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Locally Unsplit Families of Rational Curves of Large Anticanonical Degree on Fano Manifolds

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In this paper, we address Fano manifolds of dimension $n \geq 3$ with a locally unsplit dominating family of rational curves of anticanonical degree n. We first observe that their Picard number is at most 3, and then we provide a classification of all cases with maximal Picard number. We also give examples of locally unsplit dominating families of rational curves whose varieties of minimal tangents at a general point are singular.

1 Introduction

Let X be a Fano manifold, and let V be a dominating family of rational curves on X. By this, we mean that V is an irreducible component of $\mathrm{RatCurves}^n(X)$, the scheme parameterizing integral rational curves on X, and that the union of the curves parameterized by V is dense in X.

We say that V is *locally unsplit* if for general $x \in X$, the subfamily V_x of curves containing x is proper. This is true, for instance, if V is a dominating family with minimal degree with respect to some ample line bundle on X.

When V is locally unsplit, the anticanonical degree of the curves of the family can vary between 2 and $n+1$, where n is the dimension of X.

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Following Miyaoka [30], we define l_X to be the minimal anticanonical degree of a locally unsplit dominating family of rational curves in X, so that $l_X \in \{2, \dots, n+1\}$. Equivalently, l_X is the minimal anticanonical degree of a free rational curve in X (see Remark 4.2).

In this paper, we study Fano manifolds with a locally unsplit dominating family of rational curves of anticanonical degree n, including in particular Fano manifolds X with $l_X = n$.

Let us recall the following results, due, respectively, to Cho, Miyaoka, and Shepherd-Barron (see also [24]), and to Miyaoka.

Theorem 1.1 ([8])**.** Let X be a Fano manifold of dimension n. The following properties are equivalent:

(i) X has a locally unsplit dominating family of rational curves of maximal anticanonical degree $n+1$;

(ii) $X \cong \mathbb{P}^n$.

In particular, $l_X = n+1$ if and only if $X \cong \mathbb{P}^n$. □

Theorem 1.2 ([30])**.** Let X be a Fano manifold of dimension $n \geq 3$, and with Picard number $\rho_X = 1$. Then $l_X = n$ if and only if X is isomorphic to a quadric. □

On the other hand, there are also cases where $l_X = n$ and $\rho_X > 1$.

Example 1.3 ([30, Remark 4.2])**.** Let $A \subset \mathbb{P}^n$ be a smooth subvariety, of dimension $n-2$ and degree $d \in \{1, \dots, n\}$, contained in a hyperplane. Let X be the blow-up of $\mathbb{P}^n$ along A. Then X is Fano with $\rho_X = 2$ and $l_X = n$. The locally unsplit dominating family of rational curves of anticanonical degree n is given by the strict transforms of lines intersecting A in one point. □

First of all, we show that in fact these are the only examples.

Theorem 1.4. Let X be a Fano manifold of dimension $n \geq 3$, with $\rho_X > 1$ and $l_X = n$. Then there exists a smooth subvariety $A \subset \mathbb{P}^n$ of dimension $n-2$ and degree $d \in \{1, \dots, n\}$, contained in a hyperplane, such that X is isomorphic to the blow-up of $\mathbb{P}^n$ along A. □

Together with Miyaoka's result (Theorem 1.2), this gives a complete classification of Fano manifolds with $l_X = n$.

Then we turn to the case where X is a Fano manifold having a locally unsplit dominating family of rational curves of anticanonical degree n, where $n \geq 3$ is the dimension of X. Note that this assumption is easier to check than the condition $l_X = n$, as it involves only one family of rational curves.

In the toric case, these varieties have been classified by Chen, Fu, and Hwang.

Proposition 1.5 ([7, Proposition 3.8])**.** Let X be a toric Fano manifold, of dimension $n \geq 3$, having a locally unsplit dominating family of rational curves of anticanonical degree n. Then X is one of the following:

(1) the blow-up of $\mathbb{P}^n$ at a linear $\mathbb{P}^{n-2}$ (here $\rho = 2$ and $l_X = n$);
(2) $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ (here $\rho = 2$ and $l_X = 2$);
(3) the blow-up of $\mathbb{P}^n$ at $A \cup \{p\}$, where A is a linear $\mathbb{P}^{n-2}$, and p a point not in A (here $\rho = 3$ and $l_X = 2$). □

We show that if X has a locally unsplit dominating family V of rational curves of anticanonical degree n, then the Picard number of X is at most 3 (see Proposition 4.7). Moreover, we classify all cases with $\rho_X = 3$, giving a complete description of X and V. Let us describe our results.

We first construct and study a family of examples.

Example 1.6. Fix integers n, a, and d such that $n \geq 3$, $d \geq 1$, and $0 \leq a \leq d$. Let moreover $A \subset \mathbb{P}^{n-1}$ be a smooth hypersurface of degree d.

Set $Y := \mathbb{P}_{\mathbb{P}^{n-1}}(\mathscr{O}_{\mathbb{P}^{n-1}} \oplus \mathscr{O}_{\mathbb{P}^{n-1}}(a))$, and let $\hat{G}_Y \cong \mathbb{P}^{n-1} \subset Y$ be a section of the $\mathbb{P}^1$-bundle $Y \to \mathbb{P}^{n-1}$ with normal bundle $\mathcal{N}_{\hat{G}_Y/Y} \cong \mathscr{O}_{\mathbb{P}^{n-1}}(a)$.

Finally, set $A_Y := \hat{G}_Y \cap \pi^{-1}(A)$ (so that $A_Y \cong A$), and let $\sigma : X \to Y$ be the blow-up of A_Y.

Then X is smooth of dimension n and Picard number 3, and it is Fano if and only if $a \leq n-1$ and $d - a \leq n-1$. In the Fano case, these varieties appear in [31], where the author classifies Fano manifolds containing a divisor $G \cong \mathbb{P}^{n-1}$ and with negative normal bundle. □

Proposition 1.7. Let X be as in Example 1.6. Then X has a locally unsplit dominating family V of rational curves of anticanonical degree n. □

Then we show that these are all the examples with $\rho \geq 3$.

Theorem 1.8. Let X be a Fano manifold of dimension $n \geq 3$, and suppose that X has a locally unsplit dominating family V of rational curves of anticanonical degree n. Then $\rho_X \leq 3$.

If moreover $\rho_X = 3$, then X is isomorphic to one of the varieties described in Example 1.6, and the family V is unique. □

We study in more detail the family of curves given by Proposition 1.7. For $x \in X$, we denote by V_x the normalization of the closed subset of V parameterizing curves containing x.

Theorem 1.9. Let X and V be as in Proposition 1.7, and $x \in X$ a general point. Let $z \in \mathbb{P}^{n-1}$ be the image of x under the morphism $X \to \mathbb{P}^{n-1}$, and let $p_z : A \to \mathbb{P}^{n-2}$ be the degree d morphism induced by the linear projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ from z.

Then V_x is smooth and connected. If $a = 0$, then $V_x \cong \mathbb{P}^{n-2}$. If $a > 0$, then V_x is isomorphic to the relative Hilbert scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^{n-2})$ of zero-dimensional subschemes, of length a, of fibers of p_z. □

Finally, we consider the variety of minimal rational tangents (VMRT) associated to the locally unsplit family V at a general point x, defined as follows. Let

$$\tau_x : V_x \dashrightarrow \mathbb{P}(T^*_{X,x})$$

be the map defined by sending a general curve from V_x to its tangent direction at x, and define the VMRT $\mathcal{C}_x$ to be the closure of image of τ_x in $\mathbb{P}(T^*_{X,x})$. We still denote by τ_x the induced map $V_x \to \mathcal{C}_x$; this is, in fact, the normalization morphism by [21, 25].

Theorem 1.10. Let X and V be as in Proposition 1.7, and $x \in X$ a general point.

(1) The VRMT $\mathcal{C}_x \subset \mathbb{P}(T^*_{X,x})$ is an irreducible hypersurface of degree $\binom{d}{a}$.
(2) If $a \in \{0, 1, d-1, d\}$, then $\tau_x : V_x \to \mathcal{C}_x$ is an isomorphism.
(3) If $2 \leq a \leq d-2$, then $\tau_x : V_x \to \mathcal{C}_x$ is not an isomorphism. More precisely, the closed subset where τ_x is not an isomorphism has codimension 1, and the closed subset where τ_x is not an immersion has codimension 2. □

This provides the first examples of locally unsplit dominating families of rational curves whose VMRT at a general point x is singular, equivalently such that $\tau_x : V_x \to \mathcal{C}_x$ is not an isomorphism (see [18, Question 1; 26, Problem 2.20; 19; 20]). Note that if $2 \leq a \leq d-2$, then V is not a dominating family of rational curves of minimal degree (see Remark 5.4).

In [20], Hwang studies projective manifolds X having a locally unsplit dominating family V of rational curves of anticanonical degree equal to the dimension of X. Under the assumption that the VMRT of V at a general point is smooth, he gives a birational description of X, see [20, Theorem 1.4].

In order to prove Theorems 1.9 and 1.10, we are led to study relative Hilbert schemes of zero-dimensional subschemes of the projection of a smooth hypersurface from a general point. We obtain the following result, of independent interest.

Theorem 1.11. Fix integers m, a, and d, such that $m \geq 1$ and $1 \leq a \leq d$. Let $A \subset \mathbb{P}^{m+1}$ be a smooth hypersurface of degree d. Let $z \in \mathbb{P}^{m+1}$ be a general point, and let $A \to \mathbb{P}^m$ be the morphism induced by the linear projection from z, where we identify $\mathbb{P}^m$ with the variety of lines through z in $\mathbb{P}^{m+1}$.

(1) The scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is connected and smooth of dimension m, and the natural morphism $\Pi : \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m) \to \mathbb{P}^m$ is finite of degree $\binom{d}{a}$.

(2) Let $\ell \subset \mathbb{P}^{m+1}$ be a line through z, and let $[W] \in \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ be a point over $[\ell] \in \mathbb{P}^m$. Then Π is smooth at $[W]$ if and only if W is a union of irreducible components of $\ell \cap A$. □

Note that the smoothness of $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ was proved by Gruson and Peskine [16, Theorem 1.3], with different methods. Our proof is independent from Gruson and Peskine, and relies on Theorem 1.9.

1.12. **Outline of the paper.** In Section 2, we introduce notations used in the remainder of the paper, and we discuss some properties of families of rational curves.

In Section 3, we study Fano manifolds X of dimension $n \geq 3$ having a prime divisor D with $\rho_D = 1$, using techniques from the Minimal Model Program, and in particular results from [4]. It follows from [31, Proposition 5] that $\rho_X \leq 3$; we study the cases $\rho_X = 2$ and $\rho_X = 3$. In the case $\rho_X = 2$, we describe the possible extremal contractions of X (see Remark 3.2 and Proposition 3.3). Then, we give a complete classification of these varieties when $\rho_X = 3$, see Example 3.4 and Theorem 3.8. This generalizes results from [5, 31] for the case $D \cong \mathbb{P}^{n-1}$ and negative normal bundle.

In Section 4, we specialize the results of the previous section to the case where X has a locally unsplit dominating family V of rational curves of anticanonical degree n. Indeed, for general $x \in X$, any irreducible component of the locus swept out by curves of the family through x is a divisor D with $\rho_D = 1$. This yields $\rho_X \leq 3$.

We first prove Theorem 1.4 on the case $l_X = n$. Next we show that, under the assumptions of Theorem 1.8, if moreover $\rho_X = 3$, then X is isomorphic to one of the varieties described in Example 1.6, and we determine the class of the family V in $\mathcal{N}_1(X)$ (see Proposition 4.7). Finally, when $\rho_X = 2$, we describe the possible extremal contractions of X (see Lemma 4.5). In this section, we use repeatedly the characterization of the projective space given by Theorem 1.1.

In Section 5, we first construct a locally unsplit dominating family of rational curves on the varieties introduced in Example 1.6, proving Proposition 1.7. With the use of Proposition 4.7, we also show that the family is unique, completing the proof of Theorem 1.8. Finally, we show Theorems 1.9 and 1.10, using Theorem A.1 from the Appendix (Section 5).

In Section 5, we discuss the relative Hilbert scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$, where $A \to \mathbb{P}^m$ is the projection of a smooth hypersurface $A \subset \mathbb{P}^{m+1}$ from a general point. We first study local properties of the natural morphism $\Pi : \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m) \to \mathbb{P}^m$, and we show that $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is integral (see Theorem A.1). This is used in the proof of Theorem 1.9.

In Theorem A.1, we also determine the genus of the curve $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$; together with the description of the fibers of Π, this is crucial for the proof of Theorem 1.10.

At last we prove that the Hilbert scheme is smooth (Theorem 1.11), as a consequence of Theorem 1.9. □

2 Notations and Preliminaries

Throughout this paper, we work over the field of complex numbers.

We will use the definitions and apply the techniques of the Minimal Model Program frequently, without explicit references. We refer the reader to [11, 28] for background and details.

For any projective variety X, we denote by $\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) the vector space of one-cycles (respectively, Cartier divisors), with real coefficients, modulo numerical equivalence. We denote numerical equivalence by $\equiv$, for both one-cycles and $\mathbb{Q}$-Cartier divisors. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class of a curve C (respectively, of a Cartier divisor D). Moreover, $\mathrm{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves.

For any closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z, X)$ the subspace of $\mathcal{N}_1(X)$ generated by classes of curves contained in Z.

If D is a Cartier divisor in X, we set $D^{\perp} := \{\gamma \in \mathcal{N}_1(X) \mid D \cdot \gamma = 0\}$.

If X is a normal projective variety, a *contraction* of X is a surjective morphism $\varphi : X \to Y$, with connected fibers, where Y is normal and projective. The contraction is elementary if $\rho_X - \rho_Y = 1$.

Let R be an extremal ray of $\mathrm{NE}(X)$. If D is a divisor in X, the sign of $D \cdot R$ is the sign of $D \cdot \Gamma$, Γ a nonzero one-cycle with class in R.

Suppose that $K_X \cdot R < 0$, and let $\varphi : X \to Y$ be the associated elementary contraction. We set $\mathrm{Locus}(R) := \mathrm{Exc}(\varphi)$, the locus where φ is not an isomorphism.

By a $\mathbb{P}^1$-bundle we mean a smooth morphism whose fibers are isomorphic to $\mathbb{P}^1$, while a morphism is called a conic bundle if every fiber is isomorphic to a plane conic.

If $\mathscr{E}$ is a vector bundle on a variety Y, we denote by $\mathbb{P}_Y(\mathscr{E})$ the scheme $\mathrm{Proj}_Y(\mathrm{Sym}(\mathscr{E}))$.

We refer the reader to [27, §II.2 and IV.2] for the main properties of families of rational curves; we will keep the same notation as [27]. In particular, we recall that $\mathrm{RatCurves}^n(X)$ is the normalization of the open subset of $\mathrm{Chow}(X)$ parameterizing integral rational curves.

By a family of rational curves in X, we mean an irreducible component V of $\mathrm{RatCurves}^n(X)$. We say that V is a *dominating family* if its universal family dominates X. We say that V is *locally unsplit* if, for a general point $x \in X$, the subfamily of V parameterizing curves through x is proper.

Let V be a locally unsplit dominating family of rational curves on X. The class in $\mathcal{N}_1(X)$ of a curve C from V does not depend on the choice of C, and will be denoted by $[V]$.

We denote by $[C] \in V$ a point corresponding to the integral rational curve $C \subset X$. We warn the reader that this is the same notation as for the numerical equivalence class of the curve C in $\mathcal{N}_1(X)$. Unfortunately both notations are standard; however, it will be easy for the reader to understand from the context whether we are considering the point $[C]$ in V or in $\mathcal{N}_1(X)$.

For $x \in X$, we denote by V_x the *normalization* of the closed subset of V parameterizing curves through the point x, and by $\mathrm{Locus}(V_x) \subseteq X$ the union of all curves of the family V_x.

We denote by $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ the scheme parameterizing morphisms from $\mathbb{P}^1$ to X sending $0 \in \mathbb{P}^1$ to x.

We now recall some well-known properties.

Suppose that $x \in X$ is *general*. Then every curve in V_x is free. This implies that V_x is smooth, of dimension $-K_X \cdot [V] - 2$, but possibly not connected. Moreover, there is a smooth closed subset $\hat{V}_x \subset \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$, which is a union of irreducible components, containing all birational maps $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and

$f(\mathbb{P}^1)$ is a curve of the family V_x. There is an induced smooth morphism $\hat{V}_x \to V_x$, sending $[f]$ to $[f(\mathbb{P}^1)]$.

Still for a general point x, if a curve from V_x is smooth at x, then it is parameterized by a unique point of V_x.

We say that an integral rational curve $C \subset X$ is *standard* if the pull-back of $T_{X|C}$ under the normalization $\mathbb{P}^1 \to C$ is $\mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^p \oplus \mathscr{O}_{\mathbb{P}^1}^{n-1-p}$ for some $p \in \{0, \ldots, n-1\}$.

3 Fano Manifolds Containing a Prime Divisor with Picard Number One

In this section, we study Fano manifolds X having a prime divisor D with $\rho_D = 1$, or more generally $\dim \mathcal{N}_1(D, X) = 1$. The main technique here is the study of extremal rays and contractions of X.

