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Codimension One Foliations with Numerically Trivial Canonical Class on Singular Spaces II

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In this article, we give the structure of codimension one foliations with canonical singularities and numerically trivial canonical class on varieties with klt singularities. Building on recent works of Spicer, Cascini—Spicer and Spicer—Svaldi, we then describe the birational geometry of rank two foliations with canonical singularities and canonical class of numerical dimension zero on complex projective three-folds.

1 Introduction

In the last few decades, much progress has been made in the classification of complex projective (singular) varieties. The general viewpoint is that complex projective varieties should be classified according to the behavior of their canonical class. Similar ideas have been successfully applied to the study of global properties of holomorphic foliations. This led, for instance, to the birational classification of foliations by curves on surfaces ([5], [28]), generalizing most of the important results of the Enriques–Kodaira classification. However, it is well known that the abundance conjecture fails already in ambiant dimension two. In very recent works, a foliated analogue of the minimal model

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program is established for rank two foliations on projective three-folds ([32], [11] and [33]).

The foliated analogue of the minimal model program aims in particular to reduce the birational study of mildly singular foliations with numerical dimension zero on complex projective manifolds to the study of associated minimal models, that is, mildly singular foliations with numerically trivial canonical class on klt spaces. In [27], motivated by these developments, the authors describe the structure of codimension one foliations with canonical singularities (we refer to Section 2 for this notion) and numerically trivial canonical class on complex projective manifolds. This result was extended by the first-named author to the setting of projective varieties with canonical singularities in [12]. However, from the point of view of birational classification of foliations, this class of singularities is inadequate. The main result of this paper settles this problem in full generality.

Theorem 1.1. Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension one foliation on X with canonical singularities. Suppose furthermore that $K_{\mathscr{G}} \equiv 0$. Then one of the following holds.

(1) There exist a smooth complete curve C, a complex projective variety Y with canonical singularities, and $K_Y \sim_{\mathbb{Z}} 0$, as well as a quasi-étale cover $f\colon Y \times C \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times C \to C$.

(2) There exist complex projective varieties Y and Z with canonical singularities, as well as a quasi-étale cover $f\colon Y \times Z \to X$ and a foliation $\mathscr{H} \cong \mathscr{O}_Z^{\dim Z-1}$ on Z such that $f^{-1}\mathscr{G}$ is the pull-back of $\mathscr{H}$ via the projection $Y \times Z \to Z$. In addition, we have $K_Y \sim_{\mathbb{Z}} 0, Z$ is an equivariant compactification of a commutative algebraic group of dimension at least 2, and $\mathscr{H}$ is induced by a codimension one Lie subgroup.

As an immediate consequence, we prove the abundance conjecture in this setting.

Corollary 1.2. Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension one foliation on X with canonical singularities. If $K_{\mathscr{G}} \equiv 0$, then $K_{\mathscr{G}}$ is torsion.

Together with the foliated analogue of the minimal program for rank two foliations with F-dlt singularities on normal projective three-folds ([32], [11], and [33]), we obtain the following result.

Corollary 1.3. Let X be a normal complex projective three-fold, and let $\mathscr{G}$ be a codimension one foliation on X with canonical singularities. Suppose furthermore that $\nu(K_{\mathscr{G}}) = 0$. Then one of the following holds.

(1) There exist a smooth complete curve C, a complex projective variety Y with canonical singularities and $K_Y \sim_{\mathbb{Z}} 0$, as well as a generically finite rational map $f\colon Y \times C \dashrightarrow X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times C \to C$.

(2) There exist a smooth complete curve C of genus one, a complex projective surface Z with canonical singularities, as well as a generically finite rational map $f\colon C \times Z \dashrightarrow X$ and a foliation $\mathscr{H} \cong \mathscr{O}_Z$ on Z such that $f^{-1}\mathscr{G}$ is the pull-back of $\mathscr{H}$ via the projection $C \times Z \to Z$. In addition, Z is an equivariant compactification of a commutative algebraic group and $\mathscr{H}$ is induced by a one-dimensional Lie subgroup.

This paper is a sequel to the article [12] by the first-named author and follows the same general strategy. The main new (crucial) ingredients are Propositions 4.1 and 6.1. In order to prove these results, one needs to extend the Baum–Bott formula and the Camacho–Sad formula to surfaces with klt singularities.

Structure of the paper

In section 2, we recall the definitions and basic properties of foliations. We also establish some Bertini-type results. In section 3, we extend a number of earlier results to the context of quasi-projective varieties with quotient singularities. In particular, we extend the Baum–Bott formula as well as the Camacho–Sad formula to this context. Section 4 prepares for the proof of our main result. We confirm the Ekedahl–Shepherd–Barron–Taylor conjecture for mildly singular codimension one foliations with trivial canonical class on projective varieties X with $\nu(X) = -\infty$ in Section 5, and then on those with $\nu(X) = 1$ in Section 6. In section 7, we address codimension one foliations with numerically trivial canonical class defined by closed twisted rational 1-forms. Section 8 is devoted to the proof of Theorem 1.1 and Corollary 1.3.

Global conventions

Throughout the paper, we work over the complex number field.

Given a variety X, we denote by X_{reg} its smooth locus.

We will use the notions of terminal, canonical, klt, and lc singularities for pairs without further explanation or comment and simply refer to [22, Section 2.3] for a discussion and for their precise definitions.

Given a normal variety $X, m \in \mathbb{N}_{>0}$, and coherent sheaves $\mathscr{E}$ and $\mathscr{G}$ on X, write $\mathscr{E}^{[m]} := (\mathscr{E}^{\otimes m})^{**}$, $\det \mathscr{E} := (\Lambda^{\operatorname{rank}\mathscr{E}}\mathscr{E})^{**}$, and $\mathscr{E} \boxtimes \mathscr{G} := (\mathscr{E} \otimes \mathscr{G})^{**}$. Given any morphism $f\colon Y \to X$, write $f^{[*]}\mathscr{E} := (f^*\mathscr{E})^{**}$.

2 Foliations

In this section, we have gathered a number of results and facts concerning foliations that will later be used in the proofs.

Definition 2.1. A *foliation* on a normal variety X is a coherent subsheaf $\mathscr{G} \subseteq T_X$ such that

(1) $\mathscr{G}$ is closed under the Lie bracket, and
(2) $\mathscr{G}$ is saturated in T_X. In other words, the quotient $T_X/\mathscr{G}$ is torsion-free.

The *rank* r of $\mathscr{G}$ is the generic rank of $\mathscr{G}$. The *codimension* of $\mathscr{G}$ is defined as $q := \dim X - r$.

The *canonical class* $K_{\mathscr{G}}$ of $\mathscr{G}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{G}}) \cong \det \mathscr{G}$.

Let $X^\circ \subseteq X_{\mathrm{reg}}$ be the open set where $\mathscr{G}_{|X_{\mathrm{reg}}}$ is a subbundle of $T_{X_{\mathrm{reg}}}$. The *singular locus* of $\mathscr{G}$ is defined to be $X \setminus X^\circ$. A *leaf* of $\mathscr{G}$ is a maximal connected and immersed holomorphic submanifold $L \subset X^\circ$ such that $T_L = \mathscr{G}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure.

The foliation $\mathscr{G}$ is said to be *algebraically integrable* if its leaves are algebraic.

2.1 Foliations defined by q-forms

Let $\mathscr{G}$ be a codimension q foliation on an n-dimensional normal variety X. The *normal sheaf* of $\mathscr{G}$ is $\mathscr{N} := (T_X/\mathscr{G})^{**}$. The q-th wedge product of the inclusion $\mathscr{N}^* \hookrightarrow \Omega_X^{[1]}$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega_X^q \boxtimes \det \mathscr{N})$ whose zero locus has codimension at least two in X. Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{G}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X, and let $\omega \in H^0(X, \Omega_X^q \boxtimes \mathscr{L})$ be a global section whose zero locus has codimension at least two in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1} \boxtimes \mathscr{L}$ given by the contraction with ω defines a foliation of codimension q on X. These constructions are inverse of each other.

2.2 Foliations described as pull-backs

Let X and Y be normal varieties, and let $\varphi\colon X \dashrightarrow Y$ be a dominant rational map that restricts to a morphism $\varphi^\circ\colon X^\circ \to Y^\circ$, where $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$ are smooth open subsets.

Let $\mathscr{G}$ be a codimension q foliation on Y. Suppose that the restriction $\mathscr{G}^\circ$ of $\mathscr{G}$ to Y° is defined by a twisted q-form $\omega_{Y^\circ} \in H^0(Y^\circ, \Omega^q_{Y^\circ} \otimes \det \mathscr{N}_{\mathscr{G}^\circ})$. Then, ω_{Y° induces a nonzero twisted q-form

$$\omega_{X^\circ} := d\varphi^\circ(\omega_{Y^\circ}) \in H^0\big(X^\circ, \Omega^q_{X^\circ} \otimes (\varphi^\circ)^*(\det \mathscr{N}_{\mathscr{G}|Y^\circ})\big),$$

which defines a codimension q foliation $\mathscr{E}^\circ$ on X°. *The pull-back $\varphi^{-1}\mathscr{G}$ of $\mathscr{G}$ via φ* is the foliation on X whose restriction to X° is $\mathscr{E}^\circ$. We will also write $\mathscr{G}_{|X}$ instead of $\varphi^{-1}\mathscr{G}$.

2.3 Singularities of foliations

Recently, notions of singularities coming from the minimal model program have shown to be very useful when studying birational geometry of foliations. We refer the reader to [28, Section I] for an in-depth discussion. Here, we only recall the notion of canonical foliation following McQuillan ([28, Definition I.1.2]).

Definition 2.2. Let $\mathscr{G}$ be a foliation on a normal complex variety X. Suppose that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. Let $\beta\colon Z \to X$ be a projective birational morphism. Then, there are uniquely defined rational numbers $a(E, X, \mathscr{G})$ such that

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + \sum_E a(E, X, \mathscr{G})E,$$

where E runs through all exceptional prime divisors for β. The rational numbers $a(E, X, \mathscr{G})$ do not depend on the birational morphism β, but only on the valuations associated to the E. We say that $\mathscr{G}$ is *canonical* if, for all E exceptional over X, $a(E, X, \mathscr{G}) \geqslant 0$.

We finally recall the behavior of canonical singularities with respect to birational maps and finite covers.

Lemma 2.3. [12, Lemma 4.2]
Let $\beta\colon Z \to X$ be a birational projective morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X. Suppose that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier.

(1) Suppose that $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^*K_{\mathscr{G}} + E$ for some effective β-exceptional $\mathbb{Q}$-divisor on Z. If $\beta^{-1}\mathscr{G}$ is canonical, then so is $\mathscr{G}$.
(2) If $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^*K_{\mathscr{G}}$, then $\mathscr{G}$ is canonical if and only if so is $\beta^{-1}\mathscr{G}$.

Lemma 2.4. [12, Lemma 4.3]
Let $f\colon X_1 \to X$ be a quasi-finite dominant morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X with $K_{\mathscr{G}}$$\mathbb{Q}$-Cartier. Suppose that any codimension one component of the branch locus of f is $\mathscr{G}$-invariant. If $\mathscr{G}$ is canonical, then so is $f^{-1}\mathscr{G}$.

2.4 Bertini-type results

The present paragraph is devoted to the following auxiliary results.

Lemma 2.5. Let $\beta\colon X \to Y$ be a birational morphism of smooth quasi-projective varieties with $n := \dim X \geqslant 3$. Let $\mathscr{G}$ be a codimension one foliation on X given by a twisted 1-form $\omega \in H^0(X, \Omega^1_X \otimes \mathscr{L})$. Suppose that any β-exceptional prime divisor on X dominates a codimension 2 closed subset in X. If A is a general member of a very ample linear system $|A|$ on Y, then the zero set of the induced twisted 1-form $\omega_H \in H^0(H, \Omega^1_H \otimes \mathscr{L}_{|H})$ on $H := \beta^{-1}(A)$ has codimension at least two.

Proof. Set $|H| := \beta^*|A|$. Recall that there exists a composition $Z \to Y$ of a finite number of blow-ups with smooth centers such that the induced rational map $Z \dashrightarrow X$ is a morphism. This immediately implies that $d\beta$ has rank $n-1$ at the generic point of any β-exceptional divisor. Let X° be an open set in X with complement of codimension at least 2 such that $\mathscr{G}$ is regular on X° and such that $d\beta_x(\mathscr{G}_x)$ has rank at least $n-2$ at any point of all β-exceptional divisor in X°. Consider

$$I^\circ = \left\{(x, H) \in X^\circ \times |H| \text{ such that } x \in H \text{ and } \mathscr{G}_x \subseteq T_xH\right\}.$$

We denote by $p\colon I^\circ \to X^\circ$ the projection. If $x \in X^\circ \setminus \mathrm{Exc}(\beta)$, then $p^{-1}(x) \subset |H|$ is a linear subspace of dimension $\dim |H| - n$. If $x \in \mathrm{Exc}(\beta)$, then $p^{-1}(x) \subset |H|$ is a linear subspace of dimension at most $\dim|H| - (n-1)$ since $d\beta_x(\mathscr{G}_x)$ has rank at least $n-2$ by choice of X°. It follows that any irreducible component of I° has dimension at most $\dim|H|$. Thus, general fibers of the second projection $q\colon I^\circ \to |H|$ have dimension $\leqslant 0$. Our claim then follows easily. ■

Lemma 2.6. Let $\psi\colon X \to Y$ be a dominant and equidimensional morphism of smooth quasi-projective varieties with reduced fibers. Suppose in addition that $\dim Y = \dim X - 1 \geqslant 2$. Let $\mathscr{G}$ be a codimension one foliation on X given by a twisted 1-form $\omega \in H^0(X, \Omega^1_X \otimes \mathscr{L})$. Suppose that the generic fiber of ψ is not tangent to $\mathscr{G}$. If A is a general member of a very ample linear system on Y, then the zero set of the induced twisted 1-form $\omega_H \in H^0(H, \Omega^1_H \otimes \mathscr{L}_{|H})$ on $H := \psi^{-1}(A)$ has codimension at least two.

Proof. This also follows from an easy dimension count (see proof of Lemma 2.5). ■

The proof of Proposition 2.10 makes use of the following result of McQuillan.

