ADDENDUM

Niklas Müller pointed out that the following converse to [DB22, Corollary 1.7] holds.

Proposition. *Let (X, D) be a log smooth reduced pair with X projective. Suppose that there exists a smooth morphism with connected fibers $a\colon X \to A$ onto a torus quotient A. Suppose furthermore that the fibration $(X, D) \to T$ is locally trivial for the analytic topology with typical fiber F being a toric variety with boundary divisor $D_{|F}$. Then $T_X(-\log D)$ is numerically flat.*

Proof. There is an exact sequence

$$0 \to T_{X/A}(-\log D) \to T_X(-\log D) \to a^*T_A \to 0.$$

In particular, it suffices to show that $T_{X/A}(-\log D)$ is numerically flat. Since $T_F(-\log D|_F) \cong \mathscr{O}_F^{\oplus \dim F}$, there exists a locally free sheaf $\mathscr{G}$ on A such that $T_{X/A}(-\log D) \cong a^*\mathscr{G}$. Now, the group $\mathrm{Aut}^0(F, D_{|F})$ is an algebraic torus and hence a commutative algebraic group. Together with the fact that the fibration $(X, D) \to T$ is locally trivial for the analytic topology, this implies that $\mathscr{G}$ is a locally free sheaf associated to a local system of $\mathbb{C}$-vector spaces. As a consequence, $\mathscr{G}$ is equipped with a connection. By [M$\ddot{2}$2, Theorem 0.2], $\mathscr{G}$ is then numerically flat. The claim follows easily. □

References

[DB22] Stéphane Druel and Federico Lo Bianco, *Numerical characterization of some toric fiber bundles*, Math. Z. **300** (2022), no. 4, 3357–3382.

[M$\ddot{2}$2] Niklas Müller, *Locally constant fibrations and positivity of curvature*, Preprint `arXiv:2212.11530`, 2022.