The first step for the proofs of Theorems 1.4 and 3.8 is the following lemma. It is a standard application of Mori theory, in particular the proof can be adapted from [31, proof of Proposition 5], and follows the same strategy used in [5]. We give a short proof for the reader's convenience.

Lemma 3.1. Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho_X > 1$, and let $D \subset X$ be a prime divisor with $\dim \mathcal{N}_1(D, X) = 1$. Then one of the following holds:

(i) $\rho_X = 2$ and there exists a blow-up $\sigma : X \to Y$ with center $A_Y \subset Y$ smooth of codimension 2, Y is smooth and Fano, and $D \cdot R > 0$, where R is the extremal ray of $\mathrm{NE}(X)$ generated by the class of a curve contracted by σ;

(ii) $\rho_X = 2$ and there exists a conic bundle $\sigma : X \to Y$, finite on D, such that Y is smooth and Fano;

(iii) $\rho_X = 3$ and there is a conic bundle $\varphi : X \to Z$, finite on D, such that Z is smooth, Fano, and $\rho_Z = 1$. The conic bundle φ is the contraction of a face $R + \hat{R}$ of $\mathrm{NE}(X)$, where R and $\hat{R}$ both correspond to a smooth blow-up of a codimension 2 subvariety. Moreover, $D \cdot R > 0$, and we have a diagram:

$$\begin{tikzcd}
& X \arrow[dl, "\hat{\sigma}"'] \arrow[dd, "\varphi"] \arrow[dr, "\sigma"] & \\
\hat{Y} \arrow[dr, "\hat{\pi}"'] & & Y \arrow[dl, "\pi"] \\
& Z &
\end{tikzcd}$$

where σ is the contraction of R, $\hat{\sigma}$ is the contraction of $\hat{R}$, Y and $\hat{Y}$ are smooth, Y is Fano, the center $A_Y \subset Y$ of the blow-up σ is contained in $D_Y := \sigma(D)$, and π and $\hat{\pi}$ are conic bundles. □

Proof. Let R be an extremal ray of $\mathrm{NE}(X)$ with $D \cdot R > 0$, and $\sigma : X \to Y$ the associated contraction.

If $R \subset \mathcal{N}_1(D, X)$, then every curve contained in D has class in R, hence $\sigma(D)$ is a point and $D \subseteq \mathrm{Locus}(R)$. We conclude that $\mathrm{Locus}(R) = X$, because $D \cdot R > 0$. On the other hand, since $\rho_X > 1$, we can find a nontrivial fiber F of σ disjoint from D, which yields $D \cdot R = 0$, a contradiction. Therefore $R \not\subset \mathcal{N}_1(D, X)$.

This implies that σ is finite on D, therefore every nontrivial fiber of σ has dimension 1. Thus, Y is smooth and there are two possibilities: either σ is a conic bundle, or it is the blow-up of $A_Y \subset Y$ with A_Y smooth of codimension 2 (see [32, Theorem 1.2]).

If σ is a conic bundle, then $\rho_Y = 1$ because $\sigma(D) = Y$, so we are in case (ii). If σ is a blow-up and $\rho_Y = 1$, then we are in case (i).

Assume that σ is a blow-up and $\rho_Y \geq 2$, and set $D_Y := \sigma(D)$. Then D_Y is a prime divisor in Y and $\mathcal{N}_1(D_Y, Y) = \sigma_*(\mathcal{N}_1(D, X))$, hence $\dim \mathcal{N}_1(D_Y, Y) = 1$. Moreover, $A_Y \subset D_Y$.

Let $E \subset X$ be the exceptional divisor. We have $-K_X + E = \sigma^*(-K_Y)$. If $C \subset A_Y$ is an irreducible curve, and $C' \subset D_Y$ is an irreducible curve not contained in A_Y, then there exists $\lambda \in \mathbb{Q}_{>0}$ such that $C \equiv \lambda C'$, so that $-K_Y \cdot C = \lambda(-K_Y \cdot C') = \lambda(-K_X \cdot \tilde{C}) + \lambda(E \cdot \tilde{C}) > 0$, where $\tilde{C}$ is the strict transform of C' in X. This implies that Y is Fano (see [32]).

We repeat the same argument in Y and take an extremal ray $R_2 \subset \mathrm{NE}(Y)$ with $D_Y \cdot R_2 > 0$.

Similarly, as before we see that $R_2 \not\subset \mathcal{N}_1(D_Y, Y)$, so that again, if $\pi : Y \to Z$ is the contraction of R_2, π is finite on D_Y and has fibers of dimension at most one. Hence as before π is either a conic bundle or the smooth blow-up of a subvariety of codimension 2 in Z.

If π is a conic bundle, then $\rho_Z = 1$ because $\pi(D_Y) = Z$, so $\rho_X = 3$. Set $\varphi := \pi \circ \sigma : X \to Z$; then φ has a second factorization $X \xrightarrow{\hat{\sigma}} \hat{Y} \xrightarrow{\hat{\pi}} Z$. Since every fiber of φ has dimension 1, both $\hat{\sigma}$ and $\hat{\pi}$ have fibers of dimension ≤ 1. Applying [1, Theorem 4.1] we conclude that φ and $\hat{\pi}$ are conic bundles, $\hat{Y}$ is smooth, and $\hat{\sigma}$ the blow-up of a smooth subvariety of codimension 2, so we are in case (iii).

Finally, π cannot be a blow-up. Indeed if so, $\mathrm{Exc}(\pi)$ is a prime divisor which intersects D_Y, and since $\dim \mathcal{N}_1(D_Y, Y) = 1$, $\mathrm{Exc}(\pi)$ has strictly positive intersection with every curve contained in D_Y. In particular, $\mathrm{Exc}(\pi)$ must intersect A_Y, as

$\dim A_Y = n-2 \geq 1$. If F is a nontrivial fiber of π with $F \cap A_Y \neq \emptyset$, and $\tilde{F} \subset X$ is its strict transform, one has $-K_X \cdot \tilde{F} < -K_Y \cdot F = 1$, a contradiction. ■

Remark 3.2 and Proposition 3.3 describe the possible extremal contractions of X in the case $\rho_X = 2$.

Remark 3.2. Let X be a Fano manifold, and $D \subset X$ a prime divisor with $\dim \mathcal{N}_1(D, X) = 1$. If D is not nef, then there exists a unique extremal ray R such that $D \cdot R < 0$; the contraction associated to R is divisorial and sends D to a point.

Indeed, let R be an extremal ray of $\mathrm{NE}(X)$ such that $D \cdot R < 0$, and σ the associated contraction. Note that $\mathrm{Exc}(\sigma) \subseteq D$ since $D \cdot R < 0$. On the other hand, every curve contained in D has class in R since $\dim \mathcal{N}_1(D, X) = 1$. This implies that $D = \mathrm{Exc}(\sigma)$, and that $\sigma(D)$ is a point. □

Proposition 3.3. Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho_X = 2$, and let $D \subset X$ be a nef prime divisor with $\dim \mathcal{N}_1(D, X) = 1$. Then $S := D^{\perp} \cap \mathrm{NE}(X)$ is an extremal ray of X, and one of the following holds:

(i) the contraction of S is a fiber type contraction onto $\mathbb{P}^1$, having D as a fiber;

(ii) the contraction of S is divisorial, sends its exceptional divisor G to a point, and $G \cap D = \emptyset$;

(iii) the contraction of S is small, it has a flip $X \dashrightarrow X'$, X' is smooth, and there is a $\mathbb{P}^1$-bundle $\psi : X' \to Y'$. Moreover, ψ is finite on the strict transform of D in X'.

Furthermore, if there exists a smooth, irreducible subvariety $A \subset D$, of codimension 2, such that the blow-up of X along A is Fano, then (iii) cannot happen. □

Proof.

3.3.1. We first note that D is not ample, because $\mathcal{N}_1(D, X) \subsetneq \mathcal{N}_1(X)$. Indeed, the push-forward of one-cycles $\mathcal{N}_1(D) \to \mathcal{N}_1(X)$ is not surjective, so that the restriction map $\mathcal{N}^1(X) \to \mathcal{N}^1(D)$ is not injective. We have $\mathcal{N}^1(X) \cong H^2(X, \mathbb{R})$ (because X is Fano) and $\mathcal{N}^1(D) \hookrightarrow H^2(D, \mathbb{R})$, hence the restriction map $H^2(X, \mathbb{R}) \to H^2(D, \mathbb{R})$ is not injective as well. By the Lefschetz Theorem on hyperplane sections, D cannot be ample (recall that $n \geq 3$).

Since D is nef and nonample, and $\rho_X = 2$, we conclude that $D^{\perp} \cap \mathrm{NE}(X)$ is an extremal ray S of $\mathrm{NE}(X)$. Set $G := \mathrm{Locus}(S) \subseteq X$. □

3.3.2. If $S \subset \mathcal{N}_1(D, X)$, then the contraction of S sends D to a point, and hence $D \subseteq G$. On the other hand $D \cdot S = 0$, thus D is the pull-back of a Cartier divisor. Therefore, the target of the contraction of S is $\mathbb{P}^1$, D is a fiber, and we are in case (i). □

3.3.3. We assume that $S \not\subset \mathcal{N}_1(D, X)$. If $G \cap D \neq \emptyset$, then D must intersect some irreducible curve C with class in S, and this yields $C \subseteq D$, because $D \cdot C = 0$. This contradicts $S \not\subset \mathcal{N}_1(D, X)$, therefore $G \cap D = \emptyset$. In particular, the contraction of S is birational. Finally, $\mathcal{N}_1(G, X) \subseteq D^{\perp}$ has dimension 1; this implies that the contraction of S maps G to points. If S is a divisorial extremal ray, then we are in case (ii). □

3.3.4. Suppose now that the contraction of S is small; by [4, Corollary 1.4.1] the flip $X \dashrightarrow X'$ of S exists. Let $D' \subset X'$ be the strict transform of D, S' the small extremal ray of X' associated with the flip, and $G' := \mathrm{Locus}(S')$.

Note that $G' \cap D' = \emptyset$, as $G \cap D = \emptyset$.

Note also that X' has normal, $\mathbb{Q}$-factorial, and terminal singularities, and $\mathrm{Sing}(X') \subseteq G'$. We have $\rho_{X'} = \rho_X = 2$ and $K_{X'} \cdot S' > 0$; on the other hand, a curve disjoint from G' has positive anticanonical degree. In particular, by the Cone Theorem, X' has a second extremal ray T with $-K_{X'} \cdot T > 0$, and

$$\mathrm{NE}(X') = \mathbb{R}_{\geq 0} S' + \mathbb{R}_{\geq 0} T.$$

Since the flip $X \dashrightarrow X'$ is an isomorphism in a neighborhood of D, the linear subspace $\mathcal{N}_1(D', X') \cong \mathcal{N}_1(D, X)$ stays one-dimensional. Moreover $D' \cdot S' = 0$, hence we must have $D' \cdot T > 0$.

Let $\psi : X' \to Y'$ be the contraction of T. Arguing as in the proof of Lemma 3.1, we see that $T \not\subset \mathcal{N}_1(D', X')$. Since the contraction of S sends G to points, the contraction of S' sends G' to points. This implies that every curve in G' has class in the extremal ray S'.

We deduce that ψ is finite on both D' and G'.

In particular, since $D' \cdot T > 0$, every nontrivial fiber of ψ has dimension 1. □

3.3.5. Let C be an irreducible component of a nontrivial fiber of ψ. If $C \cap G' \neq \emptyset$, then:

$$-K_{X'} \cdot C > 1.$$

Indeed, if $\tilde{C} \subset X$ is the strict transform of C, we have $-K_{X'} \cdot C > -K_X \cdot \tilde{C} \geq 1$. This follows from [28, Lemma 3.38], see [6, Lemma 3.8] for an explicit proof. □

3.3.6. We show that ψ is of fiber type. By contradiction, assume that ψ is birational.

Suppose that $\mathrm{Exc}(\psi) \cap G' \neq \emptyset$, and let F_0 be an irreducible component of a fiber of ψ which intersects G'. Note that $F_0 \not\subseteq G'$ (since ψ is finite on G'), in particular

$F_0 \not\subseteq \mathrm{Sing}(X')$. We obtain $-K_{X'} \cdot F_0 \leq 1$ by [22, Lemma 1.1], and $-K_{X'} \cdot F_0 > 1$ by 3.3.5, a contradiction.

Therefore $\mathrm{Exc}(\psi) \cap G' = \emptyset$, so that $\mathrm{Exc}(\psi)$ is contained in the smooth locus of X'. By [32, Theorem 1.2], $\mathrm{Exc}(\psi)$ is a divisor. We have $\mathrm{Exc}(\psi) \cdot S' = 0$ and $\mathrm{Exc}(\psi) \cdot T < 0$, hence $-\mathrm{Exc}(\psi)$ is nef, a contradiction.

Thus, ψ is of fiber type. □

3.3.7. We show that $\psi : X' \to Y'$ is a $\mathbb{P}^1$-bundle with X' and Y' smooth, so that we are in case (iii).

Since $\mathrm{Sing}(X')$ cannot dominate Y', the general fiber of ψ is a smooth rational curve of anticanonical degree 2.

Suppose that there is a fiber F of ψ such that the corresponding one-cycle is not integral and $G' \cap F \neq \emptyset$. Then there is an irreducible component C of F, such that $-K_{X'} \cdot C \leq 1$. If $C \cap G' = \emptyset$, then $-K_{X'}$ is Cartier in a neighborhood of C, and we must have $-K_{X'} \cdot C = 1$ and $-K_{X'} \cdot (F - C) = 1$ (where we consider F as a one-cycle). Thus, up to replacing C with another irreducible component of F, we may assume that $C \cap G' \neq \emptyset$, and $-K_{X'} \cdot C \leq 1$; but this contradicts 3.3.5.

By [27, Theorem II.2.8], ψ is smooth in a neighborhood of $\psi^{-1}(\psi(G'))$. Thus $\mathrm{Sing}(X') = \psi^{-1}(\mathrm{Sing}(Y') \cap \psi(G'))$, because $\mathrm{Sing}(X') \subseteq G'$. This implies that $\mathrm{Sing}(X') = \emptyset$, since ψ is finite on G'. In particular, Y' is smooth (see [1, Theorem 4.1(2)] and references therein), ψ is a conic bundle, and either the discriminant locus Δ of ψ has pure codimension 1 or $\Delta = \emptyset$. If $\Delta \neq \emptyset$, then Δ is an ample divisor in Y', because $\rho_{Y'} = 1$. Hence $\psi(G') \cap \Delta \neq \emptyset$ (ψ is finite on G' and $\dim G' \geq 1$), a contradiction. This proves that $\psi : X' \to Y'$ is a $\mathbb{P}^1$-bundle with X' and Y' smooth. □

3.3.8. Suppose now that we are in case (iii), and that there is a smooth irreducible subvariety $A \subset D$ of codimension 2, such that the blow-up of X along A is Fano. We show that this gives a contradiction.

Let $A' \subset D'$ the strict transform of A, and let us consider the divisor $\psi^*(\psi(A'))$ in X'. Since $\psi^*(\psi(A')) \cdot T = 0$, we must have $\psi^*(\psi(A')) \cdot S' > 0$. Therefore, we find a fiber F' of ψ which intersects both A' and G'.

Let $F \subset X$ be the strict transform of F'. As in 3.3.5, we see that $-K_X \cdot F < -K_{X'} \cdot F' = 2$, so that $-K_X \cdot F = 1$. On the other hand, $F \cap A \neq \emptyset$ and $F \not\subseteq A$, hence the strict transform of F in the blow-up of X along A should have nonpositive anticanonical degree, a contradiction. ■

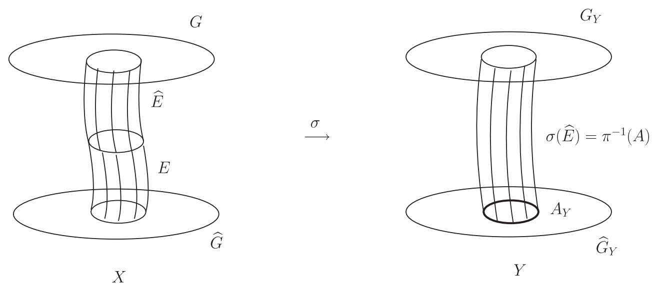

Fig. 1. The blow-up σ.

We will show that any Fano manifold X with $\rho_X = 3$ having a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$ is isomorphic to one of the varieties described below.

We first recall the following definition for the reader's convenience. Let $\mathscr{L}$ be an ample line bundle on a normal projective variety Z. Consider the $\mathbb{P}^1$-bundle $Y = \mathbb{P}_Z(\mathscr{O}_Z \oplus \mathscr{L})$, with natural projection $\pi : Y \to Z$. The tautological line bundle $\mathscr{O}_Y(1)$ is semiample on Y. For $m \gg 0$, the linear system $|\mathscr{O}_Y(m)|$ induces a birational morphism $Y \to Y_0$ onto a normal projective variety, contracting the divisor $E = \mathbb{P}_Z(\mathscr{O}_Z) \cong Z \subset Y$ corresponding to the projection $\mathscr{O}_Z \oplus \mathscr{L} \twoheadrightarrow \mathscr{O}_Z$ to a point. Following [3], we call Y_0 the *normal generalized cone over the base* $(Z, \mathscr{L})$.

