \cdots \cap H'_4$ is a reduced point $x$, $X'$ is smooth at $x$, and the local equations of $H'_1, \dots, H'_4$ at $x$ form a regular sequence in $\mathscr{O}_{X',x}$. In particular, $H'_i$ is smooth at $x$ for all $i \in \{1, \dots, 4\}$. By Bertini's Theorem again, we conclude that $Y$ is smooth.

Set $H_Y := H'_{|Y}$, and denote by $\mathscr{H}$ the codimension 1 foliation on $Y$ induced by $\mathscr{F}'$. By the adjunction formula, $-K_Y = 2H_Y$, and hence $Y$ is a del Pezzo threefold. By Paragraph 14, there exists a non-negative integer $b$ such that $-K_{\mathscr{H}} = (1+b)H_Y$. By Theorem 3, we must have $b \in \{0, 1\}$, and hence $\mathscr{H}$ is a Fano foliation. It follows from Theorem 4 and the classification of del Pezzo manifolds that $Y \cong \mathbb{P}^3$, $b = 0$, and $(H_Y)^3 = (H')^4 = 8$. Therefore, $H'$ is very ample by [22, 6.2.3], so that we can apply [20, Theorem 3] to conclude that one of the following holds:

- $X'$ is a cubic hypersurface in $\mathbb{P}^5$.
- $X'$ is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^6$.
- $X'$ is a cone over a Gorenstein del Pezzo 3-fold.
- $\dim(\mathrm{Sing}(X')) \geqslant 1$.

In the first two cases, $(H')^4 = 3$ and $(H')^4 = 4$, respectively. Since $X'$ has isolated singularities, we conclude that $X'$ must be a cone over $(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(2))$. Denote by $\pi : X \to \mathbb{P}^3$ the induced map, and by $\ell$ a general fiber of $\pi$. One computes that $L \cdot \ell = 1$. By Lemma 21, $\ell$ is tangent to $\mathscr{F}$. So, by Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}^3$. Thus

$$L_{|\ell} = (-K_{\mathscr{F}})_{|\ell} = (-K_{X/\mathbb{P}^3})_{|\ell} = -K_\ell,$$

which is absurd. This shows that $(X', L')$ cannot be as in (3b).

We show that $(X', L')$ cannot be as in (3d). Suppose to the contrary that $\varphi_{L'}$ makes $X'$ a generic $\mathbb{P}^2$-bundle over a normal surface $S$, and for a general fiber $F \cong \mathbb{P}^2$ of $\varphi_{L'}$ one has $\mathscr{O}_F(L'_{|F}) \cong \mathscr{O}_{\mathbb{P}^2}(2)$. Lemma 20 implies that $F$ is tangent to $\mathscr{F}'$. By Paragraph 12, $\mathscr{F}'$ is the pullback via $\varphi_{L'}$ of foliation by curves $\mathscr{G}$ on $S$, and thus

$$L'_{|F} = (-K_{\mathscr{F}'})_{|F} = (-K_{X'/S})_{|F},$$

which is a contradiction.

In cases (3f), (3g) and (3j), $\varphi_L : X \to X'$ factors through a factorial 4-fold $\tilde{X}$. Denote by $\tilde{L}$ the push-forward of $L$ to $\tilde{X}$. The Mukai foliation $\mathscr{F}$ induces a foliation $\tilde{\mathscr{F}}$ on $\tilde{X}$ such that $-K_{\tilde{\mathscr{F}}} \sim \tilde{L}$.

In case (3f), $\tilde{X}$ is the blowup of $\mathbb{P}^4$ along a cubic surface $S$ contained in a hyperplane $F \subset \mathbb{P}^4$. Denote by $\tilde{F} \subset \tilde{X}$ the strict transform of $F$, so that

$$N_{\tilde{F}/\tilde{X}} \cong \mathscr{O}_{\mathbb{P}^3}(-2) \quad \text{and} \quad \mathscr{O}_{\tilde{F}}(\tilde{L}_{|\tilde{F}}) \cong \mathscr{O}_{\mathbb{P}^3}(1).$$

We will reach a contradiction by exhibiting a family $H$ of rational curves on $\tilde{X}$ such that the following hold:

(1) the general member of $H$ is a curve tangent to $\tilde{\mathscr{F}}$,

(2) two general points of $\tilde{X}$ can be connected by a chain of curves from $H$ avoiding the singular locus of $\tilde{\mathscr{F}}$.