Proposition 2.7. [28, Facts I.1.8 and I.1.9] Let $\mathscr{L}$ be a foliation of rank one on a smooth complex quasi-projective surface S, and let x be a singular point of $\mathscr{L}$. Let also v be a local generator of $\mathscr{L}$ in a neighborhood of x. Then $\mathscr{L}$ is not canonical at x if and only if the linear part $D_x v$ of v at x is either nilpotent or diagonalizable with nonzero eigenvalues λ and μ satisfying in addition $\frac{\lambda}{\mu} \in \mathbb{Q}_{>0}$.

Fact 2.8. Notation as in Proposition 2.7. If $D_x v$ is nilpotent, then the following hold (see proof of [5, Theorem 1.1]). The exceptional divisor E_1 of the blow-up S_1 of S at x has discrepancy $a(E_1) \leqslant 0$. Moreover, $a(E_1) \leqslant -1$ if and only if $D_x v$ is zero.

Suppose that $a(E_1) = 0$. Then, the induced foliation $\mathscr{L}_1$ on S_1 has a unique singular point x_1 on E_1. Moreover, if v_1 is a local generator of $\mathscr{L}_1$ in a neighborhood of x_1, then $D_{x_1} v_1$ is nilpotent as well, and the exceptional divisor E_2 of the blow-up S_2 of S_1 at x_1 has discrepancy $a(E_2) \leqslant 0$.

If $a(E_2) = 0$, then the induced foliation $\mathscr{L}_2$ on S_2 has a unique singular point x_2 on E_2, and $D_{x_2} v_2$ is zero, where v_2 denotes a local generator of $\mathscr{L}_2$ in a neighborhood of x_2. Moreover, the exceptional divisor E_3 of the blow-up S_2 of S_1 at x_2 has discrepancy $a(E_3) \leqslant -1$.

Fact 2.9. Notation as in Proposition 2.7. If $D_x v$ is diagonalizable and its eigenvalues λ and μ are nonzero and satisfy $\frac{\lambda}{\mu} =: r \in \mathbb{Q}_{>0}$, then the following hold (see proof of

[5, Proposition 1.1]). Let S_1 be the blow-up of S at x with exceptional divisor E_1, and let $\mathscr{L}_1$ be the foliation on S_1 induced by $\mathscr{L}$.

Suppose that $\lambda \neq \mu$. A straightforward local computation then shows that v extends to a regular vector field v_1 on some open neighborhood of E_1 with isolated zeroes. Moreover, $\mathscr{L}_1$ has two singularities with diagonalizable linear parts and the quotients of the eigenvalues of the linear parts are $r-1$ and $\frac{1}{r}-1$. The divisor E_1 has discrepancy $a(E_1)=0$.

If $\lambda=\mu$, then E_1 has discrepancy $a(E_1)\leqslant -1$.

In either case, the Euclidean algorithm implies that there exists a divisorial valuation with center x and negative discrepancy.

Proposition 2.10. Let X be a smooth quasi-projective variety and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities. Let also B be a smooth codimension two component of the singular set of $\mathscr{G}$. Let $S \subseteq X$ be a two-dimensional complete intersection of general elements of a very ample linear system $|H|$ on X, and let $\mathscr{L}$ be the foliation of rank 1 on S induced by $\mathscr{G}$. Then, $\mathscr{L}$ has canonical singularities in a Zariski open neighborhood of the generic point of $B\cap S$.

Proof. Set $n:=\dim X$. Suppose that $\dim X\geqslant 3$. Let $U\subseteq |H|^{n-2}$ be a dense open set such that $S_u:=H_1\cap\cdots\cap H_{n-2}$ is a smooth connected surface for any $u=(H_1,\ldots,H_{n-2})\in U$, and let $\mathscr{L}_u$ be the foliation of rank one on S_u induced by $\mathscr{G}$. Let $T:=\{(x,u)\in B\times U\,|\,x\in B\cap S_u\}$, and denote by $p\colon T\to U$ and $q\colon T\to B$ the natural morphisms. Shrinking U, if necessary, we may assume that p is a finite étale cover. Let $\omega\in H^0(X,\Omega^1_X\otimes\mathscr{N})$ be a twisted 1-form defining $\mathscr{G}$. By Lemma 2.5, we may also assume that, for any $t=(x,u)\in T$, the induced twisted 1-form $\omega_u\in H^0(S_u,\Omega^1_{S_u}\otimes\mathscr{N}_{|S_u})$ on S_u has isolated zeroes. Given $t=(x,u)\in T$, let v_t be a local generator of $\mathscr{L}_u$ in a neighborhood of x. Note that x is a singular point of $\mathscr{L}_u$ since ω_u vanishes at x and has isolated zeroes.

In order to prove the proposition, we argue by contradiction and assume that the set of points $t=(x,u)\in T$ such that $\mathscr{L}_u$ is not canonical at x is dense in T.

Suppose that D_xv_t is diagonalizable with nonzero eigenvalues λ_t and μ_t satisfying $\frac{\lambda_t}{\mu_t}\in\mathbb{Q}_{>0}$ for some $t=(x,u)\in T$. Let Ω be a holomorphic 1-form defining $\mathscr{G}$ in some analytic open neighborhood W of x. Then we must have $d_x\Omega\neq 0$ since the restriction of Ω to $W\cap S_u$ defines $\mathscr{L}_{u|W}$ by choice of U. A theorem of Kupka ([26]) then says that, shrinking W, if necessary, the 2-form $d\Omega$ on W defines a codimension two foliation tangent to $\mathscr{G}_{|W}$. Therefore, shrinking W further, there exists analytic coordinates $(x_1,\ldots,x_n)$

centered at x on W such that $\mathscr{G}_{|W}$ is defined by the 1-form $a(x_1, x_2)dx_1 + b(x_1, x_2)dx_2$ for some holomorphic function a and b defined in a neighborhood of 0 in $\mathbb{C}^2$.

On the other hand, given $r \in \mathbb{Q}_{>0}$, the set of points $t = (x, u) \in T$ such that $D_x v_t$ is diagonalizable with eigenvalues $\lambda_t \neq 0$ and $r\lambda_t \neq 0$ is locally closed for the Zariski topology. Moreover, the set of points $t = (x, u) \in T$ such that the linear part $D_x v_t$ of v_t at x is nilpotent is Zariski closed.

Therefore, shrinking U again, if necessary, we may assume that one of the following holds.

(1) The linear part $D_x v_t$ is nilpotent for any $t = (x, u) \in T$.
(2) There exists $r \in \mathbb{Q}_{>0}$ such that $D_x v_t$ is diagonalizable with nonzero eigenvalues λ_t and $r\lambda_t$ for any $t = (x, u) \in T$.

Case 1. Suppose first that $D_x v_t$ is nilpotent for any $t = (x, u) \in T$. Let $\beta_1 \colon X_1 \to X$ be the blow-up of X along B with exceptional divisor E_1, and let $\mathscr{G}_1$ be the foliation on X_1 induced by $\mathscr{G}$. Notice that we have

$$K_{\mathscr{L}_u} \sim_{\mathbb{Z}} \left(K_{\mathscr{G}} + H_1 + \cdots + H_{n-2}\right)_{|S_u}. \tag{2.1}$$

Set $S_{1,u} := \beta^{-1}(H_1) \cap \cdots \cap \beta^{-1}(H_{n-2})$ and denote by $\mathscr{L}_{1,u}$ the foliation on $S_{1,u}$ induced by $\mathscr{L}_u$. Notice that $S_{1,u}$ is the blow-up of S_u along $B \cap S_u$. By Lemma 2.5, we also have

$$K_{\mathscr{L}_{1,u}} \sim_{\mathbb{Z}} \left(K_{\mathscr{G}_1} + \beta^* H_1 + \cdots + \beta^* H_{n-2}\right)_{|S_{1,u}}. \tag{2.2}$$

Let $t = (x, u) \in T$ and set $E_{1,t} := \beta^{-1}(x) \subset S_{1,u}$. From Equations (2.1) and (2.2), we conclude that

$$a(E_{1,t}, S_{1,u}, \mathscr{L}_{1,u}) = a(E, X, \mathscr{G}).$$

By Fact 2.8, we have $a(E_{1,t}, S_{1,u}, \mathscr{L}_{1,u}) \leqslant 0$. On the other hand, by assumption, we must have $a(E, X, \mathscr{G}) \geqslant 0$. It follows that

$$a(E_{1,t}, S_{1,u}, \mathscr{L}_{1,u}) = a(E, X, \mathscr{G}) = 0$$

for any $t = (x, u) \in T$. Moreover, $\mathscr{L}_{1,u}$ has a unique singular point $x_{1,t}$ on $E_{1,t}$ by Fact 2.8 again.

We claim that there exists a codimension two irreducible component $B_1 \subset E_1$ of the singular set of $\mathscr{G}_1$ dominating B. Suppose otherwise. Then, $\mathscr{G}_1$ is regular along a general fiber ℓ of the projection $E_1 \to B$. We have $K_{\mathscr{G}_1} \cdot \ell = 0$ since $K_{\mathscr{G}_1} = \beta^* K_{\mathscr{G}}$. On the other hand, we have $K_{X_1} \cdot \ell = -1$ by construction. It follows that $\mathscr{N}_{\mathscr{G}_1} \cdot \ell = 1$. This immediately implies that ℓ is tangent to $\mathscr{G}_1$ since $\deg \Omega^1_\ell \otimes \mathscr{N}_{\mathscr{G}_1|\ell} = -1$. But this contradicts Lemma 3.6. Note that B_1 is unique since $\mathscr{L}_{1,u}$ has a unique singular point $x_{1,t}$ on $E_{1,t}$ and $B_1 \cap E_{1,t}$ is contained in the singular set of $\mathscr{L}_{1,u}$.

Let $\beta_2 \colon X_2 \to X_1$ be the blow-up of X_1 along B_1 with exceptional divisor E_2, and let $\mathscr{G}_2$ be the foliation on X_2 induced by $\mathscr{G}$. Arguing as above, we see that we must have $a(E_2, X, \mathscr{G}) = 0$. Moreover, there exists a unique codimension two irreducible component $B_2 \subset E_2$ of the singular set of $\mathscr{G}_2$ dominating B.

Let now E_3 be the exceptional divisor of the blow-up of X_2 along B_2. Arguing as above, we see that must have $a(E_2, X, \mathscr{G}) \leqslant -1$ by Step 1, yielding a contradiction.

Case 2. Suppose now that there exists $r \in \mathbb{Q}_{>0}$ such that $D_x v_t$ is diagonalizable with eigenvalues λ_t and $r\lambda_t$ for any $t = (x, u) \in T$. Arguing as in Case 1, one shows that there exists a divisorial valuation with center B on X and negative discrepancy. One only needs to replace the use of Fact 2.8 by Fact 2.9. This yields again a contradiction, completing the proof of the proposition. ■

3 Basic results

In this section, we extend a number of earlier results to the context of normal quasi-projective varieties with quotient singularities. See [10] for a somewhat related result.

3.1 Algebraic and analytic $\mathbb{Q}$-structures

We will use the notions of $\mathbb{Q}$-varieties and $\mathbb{Q}$-sheaves without further explanation and simply refer to [29, Section 2].

Notation 3.1. Let X be a normal quasi-projective variety, and let $X_{\mathbb{Q}} := \big(X, \{p_\alpha : X_\alpha \to X\}_{\alpha\in A}\big)$ be a structure of $\mathbb{Q}$-variety on X. For each $\alpha \in A$, X_α is smooth and quasi-projective, and that there exist a normal variety U_α and a factorization of p_α

$$\begin{tikzcd} X_\alpha \arrow[r, "q'_\alpha\text{, Galois with group } G_\alpha"'] \arrow[rr, bend left, "p_\alpha"] & U_\alpha \arrow[r, "p'_\alpha\text{, étale}"'] & X. \end{tikzcd}$$

We will denote by $X_{\alpha\beta}$ the normalization of $X_\alpha \times_X X_\beta$, and by $p_{\alpha\beta,\alpha} \colon X_{\alpha\beta} \to X_\alpha$ and $p_{\alpha\beta,\beta} \colon X_{\alpha\beta} \to X_\beta$ the natural morphisms. Both $p_{\alpha\beta,\alpha}$ and $p_{\alpha\beta,\beta}$ are étale by the very definition of a $\mathbb{Q}$-structure.

In *loc. cit.*, Mumford constructs a *global cover* of $X_{\mathbb{Q}}$, that is, a quasi-projective normal variety $\widehat{X}$, a finite Galois cover $p : \widehat{X} \to X$ with group G, and for every $\alpha \in A$, a commutative diagram as follows:

$$\begin{tikzcd} \widehat{X}_\alpha \arrow[r, "q_\alpha\text{, Galois cover}"'] \arrow[rr, bend left, "p_{|\widehat{X}_\alpha}"] & X_\alpha \arrow[r, "p_\alpha"'] & X, \end{tikzcd}$$

where $\widehat{X}_\alpha \subseteq \widehat{X}$ is open. The finite map $(q_\alpha, q_\beta) \colon \widehat{X}_{\alpha\beta} := \widehat{X}_\alpha \cap \widehat{X}_\beta \to X_\alpha \times_X X_\beta$ factors through the normalization map $X_{\alpha\beta} \to X_\alpha \times_X X_\beta$ and induces a finite morphism $q_{\alpha\beta} \colon \widehat{X}_{\alpha\beta} \to X_{\alpha\beta}$.

Fact 3.2. Let X be a normal quasi-projective variety with only quotient singularities. Then according to Mumford ([29, Section 2]) there exists a structure of $\mathbb{Q}$-variety on X given by a collection of charts $\{p_\alpha \colon X_\alpha \to X\}_{\alpha \in A}$ where p_α is étale in codimension one for every $\alpha \in A$. We will refer to it as a *quasi-étale* $\mathbb{Q}$*-variety structure*.

Fact 3.3. Let X be a normal quasi-projective variety, and let $\{p_\alpha \colon X_\alpha \to X\}_{\alpha \in A}$ be a finite set of morphisms such that $X = \cup_{\alpha \in A} p_\alpha(X_\alpha)$. Suppose that we have a factorization of $p_\alpha = p'_\alpha \circ q'_\alpha$ as above. Then, the collection of morphisms $\{p_\alpha \colon X_\alpha \to X\}_{\alpha \in A}$ automatically defines a quasi-étale $\mathbb{Q}$-variety structure on X by purity of the branch locus.