Example 3.4. Fix integers n, a, and d, such that $n \geq 3$, $a \geq 0$, and $d \geq 1$.

Let Z be a Fano manifold of dimension $n-1$, with $\rho_Z = 1$. Let $\mathscr{O}_Z(1)$ be the ample generator of $\mathrm{Pic}(Z)$. If m is an integer, then we write $\mathscr{O}_Z(m)$ for $\mathscr{O}_Z(1)^{\otimes m}$. Let moreover $A \in |\mathscr{O}_Z(d)|$ be a smooth hypersurface.

Set $Y := \mathbb{P}_Z(\mathscr{O}_Z \oplus \mathscr{O}_Z(a))$, and let $\pi : Y \to Z$ be the $\mathbb{P}^1$-bundle.

If $a > 0$, then there is a birational contraction $Y \to Y_0$ sending a divisor G_Y to a point, where Y_0 is the normal generalized cone over $(Z, \mathscr{O}_Z(a))$. We have $G_Y \cong Z$, G_Y is a section of π, and $\mathcal{N}_{G_Y/Y} \cong \mathscr{O}_Z(-a)$.

If $a = 0$, then $Y \cong Z \times \mathbb{P}^1$. Let G_Y be a fiber of $Y \to \mathbb{P}^1$. We have $G_Y \cong Z$, G_Y is a section of π, and $\mathcal{N}_{G_Y/Y} \cong \mathscr{O}_Z$.

Let now $\hat{G}_Y \cong Z \subset Y$ be a section of π with normal bundle $\mathcal{N}_{\hat{G}_Y/Y} \cong \mathscr{O}_Z(a)$. Note that $G_Y \cap \hat{G}_Y = \emptyset$ if $a > 0$. If $a = 0$, we choose $\hat{G}_Y$ such that $G_Y \cap \hat{G}_Y = \emptyset$. Set $A_Y := \hat{G}_Y \cap \pi^{-1}(A)$. Finally, let $\sigma : X \to Y$ be the blow-up of A_Y (see Figure 1). $\square$

Then X is a smooth projective variety of dimension n, with $\rho_X = 3$.

Let $G, \hat{G} \subset X$ be the transforms of $G_Y, \hat{G}_Y \subset Y$, respectively. Then $G \cong \hat{G} \cong Z$, $\mathcal{N}_{G/X} \cong \mathscr{O}_Z(-a)$, and $\mathcal{N}_{\hat{G}/X} \cong \mathscr{O}_Z(-(d-a))$.

Table 1. Intersection table in X.

	F	$\hat{F}$	C_G	$C_{\hat{G}}$	$C_G + a\delta\hat{F}$
E	-1	1	0	$d\delta$	$a\delta$
$\hat{E}$	1	-1	$d\delta$	0	$(d-a)\delta$
G	0	1	$-a\delta$	0	0
$\hat{G}$	1	0	0	$-(d-a)\delta$	0
$-K_X$	1	1	$(i_Z-a)\delta$	$(i_Z-(d-a))\delta$	$i_Z\delta$

The composition $\varphi := \pi \circ \sigma : X \to Z$ is a conic bundle, and has a second factorization:

$$\begin{tikzcd}
& X \arrow[dl, "\hat{\sigma}"'] \arrow[dr, "\sigma"] \arrow[dd, "\varphi"] & \\
\hat{Y} \arrow[dr, "\hat{\pi}"'] & & Y \arrow[dl, "\pi"] \\
& Z &
\end{tikzcd}$$

where $\hat{Y} = \mathbb{P}_Z(\mathscr{O}_Z \oplus \mathscr{O}_Z(d-a))$. The images $\hat{\sigma}(G)$ and $\hat{\sigma}(\hat{G})$ are disjoint sections of the $\mathbb{P}^1$-bundle $\hat{\pi} : \hat{Y} \to Z$, with normal bundles $\mathcal{N}_{\hat{\sigma}(G)/\hat{Y}} = \mathscr{O}_Z(d-a)$ and $\mathcal{N}_{\hat{\sigma}(\hat{G})/\hat{Y}} = \mathscr{O}_Z(a-d)$. Moreover, $\hat{\sigma}$ is the blow-up of $\hat{Y}$ along the intersection $\hat{\sigma}(G) \cap \hat{\pi}^{-1}(A)$.

Set $E := \mathrm{Exc}(\sigma)$ and $\hat{E} := \mathrm{Exc}(\hat{\sigma})$, and let $F \subset E$ and $\hat{F} \subset \hat{E}$ be exceptional fibers of σ and $\hat{\sigma}$, respectively. Let moreover $C_Z \subset Z$ be an irreducible curve having minimal intersection with $\mathscr{O}_Z(1)$, and set $\delta := \mathscr{O}_Z(1) \cdot C_Z$. Finally, let $C_G \subset G$ and $C_{\hat{G}} \subset \hat{G}$ be curves corresponding to C_Z. We have the following relations of numerical equivalence:

$$C_G + a\delta\hat{F} \equiv C_{\hat{G}} + (d-a)\delta F, \qquad dG + a\hat{E} \equiv d\hat{G} + (d-a)E, \tag{3.1}$$

and the relevant intersections are shown in Table 1, where i_Z is the index of Z, that is, the integer defined by $\mathscr{O}_Z(-K_Z) \cong \mathscr{O}_Z(i_Z)$.

Lemma 3.5. The cone $\mathrm{NE}(X)$ is closed and polyhedral.

If $a=0$, then $\mathrm{NE}(X)$ has three extremal rays, generated by the classes of F, $\hat{F}$, and $C_{\hat{G}}$, with loci E, $\hat{E}$, and $\hat{G}$, respectively.

If $a \geq d$, then $\mathrm{NE}(X)$ has three extremal rays, generated by the classes of F, $\hat{F}$, and C_G, with loci E, $\hat{E}$, and G, respectively.

If instead $0 < a < d$, then $\mathrm{NE}(X)$ is nonsimplicial and has four extremal rays, generated by the classes of F, $\hat{F}$, C_G, and $C_{\hat{G}}$, with loci E, $\hat{E}$, G, and $\hat{G}$, respectively. □

Proof. Set $R := \mathbb{R}_{\geq 0}[F]$, $\hat{R} := \mathbb{R}_{\geq 0}[\hat{F}]$, $S := \mathbb{R}_{\geq 0}[C_G]$, and $\hat{S} := \mathbb{R}_{\geq 0}[C_{\hat{G}}]$. We already know that R and $\hat{R}$ are extremal rays of $\overline{\mathrm{NE}}(X)$, and that $R + \hat{R}$ is a face.

Since $\mathscr{E} := \mathscr{O}_Z \oplus \mathscr{O}_Z(a)$ is nef and nonample, $\mathscr{O}_Y(\hat{G}_Y) = \mathscr{O}_{\mathbb{P}_Z(\mathscr{E})}(1)$ is nef and nonample in Y, and the same holds for $\sigma^*(\hat{G}_Y) = \hat{G} + E$ in X. It is not difficult to see that $(\hat{G} + E)^{\perp} \cap \overline{\mathrm{NE}}(X) = R + S$. In particular, this shows that S is an extremal ray of $\overline{\mathrm{NE}}(X)$, so that there exists a nef divisor H such that $H^{\perp} \cap \overline{\mathrm{NE}}(X) = S$.

If $0 < a < d$, then similarly as before $\hat{\sigma}(G)$ is nef and nonample in $\hat{Y}$, so that $\hat{\sigma}^*(\hat{\sigma}(G)) = G + \hat{E}$ is nef in X. The plane $(G + \hat{E})^{\perp}$ intersects $\overline{\mathrm{NE}}(X)$ along the face $\hat{R} + \hat{S}$; in particular, $\hat{S}$ is an extremal ray.

Finally, the divisor

$$D := (-\hat{G} \cdot C_{\hat{G}})H + (H \cdot C_{\hat{G}})\hat{G}$$

is nef, it is not numerically trivial, and $D \cdot S = D \cdot \hat{S} = 0$. Therefore $D^{\perp} \cap \overline{\mathrm{NE}}(X) = S + \hat{S}$, and we obtain the statement in the case $0 < a < d$.

If instead $a \geq d$, then $\hat{\sigma}(\hat{G})$ is nef, nonample in $\hat{Y}$, so that $\hat{G} = \hat{\sigma}^*(\hat{\sigma}(\hat{G}))$ is nef, and does not intersect G. We obtain $\hat{G}^{\perp} \cap \overline{\mathrm{NE}}(X) = \hat{R} + S$, which gives the statement in the case $a \geq d$.

The case $a = 0$ follows from the case $a = d$, see Remark 3.7. ■

A straightforward consequence of Lemma 3.5 is the following.

Remark 3.6. X is Fano if and only if $a \leq i_Z - 1$ and $d - a \leq i_Z - 1$.

Indeed, since $\mathrm{NE}(X)$ is closed and polyhedral, X is Fano if and only if every extremal ray of $\mathrm{NE}(X)$ has positive intersection with the anticanonical divisor (see Table 1). □

Remark 3.7. Suppose that $a \leq d$. Then by choosing $a' = d - a$, we get a variety X' isomorphic to X, with the roles of Y and $\hat{Y}$ interchanged. □

Example 1.6 is a special case of this example, with $Z = \mathbb{P}^{n-1}$, and the additional condition $a \leq d$.

We are now in position to prove the main result of this section; see [15] for a related result.

Theorem 3.8. Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho_X = 3$, and let $D \subset X$ be a prime divisor with $\dim \mathcal{N}_1(D, X) = 1$. Then X is isomorphic to one of the varieties described in Example 3.4. □

Note that if X is as in Example 3.4, then G is a prime divisor with $\rho_G = 1$, and hence $\dim \mathcal{N}_1(G, X) = 1$.

Proof of Theorem 3.8.

3.8.1. As $\rho_X = 3$, we are in case (iii) of Lemma 3.1, and there is a conic bundle $\varphi : X \to Z$, finite on D. We keep the same notation as in Lemma 3.1; in particular, we recall the diagram:

$$\begin{tikzcd} & X \arrow[dl, "\hat{\sigma}"'] \arrow[dr, "\sigma"] \arrow[dd, "\varphi"] & \\ \hat{Y} \arrow[dr, "\hat{\pi}"'] & & Y \arrow[dl, "\pi"] \\ & Z & \end{tikzcd}$$

We set $E := \mathrm{Locus}(R)$ and $\hat{E} := \mathrm{Locus}(\hat{R})$. Note that $D \neq \hat{E}$ because φ is finite on D, hence $D \cdot \hat{R} \geq 0$. Thus, we may have $D \cdot \hat{R} > 0$ (if $D \cap \hat{E} \neq \emptyset$) or $D \cdot \hat{R} = 0$ (if $D \cap \hat{E} = \emptyset$). In the first case, $\sigma : X \to Y$ and $\hat{\sigma} : X \to \hat{Y}$ have the same properties with respect to X and D, so that their role is interchangeable, while in the second case the behavior of the two blow-ups with respect to D is different. □

3.8.2. We show that every prime divisor $B \subset X$ must intersect $E \cup \hat{E}$.

We first note that $\sigma(B) \cap \sigma(\hat{E}) \neq \emptyset$ in Y. Indeed, if $\pi(\sigma(B)) = Z$, then the claim is obvious. Otherwise, $\sigma(B) = \pi^{-1}(\varphi(B))$, and the claim follows from $\rho_Z = 1$. Thus, if $A_Y \cap \sigma(B) \neq \emptyset$, then B intersects E. Otherwise, B intersects $\hat{E}$. □

3.8.3. We show that $\pi : Y \to Z$ is a smooth morphism. Otherwise, it has a nonempty discriminant divisor $\Delta \subset Y$, whose inverse image $\sigma^{-1}(\Delta)$ must be disjoint from $E \cup \hat{E}$ since X is Fano, which contradicts 3.8.2. □

3.8.4. Let us consider the prime divisor $D_Y = \sigma(D) \subset Y$. We have $\mathcal{N}_1(D_Y, Y) = \sigma_*(\mathcal{N}_1(D, X))$, hence $\dim \mathcal{N}_1(D_Y, Y) = 1$.

We show that up to replacing D with another prime divisor $D' \subset X$, we can assume that D_Y is nef.

Suppose that D_Y is not nef. Then D_Y is the exceptional locus of a divisorial contraction $Y \to Y_0$ sending D_Y to a point, by Remark 3.2. The exceptional locus of the composite map $X \to Y_0$ is $D \cup E$, and is contracted to a point in Y_0. In particular, D is not nef either (see for instance [28, Lemma 3.39]).

Let $D' \subset X$ be the pull-back of a general prime divisor in Y_0. Then D' is a prime nef divisor in X, and $D' \cap (D \cup E) = \emptyset$. Note that the classes $[D]$ and $[E]$ in $\mathcal{N}^1(X)$ cannot be proportional, therefore the planes $D^{\perp}$ and $E^{\perp}$ in $\mathcal{N}_1(X)$ are distinct. As $\mathcal{N}_1(D', X) \subseteq D^{\perp} \cap E^{\perp}$, we obtain $\dim \mathcal{N}_1(D', X) = 1$.

By 3.8.2, we have $D' \cap \hat{E} \neq \emptyset$, hence $D' \cdot \hat{R} > 0$. Moreover, as D' is nef, we must have $\hat{\sigma}(D')$ nef too. This shows that up to replacing D with D', and R with $\hat{R}$, we can assume that D_Y is nef. □

3.8.5. By 3.8.4, we can assume that D_Y is nef. Then Proposition 3.3 applies, and $D_Y^{\perp} \cap \mathrm{NE}(Y)$ is an extremal ray S_Y of $\mathrm{NE}(Y)$. Moreover, since $A_Y \subset D_Y$ and the blow-up of Y along A_Y is Fano, case (iii) of Proposition 3.3 is excluded.

Suppose that we are in case (i) of Proposition 3.3. Then $Y \cong Z \times \mathbb{P}^1$ (see for instance [6, Lemma 4.9]), and D_Y is a fiber of $Y \to \mathbb{P}^1$. Since $A_Y \subset D_Y$, X is isomorphic to one of the varieties described in Example 3.4, with $a = 0$. This completes the proof of Theorem 3.8 in this case. □

3.8.6. We assume that we are in case (ii) of Proposition 3.3, so that the extremal ray S_Y is divisorial, its contraction sends the divisor $G_Y := \mathrm{Locus}(S_Y)$ to a point, and $D_Y \cap G_Y = \emptyset$. In particular, $\dim \mathcal{N}_1(G_Y, Y) = 1$.

Denote by $\mathcal{O}_Z(1)$ the ample generator of $\mathrm{Pic}(Z)$. By Lemma 3.9, $G_Y \subset Y$ is a section of $\pi : Y \to Z$, and there exists an integer $a > 0$ such that $\mathcal{N}_{G_Y/Y} \cong \mathcal{O}_Z(-a)$ and $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$. □

3.8.7. Suppose that there exists a section $\hat{G}_Y \subset Y$ of $\pi : Y \to Z$, disjoint from G_Y, and containing A_Y. We claim that this implies the statement. Indeed, we have

$$\mathcal{O}_Y(\hat{G}_Y) = \mathcal{O}_Y(G_Y) \otimes \pi^*(\mathcal{O}_Z(r))$$

for some $r \in \mathbb{Z}$, and since $G_Y \cap \hat{G}_Y = \emptyset$, restricting to G_Y we obtain $r = a$, and restricting to $\hat{G}_Y$ we obtain $\mathcal{N}_{\hat{G}_Y/Y} \cong \mathcal{O}_Z(a)$. Thus, X is one of the varieties described in Example 3.4, for $a > 0$. □

3.8.8. Let $G \subset X$ be the strict transform of G_Y. We have $G \cap (D \cup E) = \emptyset$ since $G_Y \cap D_Y = \emptyset$, and hence $G \cap \hat{E} \neq \emptyset$ by 3.8.2.

Let us consider now the image $\hat{\sigma}(G) \subset \hat{Y}$. Then $\hat{\sigma}(G)$ is a section of $\hat{\pi}$, so that $\hat{\sigma}(G) \cong Z$ and $\rho_{\hat{\sigma}(G)} = 1$. Moreover, $\hat{\sigma}(G)$ contains the center $A_{\hat{Y}} \subset \hat{Y}$ of the blow-up $\hat{\sigma}$. This also implies that $\hat{Y}$ is Fano, as in the proof of Lemma 3.1. $\square$

3.8.9. Suppose now that there exists a section $H \subset \hat{Y}$ of $\hat{\pi} : \hat{Y} \to Z$, disjoint from $\hat{\sigma}(G)$. Then its strict transform in Y yields a section of $\pi : Y \to Z$, disjoint from G_Y, and containing A_Y, and this implies the statement by 3.8.7.