We take $H$ to be the family of strict transforms of lines in $\mathbb{P}^4$ meeting $S$ and not contained in $F \cong \mathbb{P}^3$. It is a minimal dominating family of rational curves on $\tilde{X}$ satisfying condition (2) above. Let $\ell \subset \tilde{X}$ be a general member of $H$. One computes that

$$-K_{\tilde{X}} \cdot \ell = 4 \quad \text{and} \quad -K_{\tilde{\mathscr{F}}} \cdot \ell = \tilde{L} \cdot \ell \geqslant 3.$$

For the latter, notice that $\ell \equiv \ell_1 + 2\ell_2$, where $\ell_1 \subset \tilde{F}$ is a line under the isomorphism $\tilde{F} \cong \mathbb{P}^3$, and $\ell_2$ is a nontrivial fiber of the blowup $\tilde{X} \to \mathbb{P}^4$. Condition (1) above then follows from Lemma 21. We conclude that $(X', L')$ cannot be as in (3f).

In case (3g), $\tilde{X}$ is a conic bundle over $\mathbb{P}^3$. Moreover, there is a divisor $F \subset X$ mapping isomorphically onto its image by $X \to \tilde{X}$, and such that $(F, N_{F/X}) \cong (\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(-2))$. Denote by $\pi : X \to \tilde{X} \to \mathbb{P}^3$ the composite map, and by $\ell$ a general fiber of $\pi$. By Lemma 21,

$\ell$ is tangent to $\mathscr{F}$. So, by Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}^3$. Let $C \subset \mathbb{P}^3$ be a general line, and set $S := \pi^{-1}(C)$. Note that $S$ is smooth, $\pi_C := \pi_{|C} : S \to C$ is a conic bundle, and the foliation on $S$ induced by $\mathscr{F}$ is precisely $T_{S/C}$. Hence, by Paragraph 14, there is an effective divisor $B$ on $S$ such that

$$-K_S = L_{|S} + B. \tag{4.11}$$

On the other hand, by Paragraph 12,

$$L_{|S} = -K_{S/C} + (\pi_C)^* c_1(\mathscr{G}_{|C}). \tag{4.12}$$

Equations (4.11) and (4.12) together imply that $B = (\pi_C)^* B_C$ for some effective divisor $B_C$ on $C$, and thus $-K_S$ is ample. We will reach a contradiction by exhibiting a curve $\sigma \subset S$ such that $-K_S \cdot \sigma \leqslant 0$. We take $\sigma := F \cap S$. Using that $\mathscr{O}_F(F) \cong \mathscr{O}_{\mathbb{P}^3}(-2)$, the adjunction formula implies that $-K_S \cdot \sigma \leqslant 0$. We conclude that $(X', L')$ cannot be as in (3g).

In case (3j),

$$\tilde{X} = \mathbb{P}_{\mathbb{P}^3}(\mathscr{O}_{\mathbb{P}^3} \oplus \mathscr{O}_{\mathbb{P}^3}(-2)).$$

Denote by $\pi : X \to \tilde{X} \to \mathbb{P}^3$ the composite map, and by $\ell$ a general fiber of $\pi$. One computes that $L \cdot \ell = 1$. By Lemma 21, $\ell$ is tangent to $\mathscr{F}$. So, by Paragraph 12, $\mathscr{F}$ is the pullback via $\pi$ of a codimension 1 foliation $\mathscr{G}$ on $\mathbb{P}^3$. Thus

$$L_{|\ell} = (-K_{\mathscr{F}})_{|\ell} = (-K_{X/\mathbb{P}^3})_{|\ell} = -K_\ell,$$

which is a contradiction.