In the setting of Fact 3.2, there is a finite covering $(V_i)_{i \in I}$ of X by analytically open sets such that the following holds. For each $i \in I$, there exists $\alpha(i) \in A$ such that $V_i \subseteq p_{\alpha(i)}(X_{\alpha(i)})$ and a connected component V'_i of $(p'_{\alpha(i)})^{-1}(V_i)$ such that the restriction of $p'_{\alpha(i)}$ to V'_i induces an isomorphism onto V_i. Set $X_i := (q'_{\alpha(i)})^{-1}(V'_i)$ and $\widehat{X}_i := q^{-1}_{\alpha(i)}(X_i)$. We have a commutative diagram as follows:

$$\begin{tikzcd} \widehat{X}_i \arrow[r, hook] \arrow[d, "q_i\text{, finite}"'] & \widehat{X}_{\alpha(i)} \arrow[r, hook] \arrow[d, "q_{\alpha(i)}"] & \widehat{X} \arrow[dd, "p"] \\ X_i \arrow[r, hook] \arrow[d, "q'_i\text{, quasi-étale and Galois with group } G_i"'] \arrow[dr, "p_i"] & X_{\alpha(i)} \arrow[d, "p_{\alpha(i)}"] & \\ V_i \arrow[r, hook] & X \arrow[r, equal] & X. \end{tikzcd}$$

Note that we have $X = \cup_i p_i(X_i)$ by construction. Let X_{ij} denotes the normalization of $X_i \times_X X_j$. The natural morphisms $p_{ij,i} \colon X_{ij} \to X_i$ and $p_{ij,j} \colon X_{ij} \to X_j$ are étale by purity of the branch locus and X_{ij} is smooth. The finite map $X_{ij} \to V_i \cap V_j$ is Galois with group $G_i \times G_j$. Moreover, the finite map $(q_i, q_j) \colon \widehat{X}_{ij} := \widehat{X}_i \cap \widehat{X}_j \to X_i \times_X X_j$ induces a finite morphism $q_{ij} \colon \widehat{X}_{ij} \to X_{ij}$. We view $p \colon \widehat{X} \to X$ as a global cover for the analytic quasi-étale $\mathbb{Q}$-structure given by the collection of charts $\{p_i \colon X_i \to X\}_{i \in I}$.

Notice that the collections of open sets $(g\cdot\widehat{X}_i)_{g\in G,i\in I}$ and $(g\cdot\widehat{X}_\alpha)_{g\in G,\alpha\in A}$ both form a covering of $\widehat{X}$.

3.2 A basic formula

In the present paragraph, we extend [5, Proposition 2.2] to surfaces with quotient singularities.

Lemma 3.4. Let X be a normal quasi-projective surface with quotient singularities, and let $\mathscr{G}\subset T_X$ be a foliation of rank one. Let C be an irreducible complete curve on X and suppose that C is transverse to $\mathscr{G}$ at a general point on C. Then $K_{\mathscr{G}}\cdot C + C\cdot C \geqslant 0$.

Remark 3.5. Quotient singularities are $\mathbb{Q}$-factorial so that $K_{\mathscr{G}}$ and C are $\mathbb{Q}$-Cartier divisors.

Proof of Lemma 3.4 The proof is very similar to that of [5, Proposition 2.2] and so we leave some easy details to the reader.

Let m be a positive integer such that $mK_{\mathscr{G}}$ and mC are Cartier divisors. Let $X_{\mathbb{Q}} = \big(X, \{p_\alpha : X_\alpha \to X\}_{\alpha\in A}\big)$ be a quasi-étale $\mathbb{Q}$-structure on X (Fact 3.2). We will use the notation of paragraph 3.1. Let $B\to X$ be the normalization of C and let $\widehat{B}$ be the normalization of an irreducible component of $p^{-1}(C)$. We have a commutative diagram as follows:

$$\begin{tikzcd} \widehat{B} \arrow[r] \arrow[d] & \widehat{X} \arrow[d, "p"] \\ B \arrow[r] & X. \end{tikzcd}$$

Shrinking the X_α, if necessary, we may assume that for each $\alpha\in A$ there exists a generator v_α of $p_\alpha^{-1}\mathscr{G}$ on X_α. We may also assume without loss of generality that there exists a nowhere vanishing regular function h_α on X_α such that $h_\alpha v_\alpha^{\otimes m}$ is G_α-invariant since $mK_{\mathscr{G}}$ is a Cartier divisor by choice of m. Replacing X_α by an étale cover, if necessary, we can suppose that $h_\alpha^{\frac{1}{m}}$ is regular on X_α, and replacing v_α by $h_\alpha^{\frac{1}{m}}v_\alpha$ we may finally assume that $v_\alpha^{\otimes m}$ is G_α-invariant. Similarly, we can suppose that the ideal sheaf $p_\alpha^{-1}\mathscr{I}_C$ is generated by a regular function f_α such that f_α^m is G_α-invariant. We will denote by $p_\alpha^{-1}C$ the scheme defined by equation $\{f_\alpha = 0\}$. Notice that the corresponding $\mathbb{Q}$-sheaves $\mathscr{G}_{\mathbb{Q}}$ and $\mathscr{O}_X(-C)_{\mathbb{Q}}$ are $\mathbb{Q}$-line bundles, and the pull-back $\mathscr{M}$ of $\mathscr{G}_{\mathbb{Q}}^*\otimes\mathscr{O}_X(C)_{\mathbb{Q}}$ to $\widehat{B}$ is a genuine line bundle.

One then readily checks that the functions $(v_\alpha(f_\alpha))^m$ restricted to $p_\alpha^{-1}C$ give a nonzero global section of $\mathscr{M}^{\otimes m}$. In particular, $\mathscr{M}$ has non negative degree. Observe that the pull-back of $\mathscr{G}_{\mathbb{Q}}^{\otimes m}$ (resp. $\mathscr{O}_X(C)_{\mathbb{Q}}^{\otimes m} \cong \mathscr{O}_X(mC)_{\mathbb{Q}}$) to $\widehat{X}$ is isomorphic to $p^*\mathscr{O}_X(-mK_{\mathscr{G}})$ (resp. $p^*\mathscr{O}_X(mC)$) so that $\mathscr{M}^{\otimes m}$ is isomorphic to the pull-back of $\mathscr{O}_X(mK_{\mathscr{G}} + mC)$ to $\widehat{B}$. The lemma then follows from the projection formula. ■

3.3 The Baum–Bott partial connection

The following result generalizes [2, Corollary 3.4] to the setting of quasi-projective varieties with quotient singularities.

Lemma 3.6. Let X be a normal quasi-projective variety with quotient singularities, and let $\mathscr{G} \subset T_X$ be a codimension one foliation on X. Let also $\{p_i : X_i \to X\}_{i \in I}$ be an analytic quasi-étale $\mathbb{Q}$-structure on X and suppose that $p_i^{-1}\mathscr{G}$ is defined by a 1-form ω_i with zero set of codimension at least two such that $d\omega_i = \alpha_i \wedge \omega_i$ for some holomorphic 1-form α_i on X_i. Then the following hold.

(1) We have $c_1(\mathscr{N}_{\mathscr{G}})^2 \equiv 0$.

(2) Let $Y \subseteq X$ be a projective subvariety. Suppose that Y is not entirely contained in the union of the singular loci of X and $\mathscr{G}$. Suppose moreover that Y is tangent to $\mathscr{G}$. Then $c_1(\mathscr{N}_{\mathscr{G}})_{|Y} \equiv 0$.

Proof. We use the notation of paragraph 3.1. Set $\mathscr{L} := \mathscr{N}_{\mathscr{G}}^*$ and $\widehat{\mathscr{L}} := p^{[*]}\mathscr{L}$. By [20, Proposition 1.9], the reflexive pull-back $p_\alpha^{[*]}\mathscr{L}$ is a line bundle. It follows that $\widehat{\mathscr{L}}_{|\widehat{X}_\alpha} \cong q_\alpha^*\big(p_\alpha^{[*]}\mathscr{L}\big)$ since both sheaves are reflexive and agree over the big open set X_{reg}. This shows that $\widehat{\mathscr{L}}$ is a line bundle. Moreover, if m is a positive integer such that $\mathscr{L}^{[\otimes m]}$ is a line bundle, then we have $p^*(\mathscr{L}^{[\otimes m]}) \cong \widehat{\mathscr{L}}^{\otimes m}$.

Let $\omega \in H^0\big(X, \Omega_X^{[1]} \boxtimes \mathscr{N}_{\mathscr{G}}\big)$ be a twisted 1-form defining $\mathscr{G}$. The reflexive pull-back of ω then gives an inclusion $\widehat{\mathscr{L}} \subset p^{[*]}\Omega_X^{[1]}$, which is saturated by [1, Lemma 9.7]. Note that we have $p^{[*]}\Omega_X^{[1]} \subseteq \Omega_{\widehat{X}}^1 \subseteq \Omega_{\widehat{X}}^{[1]}$ since we have a factorization $p_{|\widehat{X}_\alpha} = p_\alpha \circ q_\alpha$ and X_α is smooth.

Shrinking the V_i, is necessary, we may assume that there exist nowhere vanishing holomorphic functions h_i on X_i such that $h_i\omega_i^{\otimes m}$ is G_i-invariant. Observe that $d\big(h_i^{\frac{1}{m}}\omega_i\big) = \big(\frac{1}{m}\frac{dh_i}{h_i} + \alpha_i\big) \wedge \big(h_i^{\frac{1}{m}}\omega_i\big)$ so that, replacing ω_i by $h_i^{\frac{1}{m}}\omega_i$ and α_i by $\frac{1}{m}\frac{dh_i}{h_i} + \alpha_i$, we can suppose that $\omega_i^{\otimes m}$ is G_i-invariant.

Now, we can write

$$\omega_{j\,|X_{ij}} = \varphi_{ij}\,\omega_{i\,|X_{ij}}$$

on X_{ij} where φ_{ij} is a nowhere vanishing holomorphic function since the 1-forms $\omega_{i|X_{ij}}$ and $\omega_{j|X_{ij}}$ both define the foliation $p_{ij,i}^{-1}p_i^{-1}\mathscr{G} = p_{ij,j}^{-1}p_j^{-1}\mathscr{G}$ on X_{ij} and their zero sets set have codimension at least two in X_{ij}. The holomorphic functions $(\varphi_{ij} \circ q_{ij})^m$ on $\widehat{X}_{ij}$ then give a cocycle with respect to the open covering $(g \cdot \widehat{X}_i)_{g\in G, i\in I}$ that corresponds to the isomorphism class of $\widehat{\mathscr{L}}^{\otimes m}$ as a complex analytic line bundle since it does over the open set X_{reg}.

Let $c \in H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big)$ be the cohomology class corresponding to the cocyle with respect to the open covering $(g \cdot \widehat{X}_i)_{g\in G, i\in I}$ induced by the $q_{ij}^*\frac{d\varphi_{ij}}{\varphi_{ij}}$. Since $d\omega_i = \alpha_i \wedge \omega_i$ for any $i \in I$, we must have

$$\Big(\frac{d\varphi_{ij}}{\varphi_{ij}} + \alpha_{i|X_{ij}} - \alpha_{j|X_{ij}}\Big) \wedge \omega_{i|X_{ij}} = 0.$$

This immediately implies that $q_{ij}^*\big(\frac{d\varphi_{ij}}{\varphi_{ij}} + \alpha_{i|X_{ij}} - \alpha_{j|X_{ij}}\big) \in H^0\big(\widehat{X}_{ij}, \widehat{\mathscr{L}}_{|\widehat{X}_{ij}}\big) \subseteq H^0\big(\widehat{X}_{ij}, p^{[*]}\Omega^{[1]}_{X\,|\widehat{X}_{ij}}\big)$ since $\widehat{\mathscr{L}}$ is saturated in $p^{[*]}\Omega_X^{[1]}$, and shows that c is the image of a cohomological class $b \in H^1\big(\widehat{X}, \widehat{\mathscr{L}}\big)$ under the natural map $H^1\big(\widehat{X}, \widehat{\mathscr{L}}\big) \to H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big)$. By construction, $c \in H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big)$ maps to $c_1\big(\widehat{\mathscr{L}}\big) \in H^1\big(\widehat{X}, \Omega^1_{\widehat{X}}\big)$ under the natural map $H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big) \to H^1\big(\widehat{X}, \Omega^1_{\widehat{X}}\big)$. On the other hand, $b \cup b = 0 \in H^1\big(\widehat{X}, \wedge^2\widehat{\mathscr{L}}\big)$ since $\widehat{\mathscr{L}}$ is a line bundle. This immediately implies that $c_1\big(\widehat{\mathscr{L}}\big)^2 = 0 \in H^2\big(\widehat{X}, \Omega^2_{\widehat{X}}\big)$, proving (1).

Let $Y \subseteq X$ be a projective subvariety, and let $\widehat{Y}$ be a resolution of some irreducible component of $p^{-1}(Y)$. We have a commutative diagram as follows:

$$\begin{tikzcd} \widehat{Y} \arrow[r, "f"] \arrow[d, "\text{generically finite}"'] & \widehat{X} \arrow[d, "p,\ \text{finite}"] \\ Y \arrow[r, hook] & X. \end{tikzcd}$$

Note that $c \in H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big)$ maps to $c_1\big(f^*\widehat{\mathscr{L}}\big) \in H^1\big(\widehat{Y}, \Omega^1_{\widehat{Y}}\big)$ under the composed map

$$H^1\big(\widehat{X}, p^{[*]}\Omega_X^{[1]}\big) \longrightarrow H^1\big(\widehat{X}, \Omega^1_{\widehat{X}}\big) \longrightarrow H^1\big(\widehat{Y}, \Omega^1_{\widehat{Y}}\big).$$

Suppose moreover that Y is not entirely contained in the union of the singular loci of X or $\mathscr{G}$, and that it is tangent to $\mathscr{G}$. Then, the composed map of sheaves

$$f^*\widehat{\mathscr{L}} \longrightarrow f^*\big(p^{[*]}\Omega_X^{[1]}\big) \longrightarrow f^*\Omega^1_{\widehat{X}} \longrightarrow \Omega^1_{\widehat{Y}}$$

vanishes. This immediately implies that $c_1\big(f^*\widehat{\mathscr{L}}\big) = 0 \in H^1\big(\widehat{Y}, \Omega^1_{\widehat{Y}}\big)$. Item (2) now follows from the projection formula. ∎

3.4 Baum–Bott formula

The next result extends the Baum–Bott formula to surfaces with quotient singularities.