In order to construct such H, we consider the divisor $D_{\hat{Y}} := \hat{\sigma}(D) \subset \hat{Y}$. Note that $\dim \mathcal{N}_1(D_{\hat{Y}}, \hat{Y}) = 1$, and that the two divisors $D_{\hat{Y}}$ and $\hat{\sigma}(G)$ are distinct, because $G \cap D = \emptyset$ in X. $\square$

3.8.10. Suppose first that $D_{\hat{Y}}$ is not nef. By Remark 3.2, $D_{\hat{Y}}$ is the exceptional locus of a divisorial contraction sending $D_{\hat{Y}}$ to a point. Then, by Lemma 3.9, $D_{\hat{Y}}$ is a section of $\hat{\pi}$.

Moreover, we have $D_{\hat{Y}} \cdot C \geq 0$ for every curve $C \subset \hat{\sigma}(G)$, because $D_{\hat{Y}} \neq \hat{\sigma}(G)$ and $\rho_{\hat{\sigma}(G)} = 1$. Since $D_{\hat{Y}}$ is not nef, the divisors $D_{\hat{Y}}$ and $\hat{\sigma}(G)$ must be disjoint, and we can set $H := D_{\hat{Y}}$. $\square$

3.8.11. We assume now that $D_{\hat{Y}}$ is nef. Then Proposition 3.3 applies, and $D_{\hat{Y}}^{\perp} \cap \mathrm{NE}(\hat{Y})$ is an extremal ray $S_{\hat{Y}}$ of $\mathrm{NE}(\hat{Y})$.

We claim that case (iii) of Proposition 3.3 cannot happen, namely that $S_{\hat{Y}}$ cannot be small. Indeed, this follows from Proposition 3.3 if $A_{\hat{Y}} \subset D_{\hat{Y}}$, namely if $D \cap \hat{E} \neq \emptyset$. If instead $D \cap \hat{E} = \emptyset$, then $\hat{\sigma}(G) \cap D_{\hat{Y}} = \emptyset$, hence the contraction of $S_{\hat{Y}}$ sends $\hat{\sigma}(G)$ to a point, and it cannot be small.

Suppose that we are in case (i) of Proposition 3.3. As in 3.8.5 we see that $\hat{Y} \cong Z \times \mathbb{P}^1$, and $D_{\hat{Y}}$ and $\hat{\sigma}(G)$ are fibers of the projection $\hat{Y} \to \mathbb{P}^1$. Thus, we can define H to be a general fiber of the projection $\hat{Y} \to \mathbb{P}^1$.

Finally, suppose that we are in case (ii) of Proposition 3.3, so that the contraction of $S_{\hat{Y}}$ is divisorial. By Lemma 3.9, $\mathrm{Locus}(S_{\hat{Y}})$ is a section of $\hat{\pi}$. So if $\mathrm{Locus}(S_{\hat{Y}}) \cap \hat{\sigma}(G) = \emptyset$, we set $H := \mathrm{Locus}(S_{\hat{Y}})$.

If instead $\mathrm{Locus}(S_{\hat{Y}}) \cap \hat{\sigma}(G) \neq \emptyset$, then we obtain $\mathrm{Locus}(S_{\hat{Y}}) = \hat{\sigma}(G)$, because $\rho_{\hat{\sigma}(G)} = 1$. Therefore by Lemma 3.9, there is a section H of $\hat{\pi}$ disjoint from $\hat{\sigma}(G)$. $\blacksquare$

The following result is certainly well known to experts. We include a proof for lack of references.

Lemma 3.9. Let Y and Z be smooth connected projective varieties, and let $\pi : Y \to Z$ be a $\mathbb{P}^1$-bundle. Let $\psi : Y \to Y_0$ be a birational morphism onto a projective variety sending an effective and reduced divisor G to points. Then $Y \cong \mathbb{P}_Z(\mathscr{O}_Z \oplus \mathscr{M})$ for some line bundle $\mathscr{M}$

on Z so that G identifies with the section of π corresponding to $\mathscr{O}_Z \oplus \mathscr{M} \twoheadrightarrow \mathscr{M}$. Moreover, $\mathscr{M}^{\otimes -1}$ is ample. $\square$

Proof. By replacing ψ with its Stein factorization, we may assume that Y_0 is normal and that ψ has connected fibers.

We show that G is a section of $Y \to Z$. Note that π is finite on G.

Let $B \subset Z$ be a general smooth connected curve, and set $S := \pi^{-1}(B)$. Then $G \cap S$ is a reduced curve; let C be an irreducible component of $G \cap S$. Moreover, let $C_0 \subset S$ be a minimal section, and $f \subset S$ a fiber of π. Then $C \neq f$. Set $e = -C_0^2$.

Suppose that $C \neq C_0$. Then $C \equiv aC_0 + bf$, where $a \in \mathbb{Z}_{>0}$, $b \geq ae$ if $e \geq 0$, and $2b \geq ae$ if $e < 0$ (see [17, Propositions V.2.20 and V.2.21]). Thus,

$$C^2 = (aC_0 + bf)^2 = -a^2e + 2ab = a(-ae + 2b) \geq 0.$$

On the other hand, the restriction of ψ to S induces a birational morphism $\psi_{|S} : S \to \psi(S)$ sending C to a point, and hence $C^2 < 0$, yielding a contradiction. Thus $C = C_0$, hence $G \cap S = C_0$. This completes the proof of the first assertion.

Set $\mathscr{G} := \pi_* \mathscr{O}_Y(G)$. Then $\mathscr{G}$ is a locally free sheaf of rank 2 that fits into a short exact sequence

$$0 \to \mathscr{O}_Z \to \mathscr{G} \to \mathscr{M} \to 0$$

corresponding to a class $\alpha \in H^1(Z, \mathscr{M}^{\otimes -1})$, Y identifies with $\mathbb{P}_Z(\mathscr{G})$, $\mathscr{O}_Y(G)$ with the tautological line bundle $\mathscr{O}_{\mathbb{P}_Z(\mathscr{G})}(1)$, and G corresponds to $\mathscr{G} \twoheadrightarrow \mathscr{M}$.

We denote by $2G$ the nonreduced closed subscheme of Y defined by the ideal sheaf $\mathscr{O}_Y(-2G)$. By [12, Lemme 3.2 and Lemme 3.3], we have

$$\mathrm{Pic}(G) \oplus H^1(Z, \mathscr{M}^{\otimes -1}) \cong \mathrm{Pic}(2G)$$

and, under the above isomorphism, $(0, \alpha)$ maps to the class of $(\mathscr{O}_{\mathbb{P}_Z(\mathscr{G})}(1) \otimes \pi^* \mathscr{M}^{\otimes -1})_{|2G}$.

Let $\mathscr{H}$ be an ample line bundle on Y_0. Then there exists $m \in \mathbb{Z}_{>0}$ such that $\psi^* \mathscr{H} \cong \mathscr{O}_{\mathbb{P}_Z(\mathscr{G})}(m) \otimes \pi^* \mathscr{M}^{\otimes -m}$. This implies that $(\mathscr{O}_{\mathbb{P}_Z(\mathscr{G})}(m) \otimes \pi^* \mathscr{M}^{\otimes -m})_{|2G} \cong \mathscr{O}_{2G}$. Hence we must have $\alpha = 0$, and $\mathscr{G} \cong \mathscr{O}_Z \oplus \mathscr{M}$. Let G' be the section of $Y \to Z$ corresponding to $\mathscr{G} \cong \mathscr{O}_Z \oplus \mathscr{M} \twoheadrightarrow \mathscr{O}_Z$. Then $G \cap G' = \emptyset$, therefore $\psi^* \mathscr{H}_{|G'}$ is ample. But $\psi^* \mathscr{H}_{|G'} \cong \mathscr{M}^{\otimes -m}$ under the isomorphism $G' \cong Z$. This completes the proof of the lemma. $\blacksquare$

4 Fano Manifolds with a Locally Unsplit Family of Rational Curves of Anticanonical Degree n

In this section, we prove Theorem 1.4 and Proposition 4.7. We start with the following observations.

Remark 4.1. Let V be a locally unsplit dominating family of rational curves on a smooth projective variety X of dimension n, and suppose that the curves of the family have anticanonical degree n. Let x be a general point, and let $\mathrm{Locus}(V_x) \subseteq X$ be the union of all curves parameterized by V_x. Then, by [27, Corollaries IV.2.6.3 and II.4.21], we have that $\mathrm{Locus}(V_x)$ is a divisor and $\mathcal{N}_1(\mathrm{Locus}(V_x), X) = \mathbb{R}[V]$. □

Remark 4.2. Let X be a Fano manifold, and recall that l_X is the minimal anticanonical degree of a locally unsplit dominating family of rational curves in X.

If $x \in X$ is a general point, then every irreducible rational curve C through x has anticanonical degree at least l_X, see [27, Theorem IV.2.4]. This implies that l_X can equivalently be defined as the minimal anticanonical degree of a dominating family of rational curves in X. □

Proof of Theorem 1.4. Since $l_X = n$, there is a locally unsplit dominating family V of rational curves of anticanonical degree n. Thus, by Remark 4.1 X contains a prime divisor D with $\dim \mathcal{N}_1(D, X) = 1$, so that we can apply Lemma 3.1.

Since $l_X = n > 2$, we know that X cannot have a conic bundle structure. Therefore, Lemma 3.1 yields that $\rho_X = 2$ and there exists $\sigma : X \to Y$ such that Y is smooth with $\dim Y = n$ and $\rho_Y = 1$, and σ is the blow-up of $A \subset Y$ smooth of codimension 2.

Let $E \subset X$ be the exceptional divisor; we have $-K_X + E = \sigma^*(-K_Y)$.

Let W be a locally unsplit dominating family of rational curves in Y, $C \subset Y$ a general curve of the family, and $\tilde{C} \subset X$ its strict transform. By [27, Proposition II.3.7], $C \cap A = \emptyset$, and hence $E \cdot \tilde{C} = 0$. Moreover, $\tilde{C}$ moves in a dominating family $\widetilde{W}$ of rational curves in X, so that $-K_X \cdot \tilde{C} \geq n$. This yields $-K_Y \cdot C \geq n$.

We show that $-K_Y \cdot [W] > n$. We argue by contradiction, and assume that $-K_Y \cdot [W] = n$. Consider a curve C as above. Then we have $-K_X \cdot \widetilde{C} = n$. Since $-K_X \cdot [\widetilde{W}] = n = l_X$, the family $\widetilde{W}$ is locally unsplit.

Now for a general point $x_1 \in X$, again by Remark 4.1, $\mathrm{Locus}(\widetilde{W}_{x_1})$ is a divisor with $\mathcal{N}_1(\mathrm{Locus}(\widetilde{W}_{x_1}), X) = \mathbb{R}[\widetilde{W}]$. Since $E \cdot [\widetilde{W}] = 0$, the divisor $\mathrm{Locus}(\widetilde{W}_{x_1})$ must be disjoint from E, thus its image in Y is disjoint from A. This gives a contradiction, because every nontrivial effective divisor in Y is ample (recall that $\rho_Y = 1$).

We conclude that W has anticanonical degree $n+1$, so that $Y \cong \mathbb{P}^n$ by Theorem 1.1.

Now, if $\ell \subset \mathbb{P}^n$ is a line intersecting A in at least two points, and $\widetilde{\ell} \subset X$ is its strict transform, then $-K_X \cdot \widetilde{\ell} < n$. We deduce that the secant variety $\mathrm{Sec}(A)$ of A is not the whole $\mathbb{P}^n$, and this implies that A is degenerate (see for instance [29, Theorem 3.4.26]). Finally, one can check that since X is Fano, the degree d of A is at most n, completing the proof. ■

We need some preliminary results for Proposition 4.7, which is the main result of this section.

Lemma 4.3. Let X and Y be smooth projective varieties, and let $\pi : X \to Y$ be a surjective morphism with connected fibers. Let V be a locally unsplit dominating family of rational curves on X. Suppose that π does not contract any curve from V.

(1) If $K_{X/Y} \cdot [V] \le 0$, then $K_{X/Y} \cdot [V] = 0$.
(2) Suppose moreover that $-K_X \cdot [V] = \dim Y + 1$, and that for a general $x \in X$, and for every $[C] \in V_x$, the map $\pi_{|C} : C \to \pi(C)$ is birational. Then $Y \cong \mathbb{P}^{\dim Y}$. □

Proof. Let x be a general point in X, and let $[C] \in V_x$ be a general curve. Set $\ell := \pi(C)$, and let m be the positive integer such that $\pi_* C = m\ell$. Note that ℓ is a free curve and hence yields a smooth point in $\mathrm{RatCurves}^n(Y)$. Let V_Y be the irreducible component of $\mathrm{RatCurves}^n(Y)$ which contains $[\ell]$. We set $y := \pi(x)$. We also set $V'_x := \{[C] \in V \mid x \in C\} \subseteq V$, and similarly we define $(V_Y)'_y \subseteq V_Y$ (recall that V_x is the normalization of V'_x, and $(V_Y)_y$ that of $(V_Y)'_y$). Note that V'_x has pure dimension $-K_X \cdot [V] - 2$, and since V_Y is a dominating family, $(V_Y)'_y$ has pure dimension $-K_Y \cdot [V_Y] - 2$.

Let us consider the morphism $\pi_* : V \to \mathrm{Chow}(Y)$ induced by the pushforward morphism $\mathrm{Chow}(X) \to \mathrm{Chow}(Y)$ (see [27, Theorem I.6.8]). We claim that π_* is finite on V'_x. Suppose otherwise, and let $T \subseteq V'_x$ be an irreducible complete curve contained in a fiber of π_*. We set $\mathrm{Locus}(T) := \cup_{t \in T} C_t \subseteq X$. Then $\dim \mathrm{Locus}(T) = 2$, and $\dim \mathcal{N}_1(\mathrm{Locus}(T)) = 1$ by [27, Corollary II.4.21]. This implies that π is finite on $\mathrm{Locus}(T)$, thus $\dim \pi(\mathrm{Locus}(T)) = 2$. On the other hand, $\ell' := \pi(C_t)$ does not depend on $t \in T$, because T is contained in a fiber of π_*. Hence $\pi(\mathrm{Locus}(T)) = \ell'$, a contradiction. This proves that π_* is finite on V'_x.

Therefore, the rational map $V'_x \dashrightarrow (V_Y)'_y$ sending $[C] \in V'_x$ to $\frac{1}{m}\pi_*[C] = [\ell] \in (V_Y)'_y$ is generically finite, and hence $\dim (V_Y)'_y \ge \dim V'_x$. We obtain

$$-K_Y \cdot [V_Y] = \dim (V_Y)'_y + 2 \ge \dim V'_x + 2 = -K_X \cdot [V].$$

Suppose from now on that $K_{X/Y} \cdot [V] \leq 0$. Then

$$-K_X \cdot [V] \geq -mK_Y \cdot [V_Y],$$

and thus $m = 1$ and $-K_X \cdot [V] = -K_Y \cdot [V_Y]$, proving (1).

We proceed to prove (2). Since $-K_Y \cdot [V_Y] = \dim Y + 1$, by Theorem 1.1 it is enough to show that V_Y is a locally unsplit family of rational curves on Y. Let X_0 be a dense open subset of X such that for every point $x \in X_0$

- for any curve $[C] \in V_x$, C is a free curve,
- V_x is proper, and
- for any curve $[C] \in V_x$, $\pi_{|C} : C \to \pi(C)$ is birational.

Let V_0 be the open subset of V consisting of points $[C]$ such that $C \cap X_0 \neq \emptyset$. Observe that $(\pi_*)_{|V_0} : V_0 \to \mathrm{Chow}(Y)$ factors through $V_Y \to \mathrm{Chow}(Y)$. We still denote by $(\pi_*)_{|V_0}$ the induced morphism $V_0 \to V_Y$.

Since V'_x is proper, to prove that V_Y is a locally unsplit family of rational curves, it is enough to show that $(\pi_*)_{|V'_x} : V'_x \to (V_Y)'_y$ is dominant.

Consider the universal families $U_0 \to V_0$ and $U_Y \to V_Y$, and the evaluation morphisms $e : U_0 \to X$ and $e_Y : U_Y \to Y$. Note that e is flat by [27, Corollary II.3.5.3 and Theorem II.2.15]. We have a commutative diagram:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & U_0 \arrow[l, "e"'] \arrow[r, "p"] \arrow[d, "\mu"] & V_0 \arrow[d, "(\pi_*)_{|V_0}"] \\ Y & U_Y \arrow[l, "e_Y"'] \arrow[r, "p_Y"] & V_Y \end{tikzcd}$$

Recall from the proof of (1) that $\dim(\pi_*)_{|V_0}(V'_x) = \dim(V_Y)'_y$. This implies that $(\pi_*)_{|V_0}$ is dominant, and hence so is μ.

On the other hand, since $V'_x = p(\mathrm{e}^{-1}(x))$ and $(V_Y)'_y = p_Y(e_Y^{-1}(y))$, we also have a commutative diagram:

$$\begin{tikzcd} e^{-1}(x) \arrow[r] \arrow[d, "\mu_{|e^{-1}(x)}"'] & V'_x \arrow[d, "(\pi_*)_{|V'_x}"] \\ e_Y^{-1}(y) \arrow[r] & (V_Y)'_y \end{tikzcd}$$

where the horizontal arrows are finite and surjective, and $(\pi_*)_{|V'_x}$ is finite by the proof of (1). Therefore, $\mu_{|\mathrm{e}^{-1}(x)} : \mathrm{e}^{-1}(x) \to e_Y^{-1}(y)$ is finite, and to show that $(\pi_*)_{|V'_x} : V'_x \to (V_Y)'_y$ is dominant, it is enough to show that $\mu_{|\mathrm{e}^{-1}(x)} : \mathrm{e}^{-1}(x) \to e_Y^{-1}(y)$ is dominant.