We conclude that $\tau(L) \geqslant n-1$.

**Step 3.** We show that if $\tau(L) = n-1$, then one of the following conditions holds:

(i) $X$ admits a structure of quadric bundle over a smooth curve. In this case, by Proposition 53, $X$ and $\mathscr{F}$ are as described in Theorem 8 (3).

(ii) $X$ admits a structure of $\mathbb{P}^{n-2}$-bundle over a smooth surface. In this case, by Section 4.4, $X$ and $\mathscr{F}$ are as described in Theorem 8 (4).

(iii) $n = 5$, $\varphi_L : X \to \mathbb{P}^5$ is the blowup of one point $P \in \mathbb{P}^5$, and $\mathscr{F}$ is induced by a pencil of hyperplanes in $\mathbb{P}^5$ containing $P$ in its base locus. This gives Theorem 8 (5).

(iv) $n = 4$, $\varphi_L : X \to \mathbb{P}^4$ is the blowup of at most eight points in general position on a plane $\mathbb{P}^2 \cong S \subset \mathbb{P}^4$, and $\mathscr{F}$ is induced by the pencil of hyperplanes in $\mathbb{P}^4$ with base locus $S$. This gives Theorem 8 (6).

(v) $n = 4$, $\varphi_L : X \to Q^4$ is the blowup of at most seven points in general position on a codimension 2 linear section $Q^2 \cong S \subset Q^4$, and $\mathscr{F}$ is induced by the pencil of hyperplanes sections of $Q^4 \subset \mathbb{P}^5$ with base locus $S$. This gives Theorem 8 (7).

Suppose that $\tau(L) = n-1$. By Theorem 40, one of the following holds:

- $X$ admits a structure of quadric bundle over a smooth curve. This is case (i).
- $X$ admits a structure of $\mathbb{P}^{n-2}$-bundle over a smooth surface. This is case (ii).
- The morphism $\varphi_L : X \to X'$ is the blowup of a smooth projective variety at finitely many points $P_1, \ldots, P_k \in X'$.

Suppose that we are in the latter case, and denote the exceptional prime divisors of $\varphi_L$ by $E_i$, $1 \leqslant i \leqslant k$. Set $L' := (\varphi_L)_*(L)$. It is an ample divisor on $X'$ and

$$L + \sum_{i=1}^{k} E_i = (\varphi_L)^* L'. \tag{4.13}$$

The Mukai foliation $\mathscr{F}$ induces a Fano foliation $\mathscr{F}'$ on $X'$ such that $-K_{\mathscr{F}'} \sim (n-3)L'$. From Theorem 19, we know that $K_{X'} + (n-3)L'$ is not nef, i.e., $\tau(L') > n-3$. On the other hand, since $K_{X'} + (n-1)L' = (\varphi_L)_*(K_X + (n-1)L)$ is ample, $\tau(L') < n-1$. It follows from Theorem 40, together with Steps 1 and 2 above, that $\rho(X') = 1$. Let $H'$ be the ample generator of $\operatorname{Pic}(X')$, and write $L' \sim \lambda H'$, $\lambda \geqslant 1$. Then $-K_{X'} = \iota_{X'} H'$, $-K_{\mathscr{F}'} = \lambda(n-3)H'$ and

$$\tau(L') = \frac{\iota_{X'}}{\lambda} < n-1. \tag{4.14}$$

By Lemma 28,

$$\iota_{X'} \geqslant \lambda(n-3) + 2. \tag{4.15}$$

Inequalities (4.14) and (4.15) together yield that $\lambda \geqslant 2$. On the other hand, by Theorem 3, $\iota_{\mathscr{F}'} = \lambda(n-3) \leqslant n-1$. Thus $(n,\lambda) \in \{(5,2),(4,3),(4,2)\}$.