Definition 3.7. Let X be a normal quasi-projective algebraic surface with quotient singularities, and let $\mathscr{L} \subset T_X$ be a foliation of rank one. Given $x \in X$, there exist an open analytic neighborhood U of x, a (not necessarily connected) smooth analytic complex manifold V and a finite Galois holomorphic map $p\colon V \to U$ that is étale outside of the singular locus. Set

$$\mathrm{BB}^{\mathbb{Q}}(\mathscr{L},x) := \frac{1}{\deg p} \sum_{y \in p^{-1}(x)} \mathrm{BB}(p^{-1}\mathscr{L}_{|U},y),$$

where $\mathrm{BB}(p^{-1}\mathscr{L}_{|U},y)$ denotes the Baum–Bott index of $p^{-1}\mathscr{L}_{|U}$ at y (we refer to [5, Section 3.1] for this notion).

Remark 3.8. One readily checks that $\mathrm{BB}^{\mathbb{Q}}(\mathscr{L},x)$ is independent of the local chart $p\colon V \to U$ at x.

Proposition 3.9. Let X be a normal projective algebraic surface with quotient singularities, and let $\mathscr{L} \subset T_X$ be a foliation of rank one. Then

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_{x \in X} \mathrm{BB}^{\mathbb{Q}}(\mathscr{L},x).$$

Proof. The proof is very similar to that of [5, Theorem 3.1] and so we leave some easy details to the reader.

Step 1. Preparation. We use the notation of paragraph 3.1. Let m be a positive integer such that $\mathscr{N}_{\mathscr{L}}^{[\otimes m]}$ is a line bundle and set $\widehat{\mathscr{N}_{\mathscr{L}}} := p^{[*]}\mathscr{N}_{\mathscr{L}}$. Shrinking V_i, if necessary, we may assume without loss of generality that ${p_i}^{-1}\mathscr{L}$ is defined by a 1-form ω_i with isolated zeroes, and that there exist a smooth $(1,0)$-form β_i on X_i and a small enough open set $W_i \Subset V_i$ such that $d\omega_i = \beta_i \wedge \omega_i$ on $X_i \setminus p_i^{-1}(\overline{W}_i)$. We may assume that W_i is the image in V_i of some small enough open ball in X_i. We may also assume that ω_i is nowhere vanishing on X_i or that it vanishes at a single point contained in $p_i^{-1}(W_i)$, and that $W_i \cap V_j = \emptyset$ if $i \neq j$. Finally, we can suppose that $\omega_i^{\otimes m}$ is G_i-invariant (see proof of Lemma 3.6) and that β_i vanishes identically on some open neighborhood of the singular locus of ω_i and some open neighborhood of the inverse image of the singular set of X in

X_i. Since ω_i is semi-invariant under G_i, replacing β_i by $\frac{1}{\sharp G_i}\sum_{g\in G_i} g^*\beta_i$, if necessary, we may assume that β_i is G_i-invariant.

We can write

$$\omega_{i|X_{ij}} = \varphi_{ij}\,\omega_{j|X_{ij}}$$

on X_{ij} where φ_{ij} is a nowhere vanishing holomorphic function since the holomorphic 1-forms $\omega_{i|X_{ij}}$ and $\omega_{j|X_{ij}}$ both define the foliation $p_{ij,i}^{-1}p_i^{-1}\mathscr{L} = p_{ij,j}^{-1}p_j^{-1}\mathscr{L}$ on X_{ij} and have isolated zeros. The holomorphic functions $(\varphi_{ij}\circ q_{ij})^m$ on $\widehat{X}_{ij}$ then give a cocycle with respect to the open covering $(g\cdot\widehat{X}_i)_{g\in G, i\in I}$ that corresponds to the isomorphism class of $\widehat{\mathscr{N}_{\mathscr{L}}}^{\otimes m}$ as a complex analytic line bundle since it does over the open set X_{reg}. Notice that the smooth form $\frac{d\varphi_{ij}}{\varphi_{ij}} + \beta_{j|X_{ij}} - \beta_{i|X_{ij}}$ vanishes identically if $i=j$. An easy computation now shows that

$$\Big(\frac{d\varphi_{ij}}{\varphi_{ij}} + \beta_{j|X_{ij}} - \beta_{i|X_{ij}}\Big)\wedge\omega_{i|X_{ij}} = 0$$

if $i\neq j$ using the fact that $W_i\cap V_j=\emptyset$. Therefore, the cocycle of smooth $(1,0)$-forms $\Big(\frac{d\varphi_{ij}}{\varphi_{ij}} + \beta_{j|X_{ij}} - \beta_{i|X_{ij}}\Big)_{ij}$ can be viewed as a cocycle of smooth sections of the $\mathbb{Q}$-line bundle $\mathscr{M}_{\mathbb{Q}}$ induced by $\mathscr{N}_{\mathscr{L}}^*$ on $X_{\mathbb{Q}}$. On the other hand, the smooth $(1,0)$-form $m\Big(\frac{d\varphi_{ij}}{\varphi_{ij}} + \beta_{j|X_{ij}} - \beta_{i|X_{ij}}\Big)$ is the pull-back of a smooth $(1,0)$-form on $V_i\cap V_j$ for $i\neq j$ and vanishes identically if $i=j$. Using a partition of unity subordinate to the open cover $(V_i)_{i\in I}$ given by [6, Proposition 1.2], we see that we may assume that there exist smooth $(1,0)$-forms γ_i on X_i such that

$$\gamma_i\wedge\omega_i = 0 \text{ on } X_i,$$

and

$$\frac{d\varphi_{ij}}{\varphi_{ij}} = \beta_i - \beta_j + \gamma_i - \gamma_j \text{ on } X_{ij}.$$

Notice that we have $d\omega_i = (\beta_i+\gamma_i)\wedge\omega_i$ on $X_i\setminus p_i^{-1}(\overline{W}_i)$. The 2-form Ω defined on X_i by $\Omega_{|X_i} := \frac{1}{2i\pi}d(\beta_i+\gamma_i)$ is a well-defined closed 2-form on $X_{\mathbb{Q}}$ whose pull-back to any resolution of $\widehat{X}$ is smooth and represents the first Chern class of the pull-back of $\widehat{\mathscr{N}_{\mathscr{L}}}$ to this resolution.

Step 2. Computation. It follows from the projection formula that

$$\deg p\cdot c_1(\mathscr{N}_{\mathscr{L}})^2 = c_1(\widehat{\mathscr{N}_{\mathscr{L}}})^2 = \int_{\widehat{X}}\widehat{\Omega}\wedge\widehat{\Omega},$$

where $\widehat{\Omega}$ denotes the 2-form induced by Ω on $\widehat{X}$. One then observes that

$$\mathrm{BB}(p_i^{-1}\mathscr{L}, x) = \frac{1}{(2i\pi)^2}\int_\Sigma \beta \wedge d\beta$$

for any $x \in X_i$, where Σ is a small enough suitably oriented 3-sphere in X_i centered at x and β is any smooth $(1,0)$-form on X_i such that $d\omega_i = \beta \wedge \omega_i$ in a neighborhood of Σ. On the other hand, on $\widehat{X}_i \setminus p^{-1}(\overline{W}_i)$, we have $\widehat{\Omega} \wedge \widehat{\Omega} = 0$ by construction. Stokes' theorem then implies that

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_i \frac{1}{(2i\pi)^2 \deg q_i'} \int_{\partial p_i^{-1}(\overline{W}_i')} (\beta_i + \gamma_i) \wedge d(\beta_i + \gamma_i),$$

where $W_i \Subset W_i' \Subset V_i$ is the image in V_i of some small enough open ball in X_i. We finally obtain

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_x \mathrm{BB}^{\mathbb{Q}}(\mathscr{L}, x),$$

completing the proof of the proposition. ■

3.5 Camacho–Sad formula

We finally observe that the Camacho–Sad formula also extends to surfaces with quotient singularities.

Definition 3.10. Let X be a normal quasi-projective algebraic surface with quotient singularities, and let $\mathscr{L} \subset T_X$ be a foliation of rank one. Let $C \subset X$ be a complete curve. Suppose that C is invariant under $\mathscr{L}$. Given $x \in X$, there exist an open analytic neighborhood U of x, a (not necessarily connected) smooth analytic complex manifold V and a finite Galois holomorphic map $p\colon V \to U$ that is étale outside of the singular locus. Set

$$\mathrm{CS}^{\mathbb{Q}}(\mathscr{L}, C, x) := \frac{1}{\deg p} \sum_{y \in p^{-1}(x)} \mathrm{CS}\big(p^{-1}\mathscr{L}_{|U}, p^{-1}(C \cap U), y\big)$$

where $\mathrm{CS}\big(p^{-1}\mathscr{L}_{|U}, p^{-1}(C \cap U), y\big)$ denotes the Camacho–Sad index (we refer to [5, Section 3.2] for this notion).

Remark 3.11. One readily checks that $\mathrm{CS}^{\mathbb{Q}}(\mathscr{L}, C, x)$ is independent of the local chart $p\colon V \to U$ at x.

Proposition 3.12. Let X be a normal quasi-projective algebraic surface with quotient singularities, and let $\mathscr{L} \subset T_X$ be a foliation of rank one. Let $C \subset X$ be a complete curve and suppose that C is invariant under $\mathscr{L}$. Then

$$C^2 = \sum_{x \in C} \mathrm{CS}^{\mathbb{Q}}(\mathscr{L}, C, x).$$

Proof. The proof is very similar to that of [5, Theorem 3.2] and so we again leave some easy details to the reader.

We maintain notation as in Step 1 of the proof of Proposition 3.9. In particular, there is a well-defined closed 2-form Ω on $X_{\mathbb{Q}}$ whose pull-back to any resolution of $\widehat{X}$ is smooth and represents the first Chern class of the pull-back of $\widehat{\mathscr{N}_{\mathscr{L}}}$ to this resolution.

Shrinking V_i, if necessary, we may assume that the ideal sheaf $p_i^{-1}\mathscr{I}_C$ is generated by a holomorphic function f_i. Notice that the corresponding analytic $\mathbb{Q}$-sheaf $\mathscr{O}_X(-C)_{\mathbb{Q}}$ is a $\mathbb{Q}$-line bundle. Let $\widehat{C}$ be the complete curve on $\widehat{X}$ whose ideal sheaf is $\mathscr{O}_{\widehat{X}}\big(-\widehat{C}\big) := p^{[*]}\mathscr{I}_C$. Set $\alpha_i := f_i^{-1}\omega_i$. The method used to construct the 2-form Ω from the ω_i in the proof of Proposition 3.9 gives a closed 2-form A on $X_{\mathbb{Q}}$ whose pull-back to any resolution of $\widehat{X}$ is smooth and represents the first Chern class of the pull-back of the line bundle $\widehat{\mathscr{N}_{\mathscr{L}}} \otimes \mathscr{O}_{\widehat{X}}\big(-\widehat{C}\big)$ to this resolution and such that $A_{|X_i} = \frac{1}{2i\pi} d\mu_i$, where μ_i is a smooth $(1,0)$-form such that $d\alpha_i = \mu_i \wedge \alpha_i$ on $X_i \setminus p_i^{-1}(\overline{W}_i)$.

Suppose that $p_i^{-1}\mathscr{L}$ is singular, and let $x \in W_i$ be its singular point. There exist holomorphic functions g_i and h_i and a holomorphic 1-form η_i such that $g_i\omega_i = h_i df_i + f_i\eta_i$ in some analytic open neighborhood of x. Then

$$\mathrm{CS}\big(p_i^{-1}\mathscr{L}, p_i^{-1}(C), x\big) = -\frac{1}{2i\pi}\int_{\sigma_i} h_i^{-1}\eta_i,$$

where $\sigma_i \subset C$ is a union of small suitably oriented circles centered at x, one for each local branch of C at x. In particular, $\int_{\sigma_i} h_i^{-1}\eta_i$ depends only on $p_i^{-1}\mathscr{L}, p_i^{-1}(C)$ and x. A straightforward computation then shows that $d\alpha_i = (h_i^{-1}\eta_i + \beta_i + \gamma_i) \wedge \alpha_i$. On the other hand, we may obviously assume that f_i and h_i are relatively prime. Then, we must have $\mu_{i|\sigma_i} = h_i^{-1}\eta_{i|\sigma_i} + \beta_{i|\sigma_i} + \gamma_{i|\sigma_i}$.

Let $\widehat{A}$ be the 2-form induced by A on $\widehat{X}$. Then

$$C^2 = \frac{1}{\deg p}\widehat{C}^2 = \frac{1}{\deg p}\int_{\widehat{C}}\widehat{\Omega} - \frac{1}{\deg p}\int_{\widehat{C}}\widehat{A}.$$

On the other hand, by Stokes' theorem, we have

$$\frac{1}{\deg p}\int_{\widehat{C}}\widehat{\Omega} - \frac{1}{\deg p}\int_{\widehat{C}}\widehat{A} = \sum_i \frac{1}{2i\pi \deg q_i'}\int_{\sigma_i}(\beta_i + \gamma_i - \mu_i) = \sum_{x\in C}\mathrm{CS}^{\mathbb{Q}}(\mathscr{L}, C, x).$$

This finishes the proof of the proposition. ∎

4 Foliations on Mori fiber spaces

In this section, we provide another technical tool for the proof of the main results.