Consider now $Z \subset U_0$ a general fiber of the composition $\pi \circ e : U_0 \to Y$; Z has pure dimension $\dim U_0 - \dim Y$.

$$\begin{tikzcd} \pi^{-1}(y) & Z \arrow[l, "e_{|Z}"'] \arrow[d, "\mu_{|Z}"] \\ & e_Y^{-1}(y) \end{tikzcd}$$

Let F be an irreducible component of $e_Y^{-1}(y)$, and let Z_0 be an irreducible component of Z dominating F under μ. Since e is flat, Z_0 must also dominate $\pi^{-1}(y)$ under e. Thus, for $x \in \pi^{-1}(y)$ general, $\mathrm{e}^{-1}(x) \cap Z_0$ is nonempty and has pure dimension $\dim Z_0 - \dim \pi^{-1}(y) = \dim \mathrm{e}^{-1}(x) = \dim e_Y^{-1}(y) = \dim F$. Finally, μ is finite on $\mathrm{e}^{-1}(x) \cap Z_0$, so that $\mathrm{e}^{-1}(x) \cap Z_0$ dominates F under μ.

We conclude that $\mu_{|e^{-1}(x)} : e^{-1}(x) \to e_Y^{-1}(y)$ is dominant, completing the proof of the lemma. ■

Lemma 4.4. Let X and Y be smooth projective varieties, and let $\pi : X \to Y$ be a $\mathbb{P}^1$-bundle. Let V be a locally unsplit dominating family of rational curves on X with $-K_X \cdot [V] = \dim(X) \geq 3$. Set $n := \dim(X)$. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$, and V is the family of lines in the $\mathbb{P}^{n-1}$'s. □

Proof. Let x be a general point in X, V'_x an irreducible component of V_x, and $U'_x \to V'_x$ the universal family. Let $U'_x \to X$ be the evaluation morphism, and $U'_x \to T$ its Stein factorization. Then T is a normal generalized cone; in particular, $\rho_T = 1$ by [27, Corollary II.4.21].

We claim that the composite map $T \to X \to Y$ is finite; in particular, it is dominant. Suppose otherwise. Then, by [27, Corollary II.4.21], π sends every curve in $D := \mathrm{Locus}(V'_x)$ to a point. Thus $-K_X \cdot [V] = 2$, yielding a contradiction.

Set $Z := T \times_Y X$, with natural morphisms $\tau : Z \to T$, and $\nu : Z \to X$.

$$\begin{tikzcd} Z \arrow[r, "\nu"] \arrow[d, "\tau"'] & X \arrow[d, "\pi"] \\ T \arrow[r] & Y \end{tikzcd}$$

Note that τ is a $\mathbb{P}^1$-bundle. Let $T_Z \subset Z$ be the section of τ induced by $T \to X$; we have $\nu(T_Z) = D$. Then $Z \cong \mathbb{P}_T(\mathscr{E})$, where $\mathscr{E} := \tau_* \mathscr{O}_Z(T_Z)$ is a rank 2 vector bundle on T that fits into a short exact sequence

$$0 \to \mathscr{O}_T \to \mathscr{E} \to \mathscr{M} \to 0,$$

with $\mathscr{M}$ a line bundle on T. Moreover, $\mathscr{E} \twoheadrightarrow \mathscr{M}$ corresponds to the section T_Z, and $\mathscr{O}_Z(T_Z)$ identifies with the tautological line bundle, so that $\mathscr{O}_Z(T_Z)_{|T_Z} \cong \tau^* \mathscr{M}_{|T_Z}$.

We prove that $K_{X/Y} \cdot [V] \leq 0$. Suppose otherwise. We have

$$\mathscr{O}_Z(K_{Z/T}) \cong \mathscr{O}_Z(-2T_Z) \otimes \tau^* \mathscr{M},$$

and therefore

$$\mathscr{O}_Z(K_{Z/T})_{|T_Z} \cong \mathscr{O}_Z(-T_Z)_{|T_Z}.$$

Let $[C] \in V'_x$, and let C_Z be an irreducible component of $\nu^{-1}(C)$ contained in T_Z. Then, by the projection formula:

$$-T_Z \cdot C_Z = K_{Z/T} \cdot C_Z = \nu^*(K_{X/Y}) \cdot C_Z = mK_{X/Y} \cdot C > 0,$$

where $m \in \mathbb{Z}_{>0}$ is such that $\nu_* C_Z = mC$. This implies that $\mathscr{M} \cdot \tau_* C_Z < 0$.

Let now $[C'] \in V$ be a general point, and C'_Z an irreducible component of $\nu^{-1}(C')$ not contained in T_Z. Then, as above, we must have $K_{Z/T} \cdot C'_Z > 0$. On the other hand, $T_Z \cdot C'_Z \geq 0$ since $C'_Z \not\subset T_Z$, and $\mathscr{M} \cdot \tau_* C'_Z < 0$ because $\rho_T = 1$. This implies that

$$K_{Z/T} \cdot C'_Z = -2T_Z \cdot C'_Z + \mathscr{M} \cdot \tau_* C'_Z < 0,$$

yielding a contradiction. Therefore $K_{X/Y} \cdot [V] \leq 0$, and Lemma 4.3(1) yields $K_{X/Y} \cdot [V] = 0$.

Let $x \in X$ be a general point, $[C] \in V_x$, and set $y := \pi(x)$ and $\ell := \pi(C)$. We show that $\pi_{|C} : C \to \ell$ is birational. Let $\mathbb{F}$ be the normalization of $\pi^{-1}(\ell)$, and let $\tilde{\ell}$ be the normalization of ℓ. Let C' be the strict transform of C in $\mathbb{F}$. By the projection formula, we have $K_{\mathbb{F}/\tilde{\ell}} \cdot C' = K_{X/Y} \cdot C = 0$. This implies that $\mathbb{F} \cong \tilde{\ell} \times \mathbb{P}^1$, and that C' is a fiber of $\mathbb{F} \to \mathbb{P}^1$, proving our claim.

Therefore $Y \cong \mathbb{P}^{n-1}$ by Lemma 4.3(2), and $T = D$ is a section of π. Since $K_{X/Y} \cdot [V] = 0$, we must have $\mathscr{M} \equiv 0$, and hence $\mathscr{M} \cong \mathscr{O}_{\mathbb{P}^{n-1}}$. Finally, $\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^{n-1}}^{\oplus 2}$, since $h^1(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}) = 0$. This completes the proof of the lemma. ∎

Lemma 4.5. Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho_X = 2$, and suppose that X has a locally unsplit dominating family of rational curves V of anticanonical degree n. Let $D \subset X$ be an irreducible component of $\mathrm{Locus}(V_x)$ for a general point $x \in X$.

Then one of the following holds:

(a) X is the blow-up of $\mathbb{P}^n$ along a linear subspace of codimension 2;

(b) $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$;

(c) $D^\perp \cap \mathrm{NE}(X)$ is an extremal ray of X, whose associated contraction is divisorial and sends its exceptional divisor to a point; the other extremal contraction of X is either a (singular) conic bundle, or the blow-up of a smooth variety along a smooth subvariety of codimension 2. □

Remark 4.6. The only examples known to the authors of Fano manifolds with dimension $n \geq 3$, Picard number $\rho_X = 2$, and having a locally unsplit dominating family of rational curves of anticanonical degree n, are $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ and the varieties from Example 1.3. These are obtained as the blow-up of $\mathbb{P}^n$ along a smooth subvariety A, of dimension $n-2$ and degree $d \in \{1, \dots, n\}$, contained in a hyperplane. If $d = 1$, this is case (a) of Lemma 4.5. If $d > 1$, this gives an example of case (c) of Lemma 4.5. □

Proof of Lemma 4.5. By Remark 4.1, $\mathcal{N}_1(D, X) = \mathbb{R}[V]$. This implies that D is nef since $D \cdot [V] \geq 0$. Let R be an extremal ray of X such that $D \cdot R > 0$, and let $\sigma : X \to Y$ be the contraction of R. By Lemma 3.1, Y is a smooth Fano variety, and either σ is a blow-up with center $A_Y \subset Y$ smooth of codimension 2, or σ is a conic bundle. Set $S := D^\perp \cap \mathrm{NE}(X)$. Then S is an extremal ray of X by Proposition 3.3.

Suppose that we are in case (i) of Proposition 3.3. We show that we are in case (a) or (b). The contraction of S is $\varphi : X \to \mathbb{P}^1$, D is a fiber, and hence $D \cdot [V] = 0$. Let F be a general fiber of φ. Then the family of rational curves from V contained in F is locally unsplit with anticanonical degree $\dim(F) + 1$, thus $F \simeq \mathbb{P}^{n-1}$ by Theorem 1.1.

Let $B_0 \subset \mathbb{P}^1$ be a dense open subset such that $X_0 := \varphi^{-1}(B_0) \to B_0$ is smooth. By Tsen's Theorem, there exists a divisor H_0 on X_0 such that $\mathscr{O}_F(H_{0|F}) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(1)$.

Let H be the closure of H_0 in X. Then H is φ-ample since φ is elementary. If $p \in \mathbb{P}^1$, then $[\varphi^*(p)] \cdot H^{n-1} = 1$. Therefore, all fibers of φ are integral, and by [14, Corollary 5.4] we have $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathscr{E})$ with $\mathscr{E}$ a vector bundle of rank n on $\mathbb{P}^1$. Since X is Fano, it is not difficult to see that either $X \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$, or X is the blow-up of $\mathbb{P}^n$ along a linear subspace of codimension 2. Thus, we get (a) or (b).

Case (ii) of Proposition 3.3 is (c).

We show that case (iii) of Proposition 3.3 does not occur. Suppose otherwise. Then the contraction φ of S is small, it has a flip $X \dashrightarrow X'$, X' is smooth, and there is a smooth $\mathbb{P}^1$-bundle $\psi : X' \to Y'$. Let $[C] \in V$ be a general point. Then $\mathrm{Exc}(\varphi) \cap C = \emptyset$ (see [27, Proposition II.3.7]). Let V' be the irreducible component of $\mathrm{RatCurves}^n(X')$ which contains C' the strict transform of C in X'.

We show that V' is a locally unsplit dominating family of rational curves on X'. Let $x \in X \setminus \mathrm{Exc}(\varphi)$ be a general point. If $\mathrm{Locus}(V_x) \cap \mathrm{Exc}(\varphi) \neq \emptyset$, then $[V] \in S$ since $\mathcal{N}_1(\mathrm{Locus}(V_x), X) = \mathbb{R}[V]$ and $D \cdot S = 0$, and hence $\mathrm{Locus}(S) = X$, yielding a contradiction. Therefore, $\mathrm{Locus}(V_x) \cap \mathrm{Exc}(\varphi) = \emptyset$, and hence $V'_x \cong V_x$ is proper. This proves that V' is a locally unsplit family of rational curves. Note also that $-K_{X'} \cdot [V'] = n$. By Lemma 4.4, $X' \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$, so that X' does not have small contractions, a contradiction. This completes the proof of the lemma. ■

Finally, we prove the main result of this section.

Proposition 4.7. Let X be a Fano manifold of dimension $n \geq 3$, and suppose that X has a locally unsplit dominating family V of rational curves of anticanonical degree n. Then $\rho_X \leq 3$.

If moreover $\rho_X = 3$, then X is isomorphic to one of the varieties described in Example 1.6, and $[V] \equiv C_{\hat{G}} + (d-a)F$, where F is a fiber of σ, and $C_{\hat{G}}$ is the strict transform of a line in $\hat{G}_Y \cong \mathbb{P}^{n-1}$ (notations as in Example 1.6). □

Proof. Let $D \subset X$ be an irreducible component of $\mathrm{Locus}(V_x)$ for a general point $x \in X$. Then $\mathcal{N}_1(D, X) = \mathbb{R}[V]$ by Remark 4.1, and by Lemma 3.1 we have $\rho_X \leq 3$. Note that D is nef, because $D \cdot [V] \geq 0$.

4.7.1. We assume that $\rho_X = 3$. By Theorem 3.8, X is as described in Example 3.4. We use the notations of Example 3.4 and Theorem 3.8; in particular, we refer the reader to Table 1 for the intersection table of X. We have to show that $Z \cong \mathbb{P}^{n-1}$, that $a \leq d$, and that $[V] \equiv C_G + a\hat{F} \equiv C_{\hat{G}} + (d-a)F$ (see (3.1)). □

4.7.2. Note that $G \cdot [V] = 0$. Indeed, if for instance $G \cap D \neq \emptyset$, as $\rho_G = 1$, then there exists $\lambda \in \mathbb{Q}_{>0}$ such that $[V] = \lambda[C_G]$. On the other hand, $G \cdot [V] \geq 0$ while $G \cdot C_G = -a\delta \leq 0$, and we conclude that $G \cdot [V] = 0$.

Set $m := -K_Z \cdot C_Z = i_Z\delta > 0$. Since $[F], [\hat{F}], [C_G]$ are a basis of $\mathcal{N}_1(X)$, and we can write:

$$[V] \equiv \alpha F + \beta \hat{F} + \gamma C_G,$$

with $\alpha, \beta, \gamma \in \mathbb{Q}$. Intersecting with G yields $\beta = a\gamma\delta$, and intersecting with $-K_X$ yields $\alpha = n - m\gamma$:

$$[V] \equiv (n - m\gamma)F + a\gamma\delta\hat{F} + \gamma C_G. \tag{4.1}$$

Note that $\gamma > 0$, because $[F]$ belongs to the extremal ray R, and $[V] \notin R$.

Moreover, $\hat{E} \cdot [V] = \gamma\delta(d - a)$, so we obtain $d - a \geq 0$.

Finally, we also have $\hat{G} \cdot [V] = n - m\gamma \geq 0$. $\square$

4.7.3. By (4.1), we have

$$K_{X/Z} \cdot [V] = K_X \cdot [V] - K_Z \cdot \varphi_*[V] = -n - K_Z \cdot \gamma C_Z = -n + \gamma m \leq 0.$$

Then Lemma 4.3(1) yields $K_{X/Z} \cdot [V] = 0$ and hence $\gamma = n/m$, and (4.1) becomes:

$$[V] \equiv \frac{n}{m}(a\delta\hat{F} + C_G). \tag{4.2}$$

$\square$

4.7.4. Let $x \in X$ be a general point. We show that φ has degree 1 on every curve of V_x.

We proceed by contradiction, and assume that for some $[C_\infty] \in V_x$, the morphism $C_\infty \to \varphi(C_\infty)$ has degree $k \geq 2$.

Let $\ell \to \varphi(C_\infty)$ be the normalization, set $\mathbb{F} := \ell \times_Z Y$, and denote by $\pi_{|\mathbb{F}} : \mathbb{F} \to \ell$ (respectively, $\nu : \mathbb{F} \to Y$) the natural morphisms.

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\nu"] \arrow[d, "\pi_{|\mathbb{F}}"'] & Y \arrow[d, "\pi"'] & X \arrow[l, "\sigma"'] \arrow[dl, "\varphi"] \\
\ell \arrow[r] & Z &
\end{tikzcd}$$

Let $C'_\infty \subset \mathbb{F}$ be the strict transform of $\sigma(C_\infty) \subset Y$. Let moreover $C_0 \subset \mathbb{F}$ be a minimal section of $\pi_{|\mathbb{F}}$, and $f \subset \mathbb{F}$ a general fiber.

We have $G_Y \cdot \sigma(C_\infty) = \sigma^*(G_Y) \cdot C_\infty = G \cdot [V] = 0$ by 4.7.2, and $\sigma(C_\infty) \not\subseteq G_Y$ because $x \in C_\infty$ is general, hence $G_Y \cap \sigma(C_\infty) = \emptyset$.

The pull-back of G_Y to $\mathbb{F}$ is precisely C_0, so that $C'_\infty \cap C_0 = \emptyset$. Moreover, $\pi_{|\mathbb{F}}$ has degree k on C'_∞.

Set $a_0 := -C_0^2 \in \mathbb{Z}_{\geq 0}$. Then $\mathbb{F} \cong \mathbb{F}_{a_0}$, and we obtain

$$C'_\infty \equiv k(C_0 + a_0 f) \quad \text{and} \quad -K_{\mathbb{F}} \cdot C'_\infty = k(a_0 + 2). \tag{4.3}$$

Consider now $\nu^*\hat{G}_Y \subset \mathbb{F}$. This is a section of $\pi_{|\mathbb{F}}$ disjoint from C_0, so that $\nu^*\hat{G}_Y \equiv C_0 + a_0 f$. We also have $\sigma^*\hat{G}_Y = \hat{G} + E$, and $\hat{G} \cdot [V] = 0$ by (4.2), so using the projection formula:

$$E \cdot C_\infty = \sigma^* \hat{G}_Y \cdot C_\infty = \hat{G}_Y \cdot \sigma(C_\infty) = \nu^* \hat{G}_Y \cdot C'_\infty = ka_0. \tag{4.4}$$

Since $A_Y \subset \hat{G}_Y$, we have $\nu^{-1}(A_Y) \subset \nu^* \hat{G}_Y$. Moreover, $C_\infty \not\subseteq E \cup \hat{E}$ because $x \in C_\infty$, hence $\varphi(C_\infty) \not\subseteq A$, and $\nu^{-1}(A_Y)$ is a zero-dimensional scheme.