Let $\omega' \in H^0(X', \Omega^1_{X'}(-K_{X'} - (n-3)L'))$ be the twisted 1-form defining $\mathscr{F}'$. The induced twisted 1-form $(\varphi_L)^*\omega' \in H^0(X, \Omega^1_X((\varphi_L)^*(-K_{X'} - (n-3)L')))$ saturates to give the twisted 1-form defining $\mathscr{F}$, $\omega_{\mathscr{F}} \in H^0(X, \Omega^1_X(-K_X - (n-3)L))$. Using (4.13), one computes that

$$-K_X - (n-3)L = (\varphi_L)^*(-K_{X'} - (n-3)L') + 2\sum_{i=1}^{k} E_i.$$

Thus $(\varphi_L)^*\omega'$ must vanish along each $E_i$ with multiplicity exactly 2.

Suppose that $(n,\lambda) = (5,2)$. Then $\iota_{\mathscr{F}'} = \operatorname{rank}(\mathscr{F}')$ and, by Theorem 3, $X' \cong \mathbb{P}^5$, and $\mathscr{F}'$ is a foliation induced by a pencil of hyperplanes in $\mathbb{P}^5$. We claim that $\varphi_L : X \to \mathbb{P}^5$ is the blowup of only one point. Indeed, if $\varphi_L : X \to \mathbb{P}^5$ blows up at least two points $P$ and $Q$, let $\ell$ be a line connecting $P$ and $Q$, and $\tilde{\ell} \subset X$ its strict transform. We get a contradiction by intersecting (4.13) with $\tilde{\ell}$, and conclude that $\varphi_L : X \to \mathbb{P}^5$ is the blowup of a single point $P \in \mathbb{P}^5$, with exceptional divisor $E$. Moreover, $(\varphi_L)^*\omega'$ vanishes along $E$ with multiplicity exactly 2. A local computation shows that this happens precisely when $P$ is in the base locus of the pencil of hyperplanes defining $\mathscr{F}'$.

Suppose that $(n,\lambda) = (4,3)$. Then $\iota_{\mathscr{F}'} = \operatorname{rank}(\mathscr{F}')$ and, by Theorem 3, $X' \cong \mathbb{P}^4$, and $\mathscr{F}'$ is a foliation induced by a pencil of hyperplanes in $\mathbb{P}^4$. Moreover, $(\varphi_L)^*\omega'$ vanishes along each $E_i$ with multiplicity exactly 2. A local computation shows that this happens precisely when the points $P_i$ all lie in the base locus of the pencil of hyperplanes defining $\mathscr{F}'$, which is a codimension 2 linear subspace. Since

$$L = (\varphi_L)^* L' - \sum_{i=1}^{k} E_i$$

is ample, we must have $k \leqslant 8$ and the points $P_i$ are in general position by Lemma 72 below.

Suppose that $(n,\lambda) = (4,2)$. Then $\mathscr{F}'$ is a codimension 1 del Pezzo foliation on $X'$. By Theorem 4, either $X' \cong \mathbb{P}^4$ and $\mathscr{F}'$ is a degree 1 foliation, or $X' \cong Q^4 \subset \mathbb{P}^5$ and $\mathscr{F}'$ is induced by a pencil of hyperplane sections.

Suppose first that $X' \cong \mathbb{P}^4$ and $\mathscr{F}'$ is a degree 1 foliation. The same argument used in the case $(n, \lambda) = (5, 2)$ shows that $\varphi_L : X \to \mathbb{P}^4$ is the blowup of only one point $P \in \mathbb{P}^4$, with exceptional divisor $E$. Moreover, $(\varphi_L)^*\omega'$ vanishes along $E$ with multiplicity exactly 2. A local computation shows that this cannot happen.

Finally suppose that $X' \cong Q^4 \subset \mathbb{P}^5$ and $\mathscr{F}'$ is induced by a pencil of hyperplane sections. Moreover, $(\varphi_L)^*\omega'$ vanishes along each $E_i$ with multiplicity exactly 2. A local computation shows that this happens precisely when the points $P_i$ all lie in the base locus of the pencil defining $\mathscr{F}'$, which is a codimension 2 linear section of $Q^4$. Since

$$L = (\varphi_L)^*L' - \sum_{i=1}^{k} E_i$$

is ample, we must have $k \leqslant 7$ and the points $P_i$ are in general position by Lemma 72 below.