Proposition 4.1. Let X be a projective variety with klt singularities and let $\psi: X \to Y$ be a Mori fiber space with $\dim Y = \dim X - 1 \geqslant 1$. Let $\mathscr{G}$ be a codimension one foliation on X with $K_{\mathscr{G}} \equiv 0$. Suppose that there exist a closed subset $Z \subset X$ of codimension at least 3 and an analytic quasi-étale $\mathbb{Q}$-structure $\{p_i : X_i \to X \setminus Z\}_{i\in I}$ on $X \setminus Z$ such that $p_i^{-1}\mathscr{G}_{|X\setminus Z}$ is defined by a closed 1-form with zero set of codimension at least two or by the 1-form $x_2dx_1 + \lambda_i x_1 dx_2$ where $(x_1, \ldots, x_n)$ are analytic coordinates on X_i and $\lambda_i \in \mathbb{Q}_{>0}$. Then, there exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least two and a finite Galois cover $g: T \to Y$ such that the following holds. Set $T^\circ := g^{-1}(Y^\circ)$ and $X^\circ := \psi^{-1}(Y^\circ)$.

1. (1) The variety T has canonical singularities and $K_T \sim_{\mathbb{Z}} 0$; T° is smooth.
2. (2) The normalization M° of the fiber product $T^\circ \times_Y X$ is a $\mathbb{P}^1$-bundle over T° and the map $M^\circ \to X^\circ$ is a quasi-étale cover.
3. (3) The pull-back of $\mathscr{G}_{|X^\circ}$ on M° yields a flat Ehresmann connection on $M^\circ \to T^\circ$.

Before we give the proof of Proposition 4.1, we need the following auxiliary result, which might be of independent interest.

Lemma 4.2. Let X be quasi-projective variety with quotient singularities and let $\psi: X \to Y$ be a dominant projective morphism with connected and reduced fibers onto a smooth quasi-projective variety Y. Suppose that ψ is equidimensional with $\dim Y = \dim X - 1 \geqslant 1$. Suppose in addition that $-K_X$ is ψ-ample. Let $\mathscr{G}$ be a codimension one foliation on X with $K_{\mathscr{G}} \equiv_\psi 0$. Suppose furthermore that there exists an analytic quasi-étale $\mathbb{Q}$-structure $\{p_i : X_i \to X\}_{i\in I}$ on X such that $p_i^{-1}\mathscr{G}$ is defined by a closed 1-form with zero set of codimension at least two or by the 1-form $x_2dx_1 + \lambda_i x_1 dx_2$ where $(x_1, \ldots, x_n)$ are analytic coordinates on X_i and $\lambda_i \in \mathbb{Q}_{>0}$. Then, there exists an open subset $Y^\circ \subseteq Y$ with the complement of codimension at least two such that $\psi^\circ := \psi_{|X^\circ}$ is a $\mathbb{P}^1$-bundle where $X^\circ := \psi^{-1}(Y^\circ)$. Moreover, $\mathscr{G}_{|X^\circ}$ yields a flat Ehresmann connection on ψ°.

Proof. Notice that $\mathscr{G}$ is regular along the generic fiber of ψ. It follows that general fibers of ψ are not tangent to $\mathscr{G}$ since $-K_X$ is ψ-ample and $K_\mathscr{G} \equiv_\psi 0$ by assumption (Lemma 3.6). We first show the following.

Claim 4.3. There exists an open set Y° with complement of codimension at least two such that, for any $y \in Y^\circ$, $\psi^{-1}(y)$ is transverse to $\mathscr{G}$ at a point of $\psi^{-1}(y)$.

Proof. We argue by contradiction and assume that there is an irreducible hypersurface D in Y such that any irreducible component of $\psi^{-1}(y)$ is tangent to $\mathscr{G}$ for all $y \in D$.

Step 1. Setup. Let $B \subset Y$ be a one-dimensional complete intersection of general members of a very ample linear system on Y passing through a general point y of D and set $S := \psi^{-1}(B) \subset X$. Notice that S is klt. Let $\mathscr{L}$ be the foliation of rank one on S induced by $\mathscr{G}$, and denote by $\pi : S \to B$ the restriction of ψ to S. By the adjunction formula, $-K_S = -K_{X|S}$ is π-ample. By general choice of B, we have $\det \mathscr{N}_\mathscr{L} \cong (\det \mathscr{N}_\mathscr{G})_{|S}$ (Lemma 2.6), and hence $K_\mathscr{L} \equiv_\pi 0$. Set $C := \psi^{-1}(y) \subset S$ and observe that $C^2 = 0$ since ψ has reduced fibers by assumption.

By general choice of B, we may also assume that the collection of charts $\{p_i : X_i \to X\}_{i\in I}$ induces an analytic quasi-étale $\mathbb{Q}$-structure $\{q_i : S_i \to S\}_{i\in I}$ on S where $S_i := p_i^{-1}(S)$ and $q_i := p_{i|S_i}$, and that either $q_i^{-1}\mathscr{L}$ is defined by the 1-form df_i where f_i is a holomorphic function on S_i such that df_i has isolated zeroes (Lemma 2.6) or it is given by the local 1-form $z_2dz_1 + \lambda_i z_1 dz_2$ where (z_1, z_2) are analytic coordinates on S_i and $\lambda_i \in \mathbb{Q}_{>0}$.

Step 2. By assumption, any irreducible component of C is invariant under $\mathscr{L}$. Let x_i be a point on S_i with $p_i(x_i) \in C$.

Suppose that $q_i^{-1}\mathscr{L}$ is defined by the 1-form df_i where f_i is a holomorphic function on S_i such that df_i has isolated zeroes. Suppose furthermore that $q_i^{-1}(C)$ is given at x_i by equation $t_i = 0$ and that $f_i(x_i) = 0$. Then, $f_i = t_i g_i$ for some local holomorphic function g_i on S_i at x_i and $-\mathrm{CS}(q_i^{-1}\mathscr{L}, q_i^{-1}(C), x_i)$ is equal to the vanishing order of $g_{i|q_i^{-1}(C)}$ at x_i. In particular, we have $\mathrm{CS}(q_i^{-1}\mathscr{L}, q_i^{-1}(C), x_i) \leqslant 0$.

Suppose now that $q_i^{-1}\mathscr{L}$ is given by the local 1-form $z_2dz_1+\lambda_i z_1 dz_2$ where (z_1, z_2) are analytic coordinates on S_i and $\lambda_i \in \mathbb{Q}_{>0}$. If x_i is not a singular point of $q_i^{-1}\mathscr{L}$, then $\mathrm{CS}(q_i^{-1}\mathscr{L}, q_i^{-1}(C), x_i) = 0$. Suppose otherwise. Then

$$\mathrm{CS}(q_i^{-1}\mathscr{L}, q_i^{-1}(C), x_i) = \begin{cases} -\lambda_i^{-1} < 0 & \text{if } q_i^{-1}(C) = \{z_1 = 0\}, \\ -\lambda_i < 0 & \text{if } q_i^{-1}(C) = \{z_2 = 0\}, \\ 2-\lambda_i-\lambda_i^{-1} = -\lambda_i(1-\lambda_i^{-1})^2 \leqslant 0 & \text{if } q_i^{-1}(C) = \{z_1z_2 = 0\}. \end{cases}$$

Together with the Camacho–Sad formula (Proposition 3.12) and using $C^2 = 0$, this shows that $\lambda_i = 1$ for all $i \in I$ as above.

Step 3. By Step 2, for any $i \in I$ such that $p_i(X_i)\cap C \neq \emptyset$, $p_i^{-1}\mathscr{G}$ is defined by a closed 1-form with zero set of codimension at least two. But then Lemma 3.6 yields a contradiction since $c_1(\mathscr{N}_{\mathscr{L}}) \cdot C = 2$. This finishes the proof of the claim. ■

Next, we show that, shrinking Y°, if necessary, $\psi^{-1}(y)$ is irreducible for every $y \in Y^\circ$. We argue by contradiction again and assume that there is an irreducible hypersurface D in Y such that $\psi^{-1}(y)$ is reducible for a general point $y \in D$. We maintain the notation of Step 1 of the proof of Claim 4.3. By Claim 4.3, there is an irreducible component C_1 of C which is not invariant under $\mathscr{L}$. Since C is reducible by assumption, we must have $C_1^2 < 0$. On the other hand, by Lemma 3.4, we have $C_1^2 = K_{\mathscr{L}} \cdot C_1 + C_1^2 \geqslant 0$, yielding a contradiction. This shows that ψ has irreducible fibers at codimension one points in Y.

Applying [24, Theorem II.2.8], we conclude that $\psi^\circ := \psi_{|X^\circ}$ is a $\mathbb{P}^1$-bundle. By choice of Y°, $F := \psi^{-1}(y) \cong \mathbb{P}^1$ is not tangent to $\mathscr{G}$ for every $y \in Y^\circ$. Since $c_1(\mathscr{N}_{\mathscr{L}}) \cdot F = 2$, we see that $\mathscr{G}$ is transverse to ψ along F, completing the proof of the lemma. ■

Proof of Proposition 4.1 We maintain notation and assumptions of Proposition 4.1. Let $(D_j)_{j\in J}$ be the possibly empty set of hypersurfaces D in Y such that ψ^*D is not integral. Since ψ is a Mori fiber space, we must have $\psi^*D_j = m_jG_j$ for some integer $m_j \geqslant 2$ and some prime divisor G_j. Let $U_j \subseteq Y$ be a Zariski open neighborhood of the generic point of D_j and let $g_j\colon V_j \to U_j$ be a cyclic cover that branches along $D_j \cap U_j$ with ramification index m_j. The normalization M_j of the fiber product $V_j \times_Y X$ has geometrically reduced fibers over general points of $g_j^{-1}(D_j \cap U_j)$ by [19, Théorème 12.2.4]. Applying Lemma 4.2 to $\psi_j\colon M_j \to V_j$, we see that ψ_j is a $\mathbb{P}^1$-bundle over a Zariski open set in V_j whose complement has codimension at least two. Moreover, $\mathscr{G}_{|M_j}$ induces a flat Ehresmann connection on this $\mathbb{P}^1$-bundle. In particular, there exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least two such that the restriction ψ° of ψ to $X^\circ :=$

$\psi^{-1}(Y^\circ)$ has irreducible fibers. Using [19, Théorème 12.2.4] together with Lemma 4.2 again, we see that we may also assume without loss of generality that ψ is a $\mathbb{P}^1$-bundle over $Y^\circ \setminus \cup_{j\in J} D_j$ and that $\mathscr{G}$ is everywhere transverse to ψ over $Y^\circ \setminus \cup_{j\in J} D_j$. Recall that Y is $\mathbb{Q}$-factorial ([23, Lemma 5.1.5]). Since codim $Y \setminus Y^\circ \geqslant 2$ and $K_{\mathscr{G}} \equiv 0$, we must have

$$K_Y + \sum_{i\in I} \frac{m_i - 1}{m_i} D_i \equiv 0.$$

On the other hand, the proof of [14, Corollary 4.5] shows that the pair $\big(Y, \sum_{i\in I} \frac{m_i-1}{m_i} D_i\big)$ is klt. Applying [30, Corollary V.4.9], we conclude that $K_Y + \sum_{i\in I} \frac{m_i-1}{m_i} D_i$ is torsion. Let $g\colon T \to Y$ be the index one cover of the pair $\big(Y, \sum_{i\in I} \frac{m_i-1}{m_i} D_i\big)$ ([31, Section 2.4]). By construction, we have

$$K_T \sim_{\mathbb{Q}} g^*\Big(K_Y + \sum_{i\in I} \frac{m_i - 1}{m_i} D_i\Big) \sim_{\mathbb{Q}} 0.$$

Replacing T by a further quasi-étale cover, we may therefore assume that $K_T \sim_{\mathbb{Z}} 0$. Then, T has canonical singularities. Shrinking Y°, if necessary, we may assume that T° is smooth, that the normalization M° of the fiber product $T^\circ \times_Y X$ has reduced and irreducible fibers over T°, and that the map $M^\circ \to X^\circ$ is a quasi-étale cover. We may finally assume that M° is a $\mathbb{P}^1$-bundle over T° and that $\mathscr{G}_{|M^\circ}$ yields a flat Ehresmann connection on $M^\circ \to T^\circ$ (Lemma 4.2). ■

5 Algebraic integrability I

The following is the main result of this section. We confirm the Ekedahl, Shepherd-Barron and Taylor conjecture ([13]) for mildly singular codimension one foliations with trivial canonical class on projective varieties with klt singularities and $\nu(X) = -\infty$.

Theorem 5.1. Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension one foliation on X. Suppose that $\mathscr{G}$ is canonical, and that it is closed under pth powers for almost all primes p. Suppose furthermore that K_X is not pseudo-effective, and that $K_{\mathscr{G}} \equiv 0$. Then $\mathscr{G}$ is algebraically integrable.

Proof. For the reader's convenience, the proof is subdivided into a number of steps.
Step 1. Arguing as in Steps 1 and 2 of the proof of [12, Theorem 9.4], we see that we may assume without loss of generality that X is $\mathbb{Q}$-factorial and that there exists a Mori fiber space $\psi\colon X \to Y$.

Step 2. We first show that $\dim X - \dim Y = 1$. We argue by contradiction and assume that $\dim X - \dim Y \geqslant 2$. Let F be a general fiber of ψ. Note that F has klt singularities, and that $K_F \sim_{\mathbb{Z}} K_{X|F}$ by the adjunction formula. Moreover, F is a Fano variety by assumption. Let $\mathscr{H}$ be the foliation on F induced by $\mathscr{G}$. Since $c_1(\mathscr{N}_{\mathscr{G}}) \equiv -K_X$ is relatively ample, we see that $\mathscr{H}$ has codimension one since otherwise $c_1(\mathscr{N}_{\mathscr{G}})$ comes from Y in a neighborhood of F. By [12, Proposition 3.6], we have $K_{\mathscr{H}} \sim_{\mathbb{Z}} K_{\mathscr{G}|F} - B$ for some effective Weil divisor B on F. Suppose that $B \neq 0$. Applying [9, Theorem 4.7] to the pull-back of $\mathscr{H}$ on a resolution of F, we see that $\mathscr{H}$ is uniruled. This implies that $\mathscr{G}$ is uniruled as well since F is general. But this contradicts [12, Proposition 4.22], and shows that $B = 0$. By [12, Proposition 4.22] applied to $\mathscr{H}$, we see that $\mathscr{H}$ is canonical. Finally, one readily checks that $\mathscr{H}$ is closed under pth powers for almost all primes p.