We see $\nu^{-1}(A_Y)$ as a zero-dimensional subscheme of $\nu^* \hat{G}_Y \cong \mathbb{P}^1$; in particular, it is determined by its support and by its multiplicity at each point. We write $\nu^{-1}(A_Y) = h_1 p_1 + \cdots + h_r p_r$. $\square$

4.7.5. Let $\varpi : \tilde{\mathbb{F}} \to \mathbb{F}$ be the blow-up of $\mathbb{F}$ along $\nu^{-1}(A_Y)$; note that there is a natural morphism $\widetilde{\nu} : \tilde{\mathbb{F}} \to X$:

$$\begin{tikzcd}
\widetilde{\mathbb{F}} \arrow[r, "\widetilde{\nu}"] \arrow[d, "\varpi"'] & X \arrow[d, "\sigma"'] \arrow[dd, bend left, "\varphi"] \\
\mathbb{F} \arrow[r, "\nu"] \arrow[d, "\pi_{|\mathbb{F}}"'] & Y \arrow[d, "\pi"'] \\
\ell \arrow[r] & Z
\end{tikzcd}$$

Observe that $\tilde{\mathbb{F}}$ is a normal surface, with local complete intersection singularities. Indeed, locally over p_i, $\tilde{\mathbb{F}}$ can be described as the blow-up of $\mathbb{A}^2_{x,y}$ at the ideal (x^{h_i}, y). In particular, $\tilde{\mathbb{F}}$ is smooth at $\varpi^{-1}(p_i)$ if $h_i = 1$; otherwise, it has a Du Val singularity of type A_{h_i-1}.

Set $F_i := \varpi^{-1}(p_i)$ (with reduced scheme structure); then

$$K_{\tilde{\mathbb{F}}} = \varpi^* K_{\mathbb{F}} + \sum_{1 \le i \le r} h_i F_i, \quad \text{and} \quad \sum_{1 \le i \le r} h_i F_i = \widetilde{\nu}^* E. \tag{4.5}$$

Let $\widetilde{C}'_\infty \subset \widetilde{\mathbb{F}}$ be the strict transform of $C'_\infty \subset \mathbb{F}$. Then $\widetilde{\nu}(\widetilde{C}'_\infty) = C_\infty$, and using (4.5), (4.3), and (4.4), we obtain

$$-K_{\tilde{\mathbb{F}}} \cdot \tilde{C}'_\infty = -K_{\mathbb{F}} \cdot C'_\infty - E \cdot C_\infty = 2k + ka_0 - ka_0 = 2k. \tag{4.6}$$

$\square$

4.7.6. The curve $\widetilde{C}'_\infty$ is an integral rational curve in $\widetilde{\mathbb{F}}$, of anticanonical degree $2k$ by (4.6). We fix a point $p_0 \in \widetilde{C}'_\infty$ such that $\widetilde{\nu}(p_0) = x$. By [27, Theorems II.1.7 and II.2.16], every irreducible component of $\mathrm{RatCurves}^n(\widetilde{\mathbb{F}}, p_0)$ containing $[\widetilde{C}'_\infty]$ has dimension $\ge 2k - 2 \ge 2$, because $k \ge 2$ by assumption.

Let $B \subseteq \mathrm{RatCurves}^n(\widetilde{\mathbb{F}}, p_0)$ be an irreducible curve containing $[\tilde{C}'_\infty]$. If B is proper, we obtain $\mathrm{Locus}(B) = \widetilde{\mathbb{F}}$ and hence $\rho_{\widetilde{\mathbb{F}}} = 1$ by [27, Corollary IV.4.21], a contradiction. If instead B is not proper, the closure of its image in $\mathrm{Chow}(\widetilde{\mathbb{F}})$ contains points corresponding to nonintegral curves. But this gives again a contradiction, because V_x is proper.

We conclude that $Z \cong \mathbb{P}^{n-1}$ by Lemma 4.3(2), $C_Z \subset Z$ is a line, $\delta = \mathscr{O}_Z(1) \cdot C_Z = 1$, $m = -K_Z \cdot C_Z = n$, and $\gamma = 1$. Finally, $[V] \equiv a\hat{F} + C_G$ by (4.2). ■

5 Examples of Locally Unsplit Families of Rational Curves of Anticanonical Degree n

Let X be one of the varieties introduced in Example 1.6; we use the same notations as in Examples 1.6 and 3.4, with $Z = \mathbb{P}^{n-1}$.

We construct a dominating family of rational curves on X, and then show that it is locally unsplit. Note that the condition $a \leq d$ is necessary to ensure the existence of such a family, see 4.7.2.

We first consider the case $a = 0$, so that $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$, and X is the blow-up of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ along $A \times \{p_0\}$, where $p_0 \in \mathbb{P}^1$ is a fixed point. The general curve of the family V is the strict transform in X of $\ell \times \{p\} \subset Y$, where $p \neq p_0$ and $\ell \subset \mathbb{P}^{n-1}$ is a line. Therefore, V is a locally unsplit dominating family of rational curves, and for a general point $x \in X$, V_x is isomorphic to the variety of lines through a fixed point in $\mathbb{P}^{n-1}$; hence $V_x \cong \mathbb{P}^{n-2}$.

Suppose from now on that $a > 0$, and set

$$X_0 := X \setminus (E \cup \hat{E} \cup G \cup \hat{G}).$$

Let $\ell \subset \mathbb{P}^{n-1}$ be a line not contained in A, and let $x \in X_0$ be such that $\varphi(x) \in \ell$. Set $\mathbb{F} := \pi^{-1}(\ell) \cong \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(a))$, and denote by $\pi_{|\mathbb{F}}$ the restriction of π to $\mathbb{F}$. We have $G_Y \cap \mathbb{F} = C_0$ the minimal section of $\pi_{|\mathbb{F}}$, and $\hat{G}_Y \cap \mathbb{F} \cong \mathbb{P}^1$ is another section with $\hat{G}_Y \cap \mathbb{F} \equiv C_0 + af$, where f is a fiber of $\pi_{|\mathbb{F}}$ (see Figure 2).

Since $A \subset \mathbb{P}^{n-1}$ is a hypersurface of degree d, and $\ell \not\subset A$, $A \cap \ell$ is a zero-dimensional scheme of length d. Moreover, $A_Y \cap \mathbb{F}$ is isomorphic to $A \cap \ell$, because $\hat{G}_Y$ is a section of π. In particular, $A_Y \cap \mathbb{F}$ can be seen as a closed subscheme of $\ell \cong \mathbb{P}^1$, hence it is determined by its support and by its multiplicity at each point.

Recall that $1 \leq a \leq d$ by assumption. Let us consider a closed subscheme W of $A_Y \cap \mathbb{F}$, of length a. Again, W is determined by its support and by its multiplicity at each point, so that, without loss of generality, we can write $W = \{p_1, \ldots, p_a\}$, where p_i are possibly equal points in $A_Y \cap \mathbb{F}$ (and each p_i appears at most h_i times, if h_i is the multiplicity of $A_Y \cap \mathbb{F}$ at p_i).

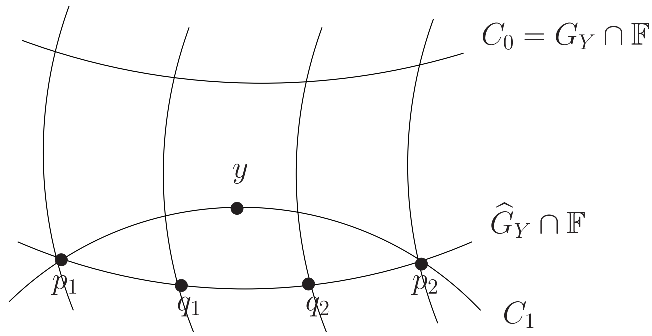

Fig. 2. The surface $\mathbb{F}$. Here $d=4$, $a=2$, $A_Y \cap \mathbb{F} = \{p_1, p_2, q_1, q_2\}$, and $W = \{p_1, p_2\}$.

Construction 5.1. We associate to (ℓ, W, x) a smooth rational curve $C \subset X$, of anticanonical degree n, and containing x. □

Set $y := \sigma(x) \in \mathbb{F} \smallsetminus (\pi^{-1}(A) \cup G_Y \cup \hat{G}_Y)$. We write $\mathscr{I}_W$ (respectively, $\mathscr{I}_y$) for the ideal sheaf of W (respectively, y) in $\mathbb{F}$. We claim that

$$h^0(\mathbb{F}, \mathscr{O}_{\mathbb{F}}(C_0 + af) \otimes \mathscr{I}_W \otimes \mathscr{I}_y) = 1,$$

and that the corresponding curve on $\mathbb{F}$ is smooth and irreducible.

Since $h^0(\mathbb{F}, \mathscr{O}_{\mathbb{F}}(C_0 + af)) = a + 2$, we must have $h^0(\mathbb{F}, \mathscr{O}_{\mathbb{F}}(C_0 + af) \otimes \mathscr{I}_W \otimes \mathscr{I}_y) \geq 1$.

Let $C_1 \in |C_0 + af|$ be a curve containing W and y. Observe first that C_1 is irreducible. Otherwise, since $C_1 \cdot f = 1$, there is a unique irreducible component C_1' of C_1 such that $C_1' \cdot f = 1$ and $C_1 \equiv C_1' + rf$ for some $r \geq 1$ (C_1' is a section of $\pi_{|\mathbb{F}}$). In particular, $C_1' \in |C_0 + bf|$ with $b < a$. By [17, Corollary V.2.18], we have $b = 0$ and $C_1' = C_0$. Thus, $C_1 = C_0 \cup f_1 \cup \cdots \cup f_a$ where the f_i's are possibly equal fibers of $\pi_{|\mathbb{F}}$.

Note that W is a subscheme of both $C_1 = C_0 \cup f_1 \cup \cdots \cup f_a$ and $\hat{G}_Y \cap \mathbb{F}$. Since $\hat{G}_Y \cap \mathbb{F}$ is disjoint from C_0, we must have

$$W = \{p_1, \ldots, p_a\} \subseteq (f_1 \cup \cdots \cup f_a) \cap \hat{G}_Y.$$

On the other hand, $\hat{G}_Y$ intersects transversally any fiber of $\pi_{|\mathbb{F}}$, thus we obtain $\{p_1, \ldots, p_a\} = \{f_1 \cap \hat{G}_Y, \ldots, f_a \cap \hat{G}_Y\}$, and up to renumbering we can assume that f_i is the fiber of $\pi_{|\mathbb{F}}$ containing p_i. This implies that $y \in C_1 \subseteq \pi^{-1}(A) \cup G_Y$, which contradicts our choice of y. Thus C_1 is irreducible, hence it is a section of $\pi_{|\mathbb{F}}$, and $C_1 \cong \mathbb{P}^1$.

To show that C_1 is unique, let $C_2 \in |C_0 + af|$ be another curve containing W and y. Then C_2 is irreducible, $C_1 \cdot C_2 = a$, and $\{p_1, \ldots, p_a, y\} \subseteq C_1 \cap C_2$, which implies that $C_1 = C_2$. This shows our claim.

We remark that

$$A_Y \cap C_1 = \hat{G}_{Y|C_1}. \tag{5.1}$$

Indeed, C_1 is not contained in $\hat{G}_Y$, because $y \notin \hat{G}_Y$. Moreover, $W \subseteq A_Y \cap C_1 \subseteq \hat{G}_Y \cap C_1$ and $\hat{G}_Y \cdot C_1 = a$, so that $W = A_Y \cap C_1 = \hat{G}_{Y|C_1}$.

We define $C \subset X$ to be the strict transform of $C_1 \subset Y$, so that C is a smooth rational curve through x. It is not difficult to see that $C \equiv C_G + a\hat{F}$, and hence $-K_X \cdot C = n$ (see Table 1).

Lemma 5.2. If $\ell \cap A$ is either reduced, or has a unique nonreduced point of multiplicity 2, then

$$T_{X|C} \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-2} \oplus \mathscr{O}_{\mathbb{P}^1},$$

hence C is a standard, free, smooth rational curve. □

Proof. Suppose first that $C \cap (E \cap \hat{E}) = \emptyset$. This implies that φ is smooth in a neighborhood of C, thus the map $T_{X|C} \to \varphi^* T_{\mathbb{P}^{n-1}}{}_{|C}$ is onto. Its kernel is a torsion-free sheaf of rank 1 on C, hence a locally free sheaf, of degree $-K_X \cdot C + \varphi^* K_{\mathbb{P}^{n-1}} \cdot C = 0$. Since $\varphi^* T_{\mathbb{P}^{n-1}}{}_{|C} \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-2}$, this implies the statement.

Suppose now that $C \cap (E \cap \hat{E}) \neq \emptyset$, and let x_1 be a point in this intersection. Set $z_1 := \varphi(x_1)$, and let $h \in \{1, 2\}$ (respectively, $k \in \{1, h\}$) be the multiplicity of z_1 in $\ell \cap A$ (respectively, in $W \subseteq \ell \cap A$ under the isomorphism $\hat{G}_Y \cong \mathbb{P}^{n-1}$).

Note that E and $\hat{E}$ are Cartier divisors on X, and they do not contain C in their support.

As σ is an isomorphism on C, we have $E_{|C} = C \cap E \cong C_1 \cap A_Y = W$. On the other hand,

$$(E + \hat{E})_{|C} = \varphi^*(A)_{|C} = \varphi^*_{|C}(A_{|\ell}),$$

therefore we have the following equalities of divisors on C:

$$C \cap \hat{E} = \hat{E}_{|C} = \varphi^*_{|C}(A_{|\ell} - W).$$

In particular, z_1 has multiplicity $h - k$ in $\varphi_*(C \cap \hat{E})$, and since $x_1 \in C \cap \hat{E}$, we have $h - k > 0$. We deduce that $h = 2$, $k = 1$, and z_1 is the unique nonreduced point in $\ell \cap A$.

As $W \cap (A_{|\ell} - W) = \{z_1\}$, we also deduce that C intersects $E \cap \hat{E}$ only in x_1. Moreover, since C is transverse to E (and to $\hat{E}$) in x_1, we have

$$C \cap E \cap \hat{E} = \{x_1\}.$$

Now let us consider the tangent morphism $d\varphi : T_X \to \varphi^* T_{\mathbb{P}^{n-1}}$, which is surjective outside $E \cap \hat{E}$, and has rank $n-2$ in every point $x \in E \cap \hat{E}$, with image $T_{A,\varphi(x)}$. Let us also consider the natural surjective morphism:

$$\xi : \varphi^* T_{\mathbb{P}^{n-1}} \longrightarrow \varphi^*(T_{\mathbb{P}^{n-1}}/T_A)_{|E\cap\hat{E}}.$$

Since E and $\hat{E}$ intersect transversally, a local computation shows that the image of $d\varphi$ is the subsheaf of $\varphi^* T_{\mathbb{P}^{n-1}}$ given by the kernel of ξ. In other words, we have an exact sequence:

$$T_X \xrightarrow{d\varphi} \varphi^* T_{\mathbb{P}^{n-1}} \xrightarrow{\xi} \varphi^*(T_{\mathbb{P}^{n-1}}/T_A)_{|E\cap\hat{E}} \longrightarrow 0.$$

Let us now restrict to the curve C. Then $\varphi^*(T_{\mathbb{P}^{n-1}}/T_A)_{|E\cap\hat{E}\cap C} = \mathbb{C}_{x_1}$ is a skyscraper sheaf, and we have an exact sequence:

$$T_{X|C} \xrightarrow{d\varphi_{|C}} \varphi^* T_{\mathbb{P}^{n-1}}{}_{|C} \xrightarrow{\xi_{|C}} \mathbb{C}_{x_1} \longrightarrow 0,$$

where $\xi_{|C}$ is just the evaluation map.

We have $\varphi^* T_{\mathbb{P}^{n-1}}{}_{|C} \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-2}$, and the factor $\mathscr{O}_{\mathbb{P}^1}(2) \cong \varphi^* T_{\ell|C} \subset \varphi^* T_{\mathbb{P}^{n-1}}{}_{|C}$ is contained in $\ker(\xi_{|C})$, because $T_{\ell,z_1} \subset T_{A,z_1}$. Therefore, we get an induced surjective morphism $\mathscr{O}_{\mathbb{P}^1}(1)^{n-2} \to \mathbb{C}_{x_1}$, whose kernel is $\mathscr{O}_{\mathbb{P}^1}(1)^{n-3} \oplus \mathscr{O}_{\mathbb{P}^1}$. This gives an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(2) \longrightarrow \ker(\xi_{|C}) \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1)^{n-3} \oplus \mathscr{O}_{\mathbb{P}^1} \longrightarrow 0,$$

which yields $\ker(\xi_{|C}) \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-3} \oplus \mathscr{O}_{\mathbb{P}^1}$.