**Step 4.** Suppose that $\tau(L) = n$. By Theorem 40, $X$ admits a structure of $\mathbb{P}^{n-1}$-bundle over a smooth curve. In this case, by Proposition 45, $X$ and $\mathscr{F}$ are as described in Theorem 8 (1) or (2). □

**Lemma 72.** *The following hold:*

(1) *Let $\pi : X \to \mathbb{P}^4$ be the blowup of finitely many points $P_1, \dots, P_k$ contained in a codimension 2 linear subspace $S \cong \mathbb{P}^2$, and denote by $E_i$ the exceptional divisor over $P_i$. Then the line bundle $\pi^*\mathscr{O}_{\mathbb{P}^4}(3) \otimes \mathscr{O}_X(-\sum_{i=1}^k E_i)$ is ample if and only if $k \leqslant 8$ and the points $P_i$ are in general position in $\mathbb{P}^2$.*

(2) *Let $\pi : X \to Q^4$ be the blowup of a smooth quadric at finitely many points $P_1, \dots, P_k$ contained in a codimension 2 linear section $S$, and denote by $E_i$ the exceptional divisor over $P_i$. Then the line bundle $\pi^*\mathscr{O}_{Q^4}(2) \otimes \mathscr{O}_X(-\sum_{i=1}^k E_i)$ is ample if and only if $k \leqslant 7$ and the points $P_i$ are in general position in $S$.*

*Proof.* Under the assumptions of (1) above, write $L = \pi^*3H - \sum_{i=1}^k E_i$, where $H$ is a hyperplane in $\mathbb{P}^4$. The divisor $L$ is ample if and only if $|mL|$ separates points in $X$ for $m \gg 1$. Notice that $|L|$ always separates points outside the strict transform $\tilde{S}$ of the plane $S \subset \mathbb{P}^4$. Moreover, for $m \geqslant 0$, any global section of $\mathscr{O}_{\tilde{S}}(mL_{|\tilde{S}})$ extends to a global section of $\mathscr{O}_X(mL)$. Hence, $L$ is ample if and only if $L_{|\tilde{S}}$ is ample. Now notice that

$$L_{|\tilde{S}} = p^*(-K_S) - \sum_{i=1}^{k} E_{i\,|\tilde{S}} = -K_{\tilde{S}},$$

where $p = \pi_{|\tilde{S}} : \tilde{S} \to S \cong \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at the points $P_1, \dots, P_k$. Therefore, $L$ is ample if and only if $-K_{\tilde{S}}$ is ample, i.e., $k \leqslant 8$ and the points $P_i$ are in general position in $\mathbb{P}^2$.

Now we proceed to prove (2). Let $\tilde{S}$ be the strict transform of the (possibly singular) irreducible quadric surface $S \subset Q^4$. Write as above $L = \pi^*2H - \sum_{i=1}^k E_i$, where $H \subset Q^4$ is a hyperplane section. Notice that $|L|$ always separates points outside $\tilde{S}$. Moreover, any global section of $\mathscr{O}_{\tilde{S}}(L_{|\tilde{S}})$ extends to a global section of $\mathscr{O}_X(L)$. Suppose that $L$ is ample, so that $L_{|\tilde{S}} = -K_{\tilde{S}}$ is ample as well. Then $k \leqslant 7$ and the points $P_i$ are in general position in $S$. Conversely, suppose that $k \leqslant 7$, and the points $P_i$ are in general position in $S$. Then $\dim(\mathrm{Bs}(-K_{\tilde{S}})) \leqslant 0$, and hence $\dim(\mathrm{Bs}(L)) \leqslant 0$. We conclude that $L$ is ample by Zariski's theorem ([36, Remark 2.1.32]). □

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Eingegangen 23. April 2014, in revidierter Fassung 18. September 2014