Let $S \subseteq F$ be a two-dimensional complete intersection of general elements of a very ample linear system $|H|$ on F. Notice that S has klt singularities and hence quotient singularities. Let $\mathscr{L}$ be the foliation of rank one on S induced by $\mathscr{H}$. By [12, Proposition 3.6], we have $\det \mathscr{N}_{\mathscr{L}} \cong (\det \mathscr{N}_{\mathscr{H}})_{|S}$. It follows that

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = K_F^2 \cdot H^{\dim F - 2} > 0. \tag{5.1}$$

Recall from [17, Proposition 9.3] that there is an open set $F^\circ \subseteq F$ with quotient singularities whose complement in F has codimension at least three. Let $\{p_\alpha \colon F_\alpha \to F^\circ\}_{\alpha \in A}$ be a quasi-étale $\mathbb{Q}$-structure on F° (Fact 3.2). Notice that $p_\alpha^{-1}\mathscr{H}$ is obviously closed under pth powers for almost all primes p. By [27, Corollary 7.8], we see that we may assume that $p_\alpha^{-1}\mathscr{H}_{|F^\circ}$ is defined at singular points (locally for the analytic topology) by the 1-form $x_2 dx_1 + \lambda x_1 dx_2$ where $(x_1, \ldots, x_n)$ are local analytic coordinates on F_α and $\lambda \in \mathbb{Q}_{>0}$. By general choice of S, we may assume that $S \subset F^\circ$ and that the collection of charts $\{p_\alpha \colon F_\alpha \to F^\circ\}_{\alpha \in A}$ induces a quasi-étale $\mathbb{Q}$-structure $\{p_{\alpha|S_\alpha} \colon S_\alpha \to S\}_{\alpha \in A}$ on S, where $S_\alpha := p_\alpha^{-1}(S)$. We may assume in addition that $(p_{\alpha|S_\alpha})^{-1}\mathscr{L}$ is defined at a singular point x (locally for the analytic topology) by a closed holomorphic 1-form with isolated zeroes or by the 1-form $x_2 dx_1 + \lambda x_1 dx_2$ where (x_1, x_2) are local analytic coordinates on S_α and $\lambda \in \mathbb{Q}_{>0}$. In the former case, we have $\mathrm{BB}\big((p_{\alpha|S_\alpha})^{-1}\mathscr{L}, x\big) = 0$, and in the latter case, we have $\mathrm{BB}\big((p_{\alpha|S_\alpha})^{-1}\mathscr{L}, x\big) = -\lambda(1-\lambda^{-1})^2$. Applying Proposition 3.9, we obtain

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_x \mathrm{BB}^{\mathbb{Q}}(\mathscr{L}, x) \leqslant 0.$$

But this contradicts inequality (5.1) and shows that $\dim X - \dim Y = 1$.

By [17, Proposition 9.3], there exists a closed subset $Z \subset X$ of codimension at least 3 such that $X \setminus Z$ has quotient singularities. Let $\{p_\beta \colon X_\beta \to X \setminus Z\}_{\beta \in B}$ be a quasi-étale $\mathbb{Q}$-structure on $X \setminus Z$ (Fact 3.2). Notice that $p_\beta^{-1}\mathscr{G}$ is obviously closed under pth powers for almost all primes p. By [27, Corollary 7.8], we see that there exists a closed subset $Z_\beta \subseteq X_\beta$ of codimension at least 3 in X_β such that $p_\beta^{-1}\mathscr{G}_{|X_\beta \setminus Z_\beta}$ is defined at singular points (locally for the analytic topology) by the 1-form $x_2 dx_1 + \lambda x_1 dx_2$ where $(x_1, \ldots, x_n)$ are local analytic coordinates on X_β and $\lambda \in \mathbb{Q}_{>0}$. Therefore, Proposition 4.1 applies. There exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least two and a finite cover $g \colon T \to Y$ such that the following holds. Set $T^\circ := g^{-1}(Y^\circ)$ and $X^\circ := \psi^{-1}(Y^\circ)$.

(1) The variety T has canonical singularities and $K_T \sim_{\mathbb{Z}} 0$; T° is smooth.
(2) The normalization M° of the fiber product $T^\circ \times_Y X$ is a $\mathbb{P}^1$-bundle over T° and the map $M^\circ \to X^\circ$ is a quasi-étale cover.
(3) The pull-back of $\mathscr{G}_{|X^\circ}$ on M° yields a flat Ehresmann connection on $M^\circ \to T^\circ$.

Step 3. By [18, Corollary 3.6] applied to T, we see that there exists an abelian variety A as well as a projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$ and augmented irregularity $\widetilde{q}(Z) = 0$ (we refer to [18, Definition 3.1] for this notion), and a quasi-étale cover $f \colon A \times Z \to T$.

Recall that f branches only on the singular set of T, so that $f^{-1}(T^\circ)$ is smooth. On the other hand, since $f^{-1}(T^\circ)$ has the complement of codimension at least two in $A \times Z_{\mathrm{reg}}$, we have $\pi_1(A \times Z_{\mathrm{reg}}) \cong \pi_1\big(f^{-1}(T^\circ)\big)$. Now, consider the representation

$$\rho : \pi_1(A \times Z_{\mathrm{reg}}) \cong \pi_1\big(f^{-1}(T^\circ)\big) \to \pi_1(T^\circ) \to \mathrm{PGL}(2, \mathbb{C})$$

induced by $\mathscr{G}_{|M^\circ}$. By [15, Theorem I], the induced representation

$$\pi_1(Z_{\mathrm{reg}}) \to \pi_1(A) \times \pi_1(Z_{\mathrm{reg}}) \cong \pi_1(A \times Z_{\mathrm{reg}}) \to \mathrm{PGL}(2, \mathbb{C})$$

has finite image. Thus, replacing Z by a quasi-étale cover, if necessary, we may assume without loss of generality that ρ factors through the projection $\pi_1(A \times Z_{\mathrm{reg}}) \to \pi_1(A)$. Let P be the corresponding $\mathbb{P}^1$-bundle over A. The natural projection $P \to A$ comes with a flat connection $\mathscr{G}_P \subset T_P$. By the GAGA theorem, P is a projective variety. By assumption, its pull-back to $A \times Z_{\mathrm{reg}}$ agrees with $f^{-1}(T^\circ) \times_{T^\circ} M^\circ$ over $f^{-1}(T^\circ)$. Moreover, the pull-backs on $A \times Z_{\mathrm{reg}}$ of the foliations $\mathscr{G}$ and $\mathscr{G}_P$ agree as well, wherever this makes sense. In particular, $\mathscr{G}$ is algebraically integrable if and only if so is $\mathscr{G}_P$. Now, one readily checks that $\mathscr{G}_P$ is closed under pth powers for almost all primes p. Theorem 5.1 then follows from [12, Proposition 9.3]. ■

6 Algebraic integrability II

In this section, we provide an algebraicity criterion for leaves of mildly singular codimension one algebraic foliations with numerically trivial canonical class on klt spaces X with $\nu(X) = 1$ (Theorem 6.4). We confirm the Ekedahl, Shepherd-Barron and Taylor conjecture in this special case.

We will need the following auxiliary result, which might be of independent interest.

Proposition 6.1. Let X be a projective variety with klt singularities and let $\beta\colon Z \to X$ be a resolution of singularities. Let also $\varphi\colon Z \to \mathfrak{H} := \mathbb{D}^N/\Gamma$ be a generically finite morphism to a quotient of the polydisc $\mathbb{D}^N$ with $N \geqslant 2$ by an arithmetic irreducible lattice in $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^N$. Let $\mathscr{H}$ be a codimension one foliation on $\mathfrak{H}$ induced by one of the tautological foliations on $\mathbb{D}^N$, and denote by $\mathscr{G}$ the induced foliation on X. Suppose that $\varphi(Z)$ is not tangent to $\mathscr{H}$. Suppose furthermore that there exists an open set $X^\circ \subseteq X$ with complement of codimension at least 3 and an analytic quasi-étale $\mathbb{Q}$-structure $\{p_i : X_i \to X^\circ\}_{i\in I}$ on X° such that either $p_i^{-1}\mathscr{G}$ is regular or it is given by the local 1-form $d(x_1x_2)$ where $(x_1,\ldots,x_n)$ are analytic coordinates on X_i. Then X is of general type.

Proof. For the reader's convenience, the proof is subdivided into a number of steps. By a result of Selberg, there exists a torsion-free subgroup Γ_1 of Γ of finite index. Set $\mathfrak{H}_1 := \mathbb{D}^N/\Gamma_1$, and denote by $\pi\colon \mathfrak{H}_1 \to \mathfrak{H}$ the natural finite morphism. Recall that $\mathfrak{H}$ has isolated quotient singularities. It follows that π is a quasi-étale cover since $N \geqslant 2$. One readily checks that there is only one separatrix for $\mathscr{H}$ at any (singular) point.
Step 1. We first show the following.

Claim 6.2. The rational map $\varphi \circ \beta^{-1}\colon X \dashrightarrow \mathfrak{H}$ is a well-defined morphism over X°.

Proof. Let Z_1 be the normalization of $Z \times_{\mathfrak{H}} \mathfrak{H}_1$, and let X_1 be the normalization of X in the function field of Z_1.

$$\begin{tikzcd}
X_1 \arrow[d, "f_1"'] & Z_1 \arrow[l, "\beta_1\text{, birational}"'] \arrow[r, "\varphi_1\text{, generically finite}"] \arrow[d, "g_1"] & \mathfrak{H}_1 \arrow[d, "\pi\text{, quasi-étale}"] \\
X & Z \arrow[l, "\beta\text{, birational}"] \arrow[r, "\varphi\text{, generically finite}"'] & \mathfrak{H}.
\end{tikzcd}$$

Let $(D_j)_{j\in J}$ be the set of codimension one irreducible components of the branched locus of f_1. Observe that φ maps $\beta_*^{-1}(D_j)$ to a (singular) point. It follows that D_j is invariant under $\mathscr{G}$ since there is only one separatrix for $\mathscr{H}$ at any point and $\varphi(Z)$ is not tangent to $\mathscr{H}$ by assumption. Let $\widehat{X}_1 \to X_1$ be a finite cover such that the induced cover $p\colon \widehat{X}_1 \to X$ is Galois. We may assume without loss of generality that p is quasi-étale away from the branch locus of f_1. Therefore, there exist positive integers $(m_j)_{j\in J}$ such that

$$K_{\widehat{X}_1} = p^*\Big(K_X + \sum_{j\in J} \frac{m_j - 1}{m_j} D_j\Big).$$

By Lemma 6.3, the pair $\big(X^\circ, \sum_{j\in J} \frac{m_j-1}{m_j} D_{j_{|X^\circ}}\big)$ is klt. It follows that $\widehat{X}_1^\circ := p^{-1}(X_1^\circ)$ has klt singularities as well.

Suppose that the rational map $\varphi \circ \beta^{-1}\colon X \dashrightarrow \mathfrak{H}$ is not a well-defined morphism on X°. By the rigidity lemma, there exist $x \in X^\circ$ such that $\dim \varphi\big(\beta^{-1}(x)\big) \geqslant 1$. Let $\widehat{Z}_1$ be the normalization of $\widehat{X}_1 \times_{X_1} Z_1$, and denote by $\widehat{\beta}_1\colon \widehat{Z}_1 \to \widehat{X}_1$ and $\widehat{\varphi}_1\colon \widehat{Z}_1 \to \mathfrak{H}_1$ the natural morphisms. Then, there is a point $x_1 \in \widehat{X}_1^\circ := p^{-1}(X_1^\circ)$ with $p(x_1) = x$ such that $\dim \widehat{\varphi}_1\big(\widehat{\beta}_1^{-1}(x)\big) \geqslant 1$. On the other hand, $\widehat{\beta}_1^{-1}(x)$ is rationally chain-connected since $\widehat{X}_1^\circ$ has klt singularities ([21, Corollary 1.6]). This yields a contradiction since $\mathfrak{H}_1$ is obviously hyperbolic. This finishes the proof of the claim. ■

Step 2. Let $F \subseteq Z$ be a prime divisor which is not β-exceptional and set $G = \beta(F)$. We show that $\dim \varphi(F) \geqslant 1$. We argue by contradiction and assume that $\dim \varphi(F) = 0$. Let $S \subseteq X$ be a two-dimensional complete intersection of general elements of a very ample linear system on X. We may assume without loss of generality that S is contained in X° and that it has klt singularities. Set $C := S \cap G$. By Step 1, the rational map $\varphi \circ \beta^{-1}$ is a well-defined morphism in a neighborhood of S. But then $C^2 < 0$ since C is contracted by the generically finite morphism $\varphi \circ \beta^{-1}{}_{|S}\colon S \to \mathfrak{H}$. On the other hand, arguing as in Step 1, we see that G must be invariant under $\mathscr{G}$. Applying Lemma 3.6, we see that $C^2 = C \cdot G = 0$, yielding a contradiction.

Step 3. We use the notation of Step 1. Recall that $\pi\colon \mathfrak{H}_1 \to \mathfrak{H}$ is étale away from finitely many points. By Step 2, the natural map $Z \times_{\mathfrak{H}} \mathfrak{H}_1 \to Z$ is a quasi-étale cover away from the exceptional locus of β. This immediately implies that f_1 is a quasi-étale cover.

Let $Z_2 \to Z_1$ be a resolution of singularities. We obtain a diagram as follows:

$$\begin{tikzcd} X_1 \arrow[d, "f_1\text{, quasi-étale}"'] & Z_2 \arrow[l, "\beta_2\text{, birational}"'] \arrow[r, "\varphi_2\text{, generically finite}"] \arrow[d, "g_2"] & \mathfrak{H}_1 \arrow[d, "\pi\text{, quasi-étale}"] \\ X & Z \arrow[l, "\beta\text{, birational}"] \arrow[r, "\varphi\text{, generically finite}"'] & \mathfrak{H}. \end{tikzcd}$$

Let $\mathscr{E}_k$ $(1 \leqslant k \leqslant N)$ be the codimension one regular foliations on $\mathfrak{H}_1$ induced by the tautological foliations on $\mathbb{D}^N$ so that $\Omega^1_{\mathfrak{H}_1} \cong \bigoplus_{1\leqslant k\leqslant N} \mathscr{N}^*_{\mathscr{E}_i}$. Set $n := \dim X$. We may assume without loss of generality that the natural map $\bigoplus_{1\leqslant k\leqslant n} \varphi_2^*\mathscr{N}^*_{\mathscr{E}_i} \to \Omega^1_{Z_2}$ is generically injective. Now observe that the line bundle $\mathscr{N}^*_{\mathscr{E}_i}$ is hermitian semipositive, so that $\varphi_2^*\mathscr{N}^*_{\mathscr{E}_i}$ is nef. On the other hand, we have $c_1(\varphi_2^*\mathscr{N}^*_{\mathscr{E}_1})\cdot\dots\cdot c_1(\varphi_2^*\mathscr{N}^*_{\mathscr{E}_n}) > 0$. This immediately implies that $\kappa(Z_2) = \nu(Z_2) = \dim Z_2$. It follows that $\kappa(X_1) = \nu(X_1) = \dim X_1$ since β_2 is a birational morphism. Applying [30, Proposition 2.7], we see that $\nu(X) = \nu(X_1) = \dim X_1 = \dim X$ since $K_{X_1} \sim_{\mathbb{Q}} f_1^*K_X$. This completes the proof of the proposition.