Thus, $d\varphi_{|C}$ yields a surjective map $T_{X|C} \to \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-3} \oplus \mathscr{O}_{\mathbb{P}^1}$. The kernel of this morphism is a torsion-free sheaf of rank 1 on C, hence a locally free sheaf, of degree $-K_X \cdot C - (n-1) = 1$. Finally, the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1) \longrightarrow T_{X|C} \xrightarrow{d\varphi_{|C}} \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{n-3} \oplus \mathscr{O}_{\mathbb{P}^1} \longrightarrow 0$$

gives the statement. ■

Lemma 5.3. Let C' be an effective one-cycle in X such that $C' \equiv C$ and $C' \cap X_0 \neq \emptyset$.

Then C' is integral and is obtained as in Construction 5.1, for some choices of $\ell' \subset \mathbb{P}^{n-1}$, $x' \in X$, and $W' \subseteq \ell' \cap A$. In particular, C' is again a smooth, connected rational curve. □

Proof. Since $\varphi_*(C') \equiv \varphi_*(C)$, it follows that $\varphi(C')$ is a line ℓ' in $\mathbb{P}^{n-1}$, and $\ell' \not\subseteq A$ because $C' \not\subseteq \varphi^{-1}(A) = E \cup \hat{E}$. Moreover, there is a unique irreducible component C'' of C' that maps onto ℓ', and $C'' \to \ell'$ is a birational morphism.

Therefore, we have

$$C' = C'' + F_1 + \cdots + F_r + \hat{F}_1 + \cdots + \hat{F}_s + e_1 + \cdots + e_h,$$

where the F_i's are (possibly equal) fibers of $\varphi_{|E}$, the $\hat{F}_i$'s are fibers of $\varphi_{|\hat{E}}$, and the e_i's are fibers of φ over $\mathbb{P}^{n-1} \smallsetminus A$. Moreover $C' \equiv C_G + a\hat{F}$, and using Table 1 we obtain

$$0 = G \cdot C' = G \cdot C'' + s + h \quad \text{and} \quad 0 = \hat{G} \cdot C' = \hat{G} \cdot C'' + r + h.$$

Since C'' is irreducible and $G \cap \hat{G} = \emptyset$, the intersections $G \cdot C''$ and $\hat{G} \cdot C''$ cannot be both negative; this yields $h = 0$. Then C'' cannot be contained in $G \cup \hat{G}$, because C' is not contained in $E \cup \hat{E} \cup G \cup \hat{G}$. We conclude that $r = s = 0$, and $C' = C''$ is integral.

Set $\mathbb{F}' := \pi^{-1}(\ell') \subset Y$, and denote by $\pi_{|\mathbb{F}'}$ the restriction of π to $\mathbb{F}'$. We consider the curve $C_1' := \sigma(C') \subset \mathbb{F}'$.

Since $\pi_{|C_1'}$ has degree 1, we have $C_1' \equiv C_0' + rf'$, where $C_0' = G_Y \cap \mathbb{F}'$ is the minimal section of $\pi_{\ell'}$, f' is one of its fibers, and $r \in \mathbb{Z}$.

On the other hand, we have

$$-K_{\mathbb{F}'/\ell'} \cdot C_1' = -K_{Y/\mathbb{P}^{n-1}} \cdot C_1' = -K_{Y/\mathbb{P}^{n-1}} \cdot C_1 = -K_{\mathbb{F}/\ell} \cdot C_1 = a,$$

where $C_1 := \sigma(C)$. Therefore $r = a$, and $C_1' \in |C_0' + af'|$.

Finally, as $E \cdot C' = a$ and C_1' is smooth, C_1' intersects A_Y in a zero-dimensional subscheme W' of length a. This shows that C' is a curve obtained via Construction 5.1. ■

Proof of Proposition 1.7. If $a = 0$, the statement is clear.

Suppose that $a > 0$. By Theorem A.5, pairs (ℓ, W) where $\ell \subset \mathbb{P}^{n-1}$ is a line not contained in A, and $W \subseteq \ell \cap A$ is a subscheme of length a, vary in an irreducible family of dimension $2n - 4$. Using this, it is not difficult to show that varying (ℓ, W, x), Construction 5.1 yields an irreducible algebraic family of smooth, connected rational curves in X, of anticanonical degree n. More precisely, we get a locally closed irreducible subvariety V_0 of $\mathrm{RatCurves}^n(X)$ of dimension $2n - 3$, whose points correspond to curves C obtained as in 5.1.

By Lemma 5.2, a general curve C from V_0 is free and hence yields a smooth point in $\mathrm{RatCurves}^n(X)$. Moreover, $\dim_{[C]} \mathrm{RatCurves}^n(X) = -K_X \cdot [V] + n - 3 = 2n - 3$. This implies that the closure V of V_0 in $\mathrm{RatCurves}^n(X)$ is an irreducible component, and that V gives a dominating family of curves of anticanonical degree n.

On the other hand, Lemma 5.3 shows that V_x is proper for every $x \in X_0$, so that V is a locally unsplit family. ■

Remark 5.4. Let $\mathscr{H}$ be an ample line bundle on X. If $2 \leq a \leq d-2$, then V is not a dominating family of rational curves of minimal degree with respect to $\mathscr{H}$. Indeed, the family W of curves on X whose points correspond to smooth fibers of φ is a locally unsplit dominating family of rational curves, and $[W] \equiv F + \hat{F}$. Thus

$$\mathscr{H} \cdot [V] = \tfrac{1}{2}(\mathscr{H} \cdot C_G + \mathscr{H} \cdot C_{\hat{G}} + a\mathscr{H} \cdot \hat{F} + (d-a)\mathscr{H} \cdot F) > \mathscr{H} \cdot [W]. \qquad \square$$

Proof of Theorem 1.8. The first part of the statement follows from Proposition 4.7. Assume that $\rho_X = 3$. Then, again by Proposition 4.7, X is isomorphic to one of the varieties described in Example 1.6, and $[V] \equiv C_{\hat{G}} + (d-a)F$ (notation as in Example 1.6 and Proposition 4.7). By (3.1), this is the same as $[V] \equiv C_G + a\hat{F}$. This means that the curves of the family V are numerically equivalent to the curves obtained in Construction 5.1. Thus, Lemma 5.3 implies that the family V coincides with the family of curves constructed in the proof of Proposition 1.7. ■

Proof of Theorem 1.9. As before, we can assume that $a > 0$. Let $x \in X_0$ be a general point, and set $z := \varphi(x) \in \mathbb{P}^{n-1}$. Let $p_z : A \to \mathbb{P}^{n-2}$ be the morphism of degree d induced by the linear projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ from z, where we see $\mathbb{P}^{n-2}$ as the variety of lines ℓ through z in $\mathbb{P}^{n-1}$. Then the pairs (ℓ, W) such that $\ell \subset \mathbb{P}^{n-1}$ is a line through z, and $W \subset A \cap \ell$ is a zero-dimensional subscheme of length a, are parameterized by the relative Hilbert scheme $\mathcal{H}_z := \mathrm{Hilb}^{[a]}(A/\mathbb{P}^{n-2})$.

Since x is general, $\mathcal{H}_z$ is integral by Theorem A.1. Therefore, using [17, Corollary III.12.9], Construction 5.1 can be made relatively over $\mathcal{H}_z$.

One first constructs a subscheme $\mathcal{C}_1 \subset Y \times \mathcal{H}_z$, such that the fiber of $\mathcal{C}_1 \to \mathcal{H}_z$ over (ℓ, W) is the curve C_1. As $\mathcal{C}_1$ intersects $A_Y \times \mathcal{H}_z$ along the Cartier divisor $\hat{G}_Y \times \mathcal{H}_z$ (see (5.1)), the strict transform

$$\mathcal{C} \subset X \times \mathcal{H}_z$$

of $\mathcal{C}_1 \subset Y \times \mathcal{H}_z$ is isomorphic to $\mathcal{C}_1$. In particular, the fiber of $\xi : \mathcal{C} \to \mathcal{H}_z$ over (ℓ, W) is the curve C.

As C is a smooth rational curve through x, $\xi : \mathcal{C} \to \mathcal{H}_z$ is a $\mathbb{P}^1$-bundle with a section $s : \mathcal{H}_z \to \mathcal{C}$, given by $s(h) = (x, h)$.

In particular, $\mathcal{C}$ is the projectivization of a rank 2 vector bundle over $\mathcal{H}_z$, and it is locally trivial in the Zariski topology.

Fix a point $0 \in \mathbb{P}^1$. Let $H_0 \subseteq \mathcal{H}_z$ be an open subset such that $\xi^{-1}(H_0) \cong \mathbb{P}^1 \times H_0$. We can assume that the section $s_{|H_0}$, under this isomorphism, is identified with the constant section $\{0\} \times H_0$.

Therefore, we get a morphism over H_0:

$$\mathbb{P}^1 \times H_0 \to X \times H_0,$$

which is an embedding on $\mathbb{P}^1 \times \{h\}$, and sends $(0, h)$ to (x, h), for every $h \in H_0$. This yields a morphism $H_0 \to \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$.

Since x is general, every curve in V_x is free, and $\mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ contains a union of smooth irreducible components $\hat{V}_x$, whose image in $\mathrm{RatCurves}^n(X, x)$ is V_x. By construction, the morphism $H_0 \to \mathrm{Hom}(\mathbb{P}^1, X, 0 \mapsto x)$ takes values in $\hat{V}_x$.

By [27, Theorem II.2.16], these morphisms glue together and yield a morphism

$$\Psi : \mathcal{H}_z \longrightarrow V_x.$$

By Lemma 5.3, the morphism Ψ is surjective. Moreover, the pair (ℓ, W) determines uniquely the curve C in Construction 5.1. As C is smooth, it corresponds to a unique point in V_x; therefore, Ψ is injective.

Finally, V_x being smooth, we conclude that Ψ is an isomorphism, $\mathcal{H}_z$ is smooth, and V_x is irreducible. ■

Note that a and $a' = d - a$ yield not only the same variety X (see Remark 3.7), but also the same family V of rational curves on X, and $V_x \cong \mathrm{Hilb}^{[a]}(A/\mathbb{P}^{n-2}) \cong \mathrm{Hilb}^{[d-a]}(A/\mathbb{P}^{n-2})$ for general $x \in X$.

The cases $a \in \{0, d\}$ and $a \in \{1, d-1\}$ are the simplest ones. For the reader's convenience, we describe explicitly the latter.

Example 5.5. If $a = 1$, then Y is the blow-up of $\mathbb{P}^n$ at a point. Thus, X is the blow-up of $\mathbb{P}^n$ along $\{p_0\} \cup A$, where A is smooth, of dimension $n-2$, degree d, contained in a hyperplane H, and $p_0 \notin H$. This is one of the few examples of Fano manifolds obtained by blowing-up a point in another manifold, see [5].

The general curve of the family V is the strict transform in X of a line in $\mathbb{P}^n$ intersecting A in one point.

Therefore, for a general point $x \in X$, we have $V_x \cong A \cong \mathrm{Hilb}^{[1]}(A/\mathbb{P}^{n-2})$.

From the point of view of the family of curves, this is essentially the same example as Example 1.3; see also [19, Example 1.7]. □

Let us consider now the morphism

$$\tau_x : V_x \longrightarrow \mathcal{C}_x \subset \mathbb{P}(T^*_{X,x}),$$

where x is a general point (notation as in the Introduction). We recall the following useful observation.

Remark 5.6. Let $[C] \in V_x$. Then τ_x is an immersion at $[C]$ if and only if C is standard, see [18, Proposition 1.4] and [2, Proposition 2.7]. □

As x is general, $\varphi : X \to \mathbb{P}^{n-1}$ is smooth at x, and the differential of φ at x induces the linear projection $\mathbb{P}(T^*_{X,x}) \dashrightarrow \mathbb{P}(T^*_{\mathbb{P}^{n-1},z})$ from the point $[T_{X/\mathbb{P}^{n-1},x}] \in \mathbb{P}(T^*_{X,x})$.

We have $[T_{X/\mathbb{P}^{n-1},x}] \notin \mathcal{C}_x$, because $\varphi_{|C}$ is an isomorphism for every C in V_x, and the projection restricts to a morphism $\Pi' : \mathcal{C}_x \to \mathbb{P}^{n-2} = \mathbb{P}(T^*_{\mathbb{P}^{n-1},z})$.

Observe that there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{H}_z \arrow[r, "\Psi"'] \arrow[rr, bend left, "\Phi"] \arrow[dr, "\Pi"'] & V_x \arrow[r, "\tau_x"'] & \mathcal{C}_x \arrow[dl, "\Pi'"] \\
& \mathbb{P}^{n-2} &
\end{tikzcd}$$

where we have set $\Phi := \tau_x \circ \Psi : \mathcal{H}_z \to \mathcal{C}_x$.

Proof of Theorem 1.10. Since Π has degree $\binom{d}{a}$, Ψ is an isomorphism, and τ_x is birational, we conclude that Π' has degree $\binom{d}{a}$, therefore $\mathcal{C}_x \subset \mathbb{P}^{n-1}$ is a hypersurface of degree $\binom{d}{a}$. Moreover, $\mathcal{C}_x$ is irreducible, because V_x is.

We have seen in the above example that τ_x is an isomorphism for $a=1$ and $a=d-1$, and the statement is clear if $a=0$ or $a=d$.

We suppose from now on that $2 \le a \le d-2$; note that in particular $d \ge 4$. Since Ψ is an isomorphism, the statement follows if we show that the closed subset where Φ is not an isomorphism (respectively, an immersion) has codimension 1 (respectively, 2).

Let $L \subset \mathbb{P}^{n-2}$ be a general line. Then $A_L := p_z^{-1}(L)$ is a smooth plane curve of degree d, and $\Pi^{-1}(L) = \mathrm{Hilb}^{[a]}(A_L/L)$ is a smooth curve of genus $1 + \frac{1}{2}\binom{d}{a}(a(d-a)-2)$ by Theorem A.1(4).

On the other hand, $(\Pi')^{-1}(L)$ is a plane curve of degree $\binom{d}{a}$, hence it has arithmetic genus $p_a = \frac{1}{2}(\binom{d}{a} - 1)(\binom{d}{a} - 2)$.

We claim that $\Phi_{|\Pi^{-1}(L)}$ cannot be an isomorphism onto its image. By contradiction, if it were, the two curves should have the same genus, so we obtain

$$\frac{\left(\binom{d}{a} - 1\right)\left(\binom{d}{a} - 2\right)}{2} = 1 + \frac{1}{2}\binom{d}{a}(a(d-a)-2).$$

This yields easily $\binom{d}{a} = a(d-a)+1$, or equivalently $d(d-1)\cdots(d-a+1) = a!(a(d-a)+1)$. Since $a \leq d-2$, we must have

$$(d-2)\cdots(d-a+1) \geq \frac{a!}{2},$$

and hence

$$d(d-1) = \frac{a!(a(d-a)+1)}{(d-2)\cdots(d-a+1)} \leq 2(a(d-a)+1).$$

Equivalently, we obtain $d^2-(2a+1)d+2a^2-2 \leq 0$. It is easy to see that this contradicts the assumption $2 \leq a \leq d-2$.

We conclude that the closed subset where Φ is not an isomorphism has codimension 1.

Let us consider again the plane curve A_L. Since $A_L \to L$ is a general projection, every nonreduced fiber contains just one double point. Hence, for every $[\ell] \in L$, the intersection $\ell \cap A$ is either reduced, or has a unique nonreduced point, of multiplicity 2. By Lemma 5.2, for every $W \subseteq \ell \cap A$, the corresponding curve $C \subset X$ is standard, therefore τ_x is an immersion at $\Psi([W])$ by Remark 5.6.

We have shown that Φ is an immersion in every point of $\Pi^{-1}(L)$, hence the closed subset where Φ is not an immersion has codimension at least 2.

Suppose now that $n \geq 4$, and let $P \subset \mathbb{P}^{n-2}$ be a general plane. Then $A_P := p_z^{-1}(P)$ is a smooth surface of degree d in $\mathbb{P}^3$, and $A_P \to P$ is the projection from a general point. Let $B \subset \mathbb{P}^2$ be the branch curve of this projection. It is classically known that B has only nodes and cusps, see [9]. Moreover, the fiber of $A_P \to P$ over a smooth point $z \in B$ contains just one double point, the fiber over a node contains two double points, and the fiber over a cusp contains one triple point (see [9, Proposition 3.7]). The number of nodes in B depends only on the degree d of A_P, and more precisely it is $d(d-1)(d-2)(d-3)/2$, see [13, Lemma 3.2(a)].

Since $d \geq 4$, B contains nodes, and we conclude that there is at least one $[\ell] \in P$ such that $A \cap \ell = 2p_1 + 2p_2 + p_3 + \cdots + p_{d-2}$. As $a \leq d-2$, we can consider the subscheme $W := p_1 + p_2 + \cdots + p_a$ and the point $[W] \in \mathcal{H}_z$. Note that the points p_1 and p_2 do appear in W, because $a \geq 2$. Then by Theorem A.1(2), the tangent space of the fiber $\Pi^{-1}([\ell])$ at $[W]$ has dimension 2.