Lemma 6.3. Let X be a variety of dimension n with quotient singularities and let $\mathscr{G}$ be a codimension one foliation on X. Let $\{p_i : X_i \to X\}_{i\in I}$ be an analytic quasi-étale $\mathbb{Q}$-structure on X. Suppose that either $p_i^{-1}\mathscr{G}$ is regular or it is given by the 1-form $x_2dx_1 + \lambda_i x_1 dx_2$ where $(x_1, \dots, x_n)$ are analytic coordinates on X_i and $\lambda_i \in \mathbb{Q}_{>0}$. Let $(D_j)_{j\in J}$ be pairwise distinct prime divisors on X. Suppose that D_j is invariant under $\mathscr{G}$ for every $j \in J$. Then the pair $\left(X, \sum_{j\in J} a_jD_j\right)$ has klt singularities for any real numbers $0 \le a_j < 1$.

Proof. It suffices to prove the statement locally on X for the analytic topology. We may therefore assume without loss of generality that there exist a (connected) smooth analytic complex manifold Y and a finite Galois holomorphic map $p\colon Y \to X$, totally branched over the singular locus and étale outside of the singular set. We may also assume that $p^{-1}\mathscr{G}$ is given by the 1-form $y_2dy_1 + \lambda y_1dy_2$, where $\lambda \in \mathbb{Q}_{>0}$ and $(y_1, \dots, y_n)$ are analytic coordinates on Y. This immediately implies that the divisor $\sum_{j\in J} C_i$ has normal crossing support, where $C_j := p^{-1}(D_j)$. Therefore, the pair $\left(Y, \sum_{j\in J} a_jC_j\right)$ has klt singularities for any real numbers $0 \le a_j < 1$. On the other hand, we have

$$K_Y + \sum_{j\in J} a_jC_j = p^*\big(K_X + \sum_{j\in J} a_jD_j\big)$$

and thus $\left(X, \sum_{j\in J} a_jD_j\right)$ has klt singularities as well. This finishes the proof of the lemma. ∎

The following is the main result of this section.

Theorem 6.4. Let X be a normal projective variety with klt singularities, and let $\mathscr{G}$ be a codimension one foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}} \equiv 0$ and that $\nu(X) = 1$. Suppose in addition that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable.

Proof. For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. Applying [12, Proposition 8.14] together with Lemma 2.4, we may assume without loss of generality that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X. To prove the statement, it then suffices to show that $\dim X = 1$.

Step 2. Arguing as in Steps 1 and 2 of the proof of [12, Theorem 10.4], we see that we may also assume that X is $\mathbb{Q}$-factorial and that K_X is movable.

Step 3. Recall from [17, Proposition 9.3] that there is an open set $X^\circ \subseteq X$ with quotient singularities whose complement in X has codimension at least three. Let $\{p_\alpha \colon X_\alpha \to X^\circ\}_{\alpha\in A}$ be a quasi-étale $\mathbb{Q}$-structure on X° (Fact 3.2). Notice that $p_\alpha^{-1}\mathscr{G}$ is obviously closed under pth powers for almost all primes p. By [27, Corollary 7.8], we see that we may assume that $p_\alpha^{-1}\mathscr{G}_{|X^\circ}$ is defined at singular points (locally for the analytic topology) by the 1-form $x_2dx_1 + \lambda x_1 dx_2$ where $(x_1, \ldots, x_n)$ are local analytic coordinates on X_α and $\lambda \in \mathbb{Q}_{>0}$.

Let $S \subseteq X$ be a two-dimensional complete intersection of general elements of a very ample linear system $|H|$ on F. We may assume that $S \subset X^\circ$ and that S has klt singularities. Let also $\mathscr{L}$ be the foliation of rank one on S induced by $\mathscr{G}$. By general choice of S, we may also assume that the collection of charts $\{p_\alpha \colon X_\alpha \to X^\circ\}_{\alpha\in A}$ induces a quasi-étale $\mathbb{Q}$-structure $\{q_\alpha \colon S_\alpha \to S\}_{\alpha\in A}$ on S, where $S_\alpha := p_\alpha^{-1}(S)$ and $q_\alpha := p_{\alpha|S_\alpha}$, and that $q_\alpha^{-1}\mathscr{L}$ is defined at a singular point x (locally for the analytic topology) by a closed holomorphic 1-form with isolated zeroes or by the 1-form $x_2dx_1 + \lambda x_1dx_2$ where (x_1, x_2) are local analytic coordinates on S_α and $\lambda \in \mathbb{Q}_{>0}$. In the former case, we have $\mathrm{BB}(q_\alpha^{-1}\mathscr{L}, x) = 0$, and in the latter case, we have $\mathrm{BB}(q_\alpha^{-1}\mathscr{L}, x) = -\lambda(1-\lambda^{-1})^2$. On the other hand, by [12, Proposition 3.6], we have $\det \mathscr{N}_{\mathscr{L}} \cong (\det \mathscr{N}_{\mathscr{G}})_{|S}$. It follows that

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = K_X^2 \cdot H^{\dim F - 2} \geqslant 0$$

since K_X is movable by Step 2. This implies that $\lambda = 1$ by Proposition 3.9.

Step 4. Let $\beta \colon Z \to X$ be a resolution of singularities with exceptional set E, and suppose that E is a divisor with simple normal crossings. Suppose in addition that the restriction of β to $\beta^{-1}(X_{\mathrm{reg}})$ is an isomorphism. Let E_1 be the reduced divisor on Z whose support is the union of all irreducible components of E that are invariant under $\beta^{-1}\mathscr{G}$. Note that $-c_1(\mathscr{N}_{\mathscr{G}}) \equiv K_X$ by assumption. By [12, Proposition 4.9] and [12, Remark 4.8], there exists a rational number $0 \leqslant \varepsilon < 1$ such that

$$\nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big) = \nu\big(-c_1(\mathscr{N}_{\mathscr{G}})\big) = 1.$$

By [34, Theorem 6] applied to $\beta^{-1}\mathscr{G}$, we may assume that there exists an arithmetic irreducible lattice Γ of $\mathrm{PSL}(2,\mathbb{R})^N$ for some integer $N \geqslant 2$, as well as a morphism $\varphi\colon Z \to \mathfrak{H} := \mathbb{D}^N/\Gamma$ of quasi-projective varieties such that $\mathscr{G} = \varphi^{-1}\mathscr{H}$, where $\mathscr{H}$ is a weakly regular codimension one foliation on $\mathfrak{H}$ induced by one of the tautological foliations on the polydisc $\mathbb{D}^N$. Note that φ is generically finite as there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X. Moreover, $\varphi(Z)$ is not tangent to $\mathscr{H}$ since $\mathscr{G}$ has codimension one.

Proposition 6.1 then says that X must be of general type. Thus, $\dim X = \nu(X) = 1$, completing the proof of the theorem. ■

The same argument used in the proof of Theorem 6.4 shows that the following holds using the fact that weekly regular foliations on smooth spaces are regular (we refer to [12, Section 5] for this notion).

Theorem 6.5. Let X be a normal projective variety with klt singularities, and let $\mathscr{G}$ be a weakly regular codimension one foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}} \equiv 0$. Suppose in addition that $\nu(X) = 1$. Then $\mathscr{G}$ is algebraically integrable.

7 Foliations defined by closed rational 1-forms

Let X be a normal projective variety, and let $\mathscr{G} \subset T_X$ be a codimension one foliation. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$. Then the twisted rational 1-form ω is not uniquely determined by $\mathscr{G}$ in general. The following result addresses this issue.

Proposition 7.1. Let X be a normal projective variety with klt singularities, and let $\mathscr{G} \subset T_X$ be a codimension one foliation with canonical singularities. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ whose zero set has codimension at least two. Suppose furthermore that K_X is not pseudo-effective, and that $K_{\mathscr{G}} \equiv 0$. Then, there exists a quasi-étale cover $f\colon X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form with zero set of codimension at least two.

To prove Proposition 7.1, we will need the following auxiliary result.

Lemma 7.2. Let X be a smooth quasi-projective variety, and let $\mathscr{G} \subset T_X$ be a codimension one foliation with canonical singularities. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle whose zero set has codimension at least two. Suppose in addition that the residue of ω at a general point of any irreducible component of its polar set is zero. Let B be a codimension two irreducible component of the singular set of $\mathscr{G}$ which is contained in the polar set of ω and let $(D_i)_{i\in I}$ be the set of irreducible components of the polar set of ω containing B. If $x \in B$ is a general point, then $\mathscr{G}$ has a first integral at x of the form $f := \prod_{i\in I} f_i^{m_i}$ where f_i is a local holomorphic equation of D_i at x and m_i is a positive integer for every $i \in I$.

Proof. Let $S \subseteq X$ be a two-dimensional complete intersection of general elements of some very ample linear system on X. Notice that S is smooth and that the foliation $\mathscr{L}$ of rank one on S induced by $\mathscr{G}$ has canonical singularities in a Zariski open neighborhood of the generic point of $B \cap S$ by Proposition 2.10.

Let now $x \in B \cap S$ be a general point. Let also f_i be a local holomorphic equation of D_i at x. By [7, Théorème III.2.1, Première partie], there exist positive integers m_i and a local holomorphic function g at x such that $\mathscr{G}$ is given in some analytic open neighborhood of x by the 1-form

$$d\Big(\frac{g}{\prod_{i\in I} f_i^{m_i}}\Big).$$

Set $f := \prod_{i\in I} f_i^{m_i}$. We may also assume that f_i and g are relatively prime at x for any $i \in I$. By general choice of S, the restrictions of g and f to S are relatively prime nonzero local holomorphic functions. In particular, $\mathscr{L}$ is given by $d\big(\frac{g}{f}\big)$ on some analytic open neighborhood of x in S. Now, recall from [28, Observation I.2.6], that $\mathscr{L}$ has only finitely many separatrices at x. This immediately implies that $g(x) \neq 0$. Let $i_0 \in I$. Then $\big(f_{i_0} g^{-\frac{1}{m_{i_0}}}\big)^{m_{i_0}} \cdot \prod_{i\in I\setminus\{i_0\}} f_i^{m_i}$ is a holomorphic first integral of $\mathscr{G}$ at x, proving the lemma. ∎

The proof of Proposition 7.1 makes use of the following minor generalization of [27, Proposition 3.11].

Lemma 7.3. Let X be a normal variety with quotient singularities and let $\mathscr{G}$ be a codimension one foliation on X with canonical singularities. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with zero set of codimension at least two. Let D be the reduced divisor on X whose support is the polar set of ω. Then (X, D) has log canonical singularities.

Proof. Let $\{p_\alpha : X_\alpha \to X\}_{\alpha \in A}$ be a quasi-étale $\mathbb{Q}$-structure on X. The claim follows easily from [27, Proposition 3.11] applied to $p_\alpha^{-1}\mathscr{G}$ on X_α for any $\alpha \in A$ together with [25, Proposition 3.16]. ■

We will also need the following generalization of [27, Corollary 3.10].

Lemma 7.4. Let X be a normal projective variety with klt singularities, and let $\mathscr{G} \subset T_X$ be a codimension one foliation with canonical singularities. Suppose that $K_{\mathscr{G}} \equiv 0$. Suppose furthermore that X is uniruled and let $X \dashrightarrow R$ be the maximal rationally chain connected fibration. Then any algebraic leaf of $\mathscr{G}$ dominates R.

Proof. Consider a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\beta,\ \text{birational}"] \arrow[d, "f_1"'] & X \arrow[d, dashed] \\ R_1 \arrow[r, "\text{birational}"] & R, \end{tikzcd}$$

where X_1 and R_1 are smooth projective varieties. Applying [16, Theorem 1.1], we see that R_1 is not uniruled. This in turn implies that K_{R_1} is pseudo-effective by [4, Corollary 0.3]. Set $\mathscr{M} := f_1^*\mathscr{O}_{R_1}(-K_{R_1})$ and $q := \dim R_1$, and let $\omega \in H^0\big(X_1, \Omega^q_{X_1} \otimes \mathscr{M}\big)$ be the twisted q-form induced by df_1. Let $\mathscr{G}_1$ be the pull-back of $\mathscr{G}$ on X_1. Then, ω yields a nonzero section $\sigma \in H^0\big(X_1, \wedge^q(\mathscr{G}_1^*) \otimes \mathscr{M}\big)$ by [12, Proposition 4.22]. On the other hand, $\wedge^q\mathscr{G}_1^* \otimes \mathscr{M}$ is semistable with respect to the pull-back on X_1 of any ample divisor on X by [12, Lemma 8.14] together with [8, Theorem 5.1]. This immediately implies that codimension one zeroes of σ are β-exceptional. Now, let F be the closure of an algebraic leaf of $\mathscr{G}$ and denote by F_1 its proper transform on X_1. If F_1 does not dominate R_1, then σ must vanish along F_1, yielding a contradiction. ■

Proof of Proposition 7.1 For the reader's convenience, the proof is subdivided into a number of steps. By assumption, there exist prime divisors $(D_i)_{1\leqslant i\leqslant r}$ on X and positive integers $(m_i)_{1\leqslant i\leqslant r}$ such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X\big(\sum_{1\leqslant i\leqslant r} m_iD_i\big) \otimes \mathscr{L}$.

Step 1. Let $X \dashrightarrow R$ be the maximal rationally chain connected fibration. Recall that it is an almost proper map and that its general fibers are rationally chain connected. Consider a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\beta,\ \text{birational}"] \arrow[d] & X \arrow[d, dashed] \\ R_1 \arrow[r, "\text{birational}"] & R, \end{tikzcd}$$

where X_1 and R_1 are smooth projective varieties. Notice that general fibers of $X_1 \to R_1$ are rationally chain connected by [21, Theorem 1.2] and recall from [7, Proposition III. 1.1, Première partie], that the hypersurfaces D_i are invariant under $\mathscr{G}$. By Lemma 7.4, we conclude that D_i dominates R for every $1 \leqslant i \leqslant r$. Now, ω induces a closed rational 1-form ω_1 on X_1 with values in the flat line bundle $\beta^*\mathscr{L}$.

Arguing as in [27, Section 8.2.1], one concludes that $\beta^*\mathscr{L}$ is torsion if the residue of ω_1 at a general point of D_i is nonzero for some $1 \leqslant i \leqslant r$.

Suppose from now on that the residue at a general point of D_i is zero for any $1 \leqslant i \leqslant r$.

Step 2. Arguing as in Steps 1 and 2 of the proof of [12, Proposition 11.6], we may assume without loss of generality that the following holds.

(1) There is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point in X.

(2) The variety X is $\mathbb{Q}$-factorial.

Step 3. Since K_X is not pseudo-effective by assumption, we may run a minimal model program for X and end with a Mori fiber space ([3, Corollary 1.3.3]). Therefore, there exists a sequence of maps

$$\begin{tikzcd} X := X_0 \arrow[r, dashed, "\varphi_0"] & X_1 \arrow[r, dashed, "\varphi_1"] & \cdots \arrow[r, dashed, "\varphi_{i-1}"] & X_i \arrow[r, dashed, "\varphi_i"] & X_{i+1} \arrow[r, dashed, "\varphi_{i+1}"] & \cdots \arrow[r, dashed, "\varphi_{m-1}"] & X_m \arrow[d, "\psi_m"] \\ & & & & & & Y \end{tikzcd}$$

where the φ_i are either divisorial contractions or flips, and ψ_m is a Mori fiber space. The spaces X_i are normal, $\mathbb{Q}$-factorial, and X_i has klt singularities for all $0 \leqslant i \leqslant m$. Let $\mathscr{G}_i$ be the foliation on X_i induced by $\mathscr{G}$. Arguing as in Step 2 of the proof of [12, Theorem 9.4], we see that $K_{\mathscr{G}_i} \equiv 0$ and that $\mathscr{G}_i$ has canonical singularities. Moreover, ω induces a closed rational 1-form ω_m on X_m with values in a flat line bundle $\mathscr{L}_m$, whose zero set has codimension at least two. By construction, $\mathscr{G}_m$ is given by ω_m, and if $\mathscr{L}_m$ is a torsion flat line bundle then so is $\mathscr{L}$.

We will show in Steps 4 – 6 that either $\mathscr{L}_m$ is torsion, or X_m is smooth, the polar locus of ω_m is a smooth connected hypersurface, say $D_{1,m} := (\varphi_{m-1} \circ \cdots \circ \varphi_0)_* D_1$, and $\mathscr{G}_m$ can be defined by a nowhere vanishing (closed) logarithmic 1-form with poles along $D_{1,m} \sqcup D'_{1,m}$ for some smooth (connected) hypersurface $D'_{1,m}$. Taking this for granted, one concludes that the conclusion of Proposition 7.1 holds for X as in Step 3 of the proof of [12, Proposition 11.6].

For simplicity of notation, we will assume in the following that $X = X_m$, writing $\psi := \psi_m$.

Step 4. Arguing as in Step 5 of the proof of [12, Proposition 11.6], we see that the following additional properties hold.

(1) We have $\dim Y = \dim X - 1$.

(2) Moreover, $\psi(D_1) = Y$, $m_1 = 2$ and $D_1 \cdot F = 1$, where F denotes a general fiber of ψ. Moreover, $\psi(D_i) \subsetneq Y$ for any $2 \leqslant i \leqslant r$.

Step 5. Recall from [17, Proposition 9.3] that klt spaces have quotient singularities in codimension two.

By Lemma 7.3, $\big(X, \sum_{1\leqslant i\leqslant r} D_i\big)$ has log canonical singularities in codimension two. Let $2 \leqslant i \leqslant r$. Lemma 7.5, then says that there exists a Zariski open set Y_i° containing the generic point of $\psi(D_i)$ and a finite cover $g_i^\circ \colon T_i^\circ \to Y_i^\circ$ of smooth varieties such that the normalization M_i° of the fiber product $T_i^\circ \times_Y X$ is a $\mathbb{P}^1$-bundle over T_i° and such that the map $M_i^\circ \to X_i^\circ$ is a quasi-étale cover, where $X_i^\circ := \psi^{-1}(Y_i^\circ)$. By Lemma 2.4, the pull-back $\mathscr{G}_{M_i^\circ}$ of $\mathscr{G}_{|X_i^\circ}$ to M_i° has canonical singularities. Applying Lemma 7.2 and shrinking Y_i°, if necessary, we see that $\mathscr{G}_{M_i^\circ}$ is defined locally for the analytic topology by a closed 1-form with zero set of codimension at least two or by the 1-form $x_2dx_1 + \lambda x_1 dx_2$ where $(x_1, \ldots, x_n)$ are analytic coordinates on X_i and $\lambda \in \mathbb{Q}_{>0}$. On the other hand, the restriction of $\mathscr{G}$ to $X \setminus \cup_{1\leqslant i\leqslant r} D_i$ is locally defined by closed 1-forms with zero set of codimension at least two by assumption. Therefore, there exist a closed subset $Z \subset X$ of codimension at least three and an analytic quasi-étale $\mathbb{Q}$-structure $\{p_j \colon X_j \to X \setminus Z\}_{j\in J}$ on $X \setminus Z$ such that $p_j^{-1}\mathscr{G}$ is defined by a closed 1-form with zero set of codimension at least two or by the 1-form $x_2dx_1 + \lambda_j x_1 dx_2$ where $(x_1, \ldots, x_n)$ are analytic coordinates on X_i and $\lambda_j \in \mathbb{Q}_{>0}$ (Fact 3.3). Proposition 4.1 then says that there exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least two and a finite cover $g \colon T \to Y$ such that the following holds. Set $T^\circ := g^{-1}(Y^\circ)$ and $X^\circ := \psi^{-1}(Y^\circ)$.

(1) The variety T has canonical singularities and $K_T \sim_{\mathbb{Z}} 0$; T° is smooth.

(2) The normalization M° of the fiber product $T^\circ \times_Y X$ is a $\mathbb{P}^1$-bundle over T° and the map $M^\circ \to X^\circ$ is a quasi-étale cover.

(3) The pull-back of $\mathscr{G}_{|X^\circ}$ yields a flat Ehresmann connection on $M^\circ \to T^\circ$.

Moreover, there exists a $\mathbb{Q}$-divisor B on Y such that $K_Y + B$ is torsion (see the proof of Proposition 4.1).

Step 6. Suppose from now on that $\mathscr{L}$ is not torsion, and let $a_Y\colon Y \to \mathrm{A}(Y)$ be the Albanese morphism. Arguing as in Step 7 of the proof of [12, Proposition 11.6], we conclude that a_Y is generically finite.

If $B \neq 0$, then Y is uniruled. But this contradicts the fact that a_Y is generically finite. Therefore, $B = 0$ and K_Y is torsion. The argument used in Step 7 of the proof of [12, Proposition 11.6] then shows that X and D_1 are smooth and that $\mathscr{G}$ can be defined by a nowhere vanishing (closed) logarithmic 1-form with poles along $D_1 \sqcup D_1'$ for some smooth (connected) hypersurface D_1'. This finishes the proof of the proposition. ■

Lemma 7.5. Let X be a quasi-projective variety with klt singularities and let $\psi\colon X \to Y$ be a Mori fiber space where Y is a smooth quasi-projective variety. Suppose in addition that $\dim Y = \dim X - 1$ and that there exists a section $D \subset X$ of ψ. Let $G \subset Y$ be an irreducible hypersurface and set $E := \psi^{-1}(G)$. Suppose furthermore that $(X, D+E)$ has log canonical singularities over the generic point of G. Then, there exist a Zariski open neighborhood Y° of the generic point of G in Y and a finite Galois cover $g^\circ\colon T^\circ \to Y^\circ$ of smooth varieties such that the normalization M° of the fiber product $T^\circ \times_Y X$ is a $\mathbb{P}^1$-bundle over T° and such that the map $M^\circ \to X^\circ$ is a quasi-étale cover, where $X^\circ := \psi^{-1}(Y^\circ)$.

Proof. By [17, Proposition 9.3], klt spaces have quotient singularities in codimension two. Shrinking Y, if necessary, we may therefore assume that X has quotient singularities. In particular, X is $\mathbb{Q}$-factorial. Since ψ is a Mori fiber space with $\mathbb{Q}$-factorial singularities and D is a section of ψ, any irreducible component of $\psi^{-1}(y)$ meets D at the unique intersection point of D and $\psi^{-1}(y)$. Let $B \subset Y$ be a one-dimensional complete intersection of general members of a very ample linear system passing through a general point y of G and set $S =: \psi^{-1}(B) \subset X$. Notice that S is klt and that $(S, D\cap S + E\cap S)$ is log canonical. This easily implies that the support of $D\cap S + E\cap S$ has at most two analytic branches at any point. This in turn implies that $\psi^{-1}(y)$ is irreducible and proves that ψ has irreducible fibers in a neighborhood of the generic point of G.

Write $\psi^*G = mE$ for some positive integer. Let Y° be a Zariski open neighborhood of the generic point of G in Y and let $g^\circ\colon T^\circ \to Y^\circ$ be a finite Galois cover of smooth varieties with ramification index m at any point over the generic point of G. Let M° be the normalization of the fiber product $T^\circ \times_Y X$. Shrinking Y°, if necessary, we may assume without loss of generality that the map $\varphi^\circ\colon M^\circ \to T^\circ$ has reduced fibers ([19, Théorème 12.2.4]). Observe that the map $f^\circ\colon M^\circ \to X^\circ$ is a quasi-étale cover, where $X^\circ := \psi^{-1}(Y^\circ)$. It follows that $\big(M^\circ, (f^\circ)^{-1}(D\cap X^\circ) + (f^\circ)^{-1}(E\cap X^\circ)\big)$ is log canonical. Moreover, $(f^\circ)^{-1}(D\cap X^\circ)$ is a section of φ° and any irreducible component of any fiber of φ° meets this section by construction. Arguing as above, we conclude that φ° has irreducible fibers over general points in $\varphi^\circ\big((f^\circ)^{-1}(E\cap X^\circ)\big)$. The lemma then follows from [24, Theorem II.2.8]. ■

8 Proof of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1 We maintain notation and assumptions of Theorem 1.1. By [12, Lemma 12.5], we have $\nu(X) \leqslant 1$. If $\nu(X) = 0$, then Theorem 1.1 follows from [12, Lemma 10.1] and [12, Proposition 10.2].

We may therefore assume from now on that $\nu(X) \in \{-\infty, 1\}$.

By [12, Proposition 12.3] together with [12, Lemma 8.15], either $\mathscr{G}$ is closed under pth powers for almost all primes p, or it is given by a closed rational 1-form with values in a flat line bundle whose zero set has codimension at least two ([12, Remark 12.4]).

If $\mathscr{G}$ is closed under pth powers for almost all primes p, then it is algebraically integrable by Theorems 5.1 and 6.4. The statement then follows from [12, Theorem 1.5].

Suppose now that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ whose zero set has codimension at least two. By assumption, there exists an effective divisor D on X such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X(D)\otimes\mathscr{L}$. On the other hand, $c_1(\mathscr{N}_{\mathscr{G}}) \equiv -K_X$ since $K_{\mathscr{G}} \equiv 0$. This immediately implies that $\nu(X) = -\infty$ since $\nu(X) \in \{-\infty, 1\}$. By Proposition 7.1, there exists a quasi-étale cover $f\colon X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form with zero set of codimension at least two. Note that X_1 has klt singularities. Moreover, $f^{-1}\mathscr{G}$ is canonical with $K_{f^{-1}\mathscr{G}} \equiv 0$ by Lemma 2.4. Theorem 1.1 follows from [12, Theorem 11.3] in this case. This finishes the proof of Theorem 1.1. ■

Proof of Corollary 1.3 We maintain notation and assumptions of Corollary 1.3. By [11, Theorem 8.1] (see also [33, Theorem 1.7]), there exists a projective birational morphism $\beta\colon X_1 \to X$ such that

(1) X_1 is $\mathbb{Q}$-factorial with klt singularities,
(2) we have $K_{\beta^{-1}\mathscr{G}} = \beta^* K_{\mathscr{G}}$ and
(3) $\mathscr{G}_1 := \beta^{-1}\mathscr{G}$ is F-dlt we refer to ([11, Paragraph 3.2] for this notion).

By [30, Proposition V.2.7] and property (2), we have $\nu(K_{\mathscr{G}_1}) = \nu(K_{\mathscr{G}}) = 0$. Moreover, $\mathscr{G}_1$ has canonical singularities by Lemma 2.3. Thus, we may run a minimal model program for $\mathscr{G}_1$ and end with a minimal model. There exist a projective three-fold X_2 with $\mathbb{Q}$-factorial klt singularities and a $K_{\mathscr{G}_1}$-negative birational map $\varphi\colon X_1 \dashrightarrow X_2$ such that $K_{\mathscr{G}_2}$ is nef, where $\mathscr{G}_2$ denotes the foliation on X_2 induced by $\mathscr{G}_1$. Since φ is $K_{\mathscr{G}_1}$-negative and using [30, Proposition V.2.7] again, we see that $\nu(K_{\mathscr{G}_2}) = 0$. Since $K_{\mathscr{G}_2}$ is nef, we must have $K_{\mathscr{G}_2} \equiv 0$. On the other hand, using the fact that φ is $K_{\mathscr{G}_1}$-negative together with Lemma 2.3, we conclude that $\mathscr{G}_2$ has canonical singularities as well. The corollary now follows from Theorem 1.1. ■

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