On the other hand, as Π' is a linear projection from a point, the tangent space of the fiber $(\Pi')^{-1}([\ell])$ at $\Phi([W])$ has dimension at most 1. This shows that Φ is not an immersion at $[W]$, and hence that the closed subset where Φ is not an immersion has codimension 2. ∎

Corollary 5.7. Let X and V be as in Proposition 1.7. Assume that $2 \leq a \leq d-2$, and that $n \geq 4$. Then nonstandard curves of the family V cover X. □

Proof. This follows from Remark 5.6 and Theorem 1.10. ■

Example 5.8 (Fano 3-folds)**.** Let X and V be as in Proposition 1.7, and consider the case $n=3$. Then X is Fano if and only if $a \leq 2$ and $d-a \leq 2$, in particular $d \leq 4$. Thus, the only case where X is Fano and τ_x is not an isomorphism is for $d=4$ and $a=2$. This Fano 3-fold X is N. 9 in [23, §12.4].

In this case, $A \subset \mathbb{P}^2$ is a smooth quartic, $V_x \cong \mathcal{H}_z = \mathrm{Hilb}^{[2]}(A/\mathbb{P}^1)$ is a smooth connected curve of genus 7, and $\Pi : \mathcal{H}_z \to \mathbb{P}^1$ has degree 6. Using Theorem A.1(2), one can describe precisely the ramification of Π.

On the other hand, $\mathcal{C}_x \subset \mathbb{P}^2$ is an irreducible curve of degree 6 and arithmetic genus 10. The normalization $\mathcal{H}_z \to \mathcal{C}_x$ is an immersion, but it is not injective. □

Appendix: The Relative Hilbert Scheme

The proof of Theorem 1.9 relies on the following results of independent interest.

Theorem A.1. Fix integers m, a, and d, such that $m \geq 1$ and $1 \leq a \leq d$. Let $A \subset \mathbb{P}^{m+1}$ be a smooth hypersurface of degree d, $z \in \mathbb{P}^{m+1} \smallsetminus A$ a general point, and $p_z : A \to \mathbb{P}^m$ the linear projection from z (where we identify $\mathbb{P}^m$ with the variety of lines through z in $\mathbb{P}^{m+1}$).

(1) The relative Hilbert scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is an integral local complete intersection scheme of dimension m, and the natural morphism

$$\Pi : \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m) \to \mathbb{P}^m$$

is flat and finite of degree $\binom{d}{a}$.

(2) Let $[\ell] \in \mathbb{P}^m$ and $[W] \in \Pi^{-1}([\ell])$. Write $\ell \cap A = h_1 p_1 + \cdots + h_r p_r$ with $h_i \geq 1$ and $p_i \neq p_j$ for $i \neq j$, and $W = k_1 p_1 + \cdots + k_r p_r$ with $0 \leq k_i \leq h_i$. The Zariski tangent space of the fiber $\Pi^{-1}([\ell])$ at $[W]$ has dimension $\sum_{i=1}^{r} \min(k_i, h_i - k_i)$.

(3) Π is smooth at $[W]$ if and only if W is a union of irreducible components of $\ell \cap A$, equivalently: $W \cap (\ell \cap A - W) = \emptyset$.

(4) Suppose that $m=1$. Then the curve $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ is smooth of genus $g = 1 + \frac{1}{2}\binom{d}{a}(a(d-a)-2)$. □

The following results will be used in the proof of Theorem A.1.

Lemma A.2. Let h be a positive integer, and set:

$$\Lambda := \frac{\mathbb{C}[t]}{(t^h)} \quad \text{and} \quad F := \operatorname{Spec}\Lambda.$$

Let W be the nonempty closed subscheme of F with ideal $I := t^k\Lambda \subseteq \Lambda$, where $k \in \{1, \dots, h\}$ is an integer. Then:

(1) $\dim_{\mathbb{C}} \operatorname{Hom}_F(\mathscr{I}_W, \mathscr{O}_W) = \dim_{\mathbb{C}} \operatorname{Ext}^1_F(\mathscr{I}_W, \mathscr{O}_W) = \min(k, h-k)$;

(2) $\operatorname{Hilb}(F)$ has dimension zero, $\operatorname{Obs}(W) = \operatorname{Ext}^1_F(\mathscr{I}_W, \mathscr{O}_W)$, and the Zariski tangent space of $\operatorname{Hilb}(F)$ at $[W]$ has dimension $\min(k, h-k)$;

(3) $\operatorname{Hilb}(F)$ is smooth at $[W]$ if and only if $W = F$. □

Proof of Lemma A.2. We have a short sequence of Λ-modules:

$$0 \longrightarrow t^{h-k}\Lambda \longrightarrow \Lambda \longrightarrow t^k\Lambda = I \longrightarrow 0,$$

where the second morphism is given by $1 \mapsto t^k$, so that $I \cong \Lambda/t^{h-k}\Lambda$ as Λ-modules. Using this isomorphism, a direct computation shows that $\dim_{\mathbb{C}} \operatorname{Hom}_\Lambda(I, \Lambda/I) = \min(k, h-k)$.

Applying the functor $\operatorname{Hom}_\Lambda(-, \Lambda/I)$ to the above sequence, and using the vanishing of $\operatorname{Ext}^1_\Lambda(\Lambda, \Lambda/I)$, we get the exact sequence of Λ-modules:

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_\Lambda(I, \Lambda/I) &\longrightarrow \operatorname{Hom}_\Lambda(\Lambda, \Lambda/I) \\ &\longrightarrow \operatorname{Hom}_\Lambda(t^{h-k}\Lambda, \Lambda/I) \longrightarrow \operatorname{Ext}^1_\Lambda(I, \Lambda/I) \longrightarrow 0. \end{aligned}$$

Similarly as before, observe that $t^{h-k}\Lambda \cong \Lambda/I$ as Λ-modules. Moreover, if $\pi : \Lambda \to \Lambda/I$ is the quotient map, it is easy to see that $\pi^* : \operatorname{Hom}_\Lambda(\Lambda/I, \Lambda/I) \to \operatorname{Hom}_\Lambda(\Lambda, \Lambda/I)$ is an isomorphism of Λ-modules. Therefore

$$\operatorname{Hom}_\Lambda(t^{h-k}\Lambda, \Lambda/I) \cong \operatorname{Hom}_\Lambda(\Lambda/I, \Lambda/I) \cong \operatorname{Hom}_\Lambda(\Lambda, \Lambda/I),$$

and we obtain (1):

$$\dim_{\mathbb{C}} \operatorname{Hom}_\Lambda(I, \Lambda/I) = \dim_{\mathbb{C}} \operatorname{Ext}^1_\Lambda(I, \Lambda/I) = \min(k, h-k).$$

The Hilbert scheme of F is supported on finitely many points, thus it has dimension zero. Recall that by [27, Definition I.2.6], the obstruction space $\operatorname{Obs}(W)$ is a subspace of $\operatorname{Ext}^1_F(\mathscr{I}_W, \mathscr{O}_W) \cong \operatorname{Ext}^1_\Lambda(I, \Lambda/I)$. Then (2) follows from (1) and [27, Theorems I.2.8.1 and I.2.8.4].

Finally, (3) follows from (2). ■

Lemma A.3. Let $p\colon A \to T$ be a finite morphism between smooth quasi-projective varieties. Let F be a fiber of p, and $W \subseteq F$ a reduced subscheme of length a. Let $\Pi\colon \mathrm{Hilb}^{[a]}(A/T)_{red} \to T$ be the natural morphism. Suppose that F has a unique nonreduced point at z_1, and that $z_1 \in \mathrm{Supp}(W)$.

Then the Hilbert scheme $\mathrm{Hilb}^{[a]}(A/T)_{\mathrm{red}}$ is smooth at $[W]$, and there exists a neighborhood for the Euclidean topology $U \subset \mathrm{Hilb}^{[a]}(A/T)_{\mathrm{red}}$ (respectively, $U_1 \subset A$) of $[W]$ (respectively, z_1) and an isomorphism $\iota\colon U_1 \cong U$ (of complex manifolds) such that the diagram:

$$\begin{tikzcd} U_1 \arrow[rr, "\iota"] \arrow[dr, "p_{|U_1}"'] & & U \arrow[dl, "\Pi_{|U}"] \\ & T & \end{tikzcd}$$

commutes. □

Proof. Recall that the Hilbert–Chow morphism $\mathrm{Hilb}^{[a]}(A) \to A^{(a)} := A^a/\mathbb{S}_a$ maps a zero-dimensional subscheme of length a of A to the associated effective 0-cycle of degree a. Note that $[W]$ is contained in the open subset of $\mathrm{Hilb}^{[a]}(A)$ where the Hilbert–Chow morphism is an isomorphism, and that $\mathrm{Hilb}^{[a]}(A)$ is smooth at $[W]$ since $[W]$ is reduced.

Let $V \subset T$ be an open neighborhood of $p(z_1)$ for the Euclidean topology such that $p^{-1}(V) = U_1 \cup \cdots \cup U_r$, $U_i \cap U_j = \emptyset$ if $i \neq j$, $U_1 \subset A$ is an open neighborhood of z_1, $p_{|U_i}\colon U_i \to V$ is an isomorphism (of complex manifolds) for $i \geq 2$, and $W \cap U_i \neq \emptyset$ if and only if $1 \leq i \leq a$.

Let us consider the map $f\colon U_1 \to A^a$ given by

$$f(z) = (z, (p_{|U_2})^{-1}(p(z)), \ldots, (p_{|U_a})^{-1}(p(z))).$$

Then f is a holomorphic immersion. Moreover, for every $z \in U_1$, the points $z, (p_{|U_2})^{-1}(p(z)), \ldots, (p_{|U_a})^{-1}(p(z)) \in A$ are pairwise distinct, so that the composition of f with the quotient map $A^a \to A^{(a)}$ is still an immersion. This yields a holomorphic immersion

$$\iota\colon U_1 \longrightarrow \mathrm{Hilb}^{[a]}(A),$$

such that $U := \iota(U_1) \subset \mathrm{Hilb}^{[a]}(A/T)_{red}$. Moreover, ι is injective, $\iota(z_1) = [W]$, and $p_{|U_1} = \Pi_{|U} \circ \iota$. Our claim follows easily. ■

Proof of Theorem A.1. Let $\ell \subset \mathbb{P}^{m+1}$ be a line passing through z, and set $F := \ell \cap A$, so that F is a zero-dimensional subscheme of $\ell \setminus \{z\} \cong \mathbb{A}^1$. We have

$$\Pi^{-1}([\ell]) = \mathrm{Hilb}^{[a]}(F);$$

in particular, Π is a finite morphism, and $\dim \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m) \le m$.

Let $[W] \in \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ be a point over $[\ell] \in \mathbb{P}^m$. Applying Lemma A.2 to every connected component of $\Pi^{-1}([\ell])$, we get (2) and (3), and also that $\dim_{\mathbb{C}} \mathrm{Hom}_F(\mathscr{I}_W, \mathscr{O}_W) = \dim_{\mathbb{C}} \mathrm{Obs}(W)$. Thus, by [27, Theorems I.2.10.3 and I.2.10.4], any irreducible component of $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ through $[W]$ has dimension m, and Π is a local complete intersection morphism. In particular, $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is a local complete intersection scheme, and Π is a flat finite morphism.

By (3), Π is étale over $[\ell]$ if and only if $\ell \cap A$ is reduced, that is, p_z is étale over $[\ell]$. Therefore, $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is generically smooth over $\mathbb{P}^m$. In particular, $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is generically reduced and hence reduced as it is a Cohen–Macaulay scheme.

We proceed to show that the scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is irreducible. This follows from the fact that since z is general, the monodromy group of the projection $p_z : A \to \mathbb{P}^m$ is the whole symmetric group $\mathbb{S}_d$, see [10, Proposition 2.3].

Let $U \subset \mathbb{P}^m$ be a dense open subset such that p_z and Π are étale over U. Set $A_0 := p_z^{-1}(U) \subseteq A$ and $H_0 := \Pi^{-1}(U) = \mathrm{Hilb}^{[a]}(A_0/U)$. Since H_0 is dense in $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$, and H_0 is smooth, we are reduced to show that H_0 is connected.

Note that H_0 is a closed subscheme of $\mathrm{Hilb}^{[a]}(A_0)$, and that every $[W]$ in H_0 is a reduced subscheme of A_0, so that H_0 is contained in the open subset of $\mathrm{Hilb}^{[a]}(A_0)$ where the Hilbert–Chow morphism $\mathrm{Hilb}^{[a]}(A_0) \to (A_0)^{(a)}$ is an isomorphism.

Let $[W_1]$ and $[W_2]$ be two points in H_0 that map to a given point in U. Since any irreducible component of H_0 maps onto U, it is enough to prove that there is a path in H_0 joining $[W_1]$ to $[W_2]$.

Let $[\hat{W}_i] \in (A_0)^a$ mapping to $[W_i]$. Here, we are considering the natural map $(A_0)^a \to (A_0)^{(a)}$, composed with the inverse of the Hilbert–Chow morphism. Since the monodromy group of p_z is the symmetric group $\mathbb{S}_d$, there is a path $\gamma : [0, 1] \to (A_0)^a$ joining $[\hat{W}_1]$ to $[\hat{W}_2]$, such that the points in $\gamma(t)$ are distinct and contained in a fiber of p_z, for every $t \in [0, 1]$. This yields a path joining $[W_1]$ to $[W_2]$ in H_0, and proves that $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is integral.

Finally, suppose that $m=1$. We show that $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ is a smooth curve of genus

$$g=1+\frac{1}{2}\binom{d}{a}(a(d-a)-2).$$

Note that since the projection $p_z: A \to \mathbb{P}^1$ is general, there are precisely $d(d-1)$ nonreduced fibers, and every nonreduced fiber contains just one nonreduced point, with multiplicity 2.

Let $[W] \in \mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ be a point over $[\ell] \in \mathbb{P}^1$. By (3), either Π is étale at $[W]$, or $\ell \cap A$ has a double point, W is reduced, and contains this point in its support. Note that there are exactly $\binom{d-2}{a-1}$ such $[W]$'s, and that Π has ramification index 2 at any of these points by Lemma A.3.

If Π is étale at $[W]$, then $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ is obviously smooth at $[W]$. Otherwise, $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ is smooth at $[W]$ by Lemma A.3. This shows that $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^1)$ is a smooth curve.

By the Hurwitz formula, we have

$$2g-2=-2\binom{d}{a}+d(d-1)\binom{d-2}{a-1}=\binom{d}{a}(a(d-a)-2).$$

This completes the proof of the theorem. ■

Remark A.4. The same proof shows that for *every* $z \in \mathbb{P}^{m+1} \smallsetminus A$, the relative Hilbert scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ is a reduced local complete intersection scheme of dimension m, and that the natural morphism $\Pi: \mathrm{Hilb}^{[a]}(A/\mathbb{P}^m) \to \mathbb{P}^m$ is flat and finite of degree $\binom{d}{a}$. Moreover, (2) and (3) hold true. □

We need also a slightly more general version of the previous construction, as follows. Let $A \subset \mathbb{P}^{m+1}$ be a smooth hypersurface of degree $d \geq 1$, and fix $a \in \{1, \ldots, d\}$. Set

$$\mathcal{G} := \{[\ell] \in G(1, m+1) \mid \ell \text{ is not contained in } A\},$$

with its universal family $\mathcal{U} := \{([\ell], z) \in \mathcal{G} \times \mathbb{P}^{m+1} \mid z \in \ell\}$. Let us consider the intersection:

$$\mathcal{I} := \mathcal{U} \cap (\mathcal{G} \times A).$$

The induced morphism $\mathcal{I} \to \mathcal{G}$ is finite and flat, of degree d; the fiber over a line $[\ell]$ is $\ell \cap A$. The relative Hilbert scheme $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G})$ parameterizes pairs (ℓ, W), where $\ell \subset \mathbb{P}^{m+1}$ is a line not contained in A, and W is a subscheme of length a of $\ell \cap A$.

Theorem A.5. The Hilbert scheme $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G})$ is an integral scheme of dimension $2m$. □

Proof. The proof is very similar to that of Theorem A.1, and so we leave some details to the reader. First, one shows that $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G})$ is a reduced local complete intersection scheme of dimension $2m$ equipped with a finite flat morphism $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G}) \to \mathcal{G}$.

We remark that if $z \in \mathbb{P}^{m+1}$ is a general point, and $P := \{[\ell] \in \mathcal{G} \mid z \in \ell\}$, then $P \cong \mathbb{P}^m$, and the inverse image of P in $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G})$ is the relative Hilbert scheme $\mathrm{Hilb}^{[a]}(A/\mathbb{P}^m)$ of the projection $p_z : A \to \mathbb{P}^m$ from z. Then, the same argument used in the proof of Theorem A.1 shows that $\mathrm{Hilb}^{[a]}(\mathcal{I}/\mathcal{G})$ is irreducible. This completes the proof of the theorem. ■

Proof of Theorem 1.11. Let X be as in Example 3.1, with $n = m + 2$. Then the statement follows from Proposition 1.7, Theorem 1.8, and Theorem A.1. ■

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