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# On Foliations with Semi-positive Anti-canonical Bundle

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**Abstract**
In this note, we describe the structure of regular foliations with semi-positive anti-canonical bundle on smooth projective varieties.

**Keywords** Foliation · Compact leaf · Algebraically integrable foliation

**Mathematics Subject Classification** 37F75

## Contents



## 1 Introduction

The purpose of this paper is to prove the following result that reduces the study of regular foliations with semi-positive anti-canonical bundle to the study of regular foliations with numerically trivial canonical class. Recall that a line bundle on a complex projective manifold is said to be *semi-positive* if it admits a smooth hermitian metric with semi-positive curvature.

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**Theorem 1.1** *Let $X$ be a complex projective manifold, and let $\mathscr{G}$ be a foliation on $X$. Suppose that $\mathscr{G}$ is regular, or that $\mathscr{G}$ has a compact leaf. Suppose in addition that $-K_{\mathscr{G}}$ is semi-positive. Then there exist a smooth morphism $\varphi\colon X \to Y$ onto a complex projective manifold $Y$ and a foliation $\mathscr{E}$ on $Y$ with $K_{\mathscr{E}} \equiv 0$ such that $\mathscr{G} = \varphi^{-1}\mathscr{E}$.*

In Touzet (2008), Touzet obtained a foliated version of the Beauville–Bogomolov decomposition theorem for codimension one regular foliations with numerically trivial canonical bundle on compact Kähler manifolds. Pereira and Touzet then addressed regular foliations $\mathscr{G}$ with $c_1(\mathscr{G}) = 0$ and $c_2(\mathscr{G}) = 0$ on complex projective manifolds in Pereira and Touzet (2013). The structure of codimension two foliations with numerically trivial canonical bundle on complex projective uniruled manifolds is given in Druel (2017c).

As a consequence of Theorem 1.1, we describe codimension one regular foliations with semi-positive anti-canonical bundle on complex projective manifolds.

**Corollary 1.2** *Let $X$ be a complex projective manifold, and let $\mathscr{G}$ be a codimension one foliation on $X$. Suppose that $\mathscr{G}$ is regular, and that $-K_{\mathscr{G}}$ is semi-positive. Then there exists a smooth morphism $\psi\colon Z \to Y$ of complex projective manifolds, as well as a finite étale cover $f\colon Z \to X$ and a foliation $\mathscr{E}$ on $Y$ such that $f^{-1}\mathscr{G} = \psi^{-1}\mathscr{E}$. In addition, either* $\dim Y = 1$ *and $\mathscr{G}$ is algebraically integrable, or $Y$ is an abelian variety and $\mathscr{E}$ is a linear foliation, or $Y$ is a $\mathbb{P}^1$-bundle over an abelian variety $A$ and $\mathscr{E}$ is a flat Ehresmann connection on $Y \to A$.*

We also prove that foliations with semi-positive anti-canonical bundle having a compact leaf are automatically regular.

**Corollary 1.3** *Let $X$ be a complex projective manifold, and let $\mathscr{G}$ be a foliation on $X$. Suppose that $\mathscr{G}$ has a compact leaf, and that $-K_{\mathscr{G}}$ is semi-positive. Then $\mathscr{G}$ is regular.*

### Outline of the Proof

The main steps for the proof of Theorem 1.1 are as follows. In the setup of Theorem 1.1, one readily checks that the general leaves of the algebraic part of $\mathscr{G}$ (we refer to Sect. 1 for this notion) are compact. We then consider the induced relative maximal rationally connected fibration. This gives a foliation $\mathscr{H}$ on $X$ whose general leaves are projective and rationally connected. Moreover, $\mathscr{G}$ is the pull-back of a foliation $\mathscr{E}$ on the space of leaves of $\mathscr{H}$. A result of Campana and Păun together with a theorem of Graber, Harris and Starr then implies that $K_{\mathscr{E}}$ is pseudo-effective. Using Viehweg's weak positivity theorem, one then concludes that $-K_{\mathscr{H}} \equiv -K_{\mathscr{G}}$ so that $-K_{\mathscr{H}}$ is semi-positive as well. Applying a criterion for regularity of foliations with semi-positive anti-canonical bundle, which is established in Sect. 2, we obtain that $\mathscr{H}$ is regular. The holomorphic version of Reeb stability theorem then implies that $\mathscr{H}$ is induced by a smooth morphism onto a projective manifold, finishing the proof of Theorem 1.1.

## 2 Foliations

In this section, we have gathered a number of results and facts concerning foliations which will later be used in the proofs.

### 2.1 Definitions

A *foliation* on a normal complex variety $X$ is a coherent subsheaf $\mathscr{G} \subseteq T_X$ such that

1. $\mathscr{G}$ is closed under the Lie bracket, and
2. $\mathscr{G}$ is saturated in $T_X$. In other words, the quotient $T_X/\mathscr{G}$ is torsion-free.

The *rank* $r$ of $\mathscr{G}$ is the generic rank of $\mathscr{G}$. The *codimension* of $\mathscr{G}$ is defined as $q := \dim X - r$.

The *canonical class* $K_{\mathscr{G}}$ of $\mathscr{G}$ is any Weil divisor on $X$ such that $\mathscr{O}_X(-K_{\mathscr{G}}) \cong \det \mathscr{G}$.

Let $X^\circ \subseteq X_{\mathrm{reg}}$ be the open set where $\mathscr{G}_{|X_{\mathrm{reg}}}$ is a subbundle of $T_{X_{\mathrm{reg}}}$. A *leaf* of $\mathscr{G}$ is a maximal connected and immersed holomorphic submanifold $L \subset X^\circ$ such that $T_L = \mathscr{G}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure.

The foliation $\mathscr{G}$ is said to be *algebraically integrable* if its leaves are algebraic.

### 2.2 Foliations Described as Pull-Backs

Let $X$ and $Y$ be normal complex varieties, and let $\varphi\colon X \dashrightarrow Y$ be a dominant rational map that restricts to a smooth morphism $\varphi^\circ\colon X^\circ \to Y^\circ$, where $X^\circ \subset X$ and $Y^\circ \subset Y$ are smooth open subsets. Let $\mathscr{G}$ be a foliation on $Y$. *The pull-back* $\varphi^{-1}\mathscr{G}$ *of* $\mathscr{G}$ *via* $\varphi$ is the foliation on $X$ whose restriction to $X^\circ$ is $(d\varphi^\circ)^{-1}\big(\mathscr{G}_{|Y^\circ}\big)$.

### 2.3 The Family of Leaves

We refer the reader to Araujo and Druel (2014), Remark 3.12 for a more detailed explanation. Let $X$ be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on $X$. There is a unique normal complex projective variety $Y$ contained in the normalization of the Chow variety of $X$ whose general point parametrizes the closure of a general leaf of $\mathscr{G}$ (viewed as a reduced and irreducible cycle in $X$). Let $Z \to Y \times X$ denotes the normalization of the universal cycle. It comes with morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ \big\downarrow{\scriptstyle\psi} & & \\ Y & & \end{array}$$

where $\beta\colon Z \to X$ is birational and, for a general point $y \in Y$, $\beta\big(\psi^{-1}(y)\big) \subseteq X$ is the closure of a leaf of $\mathscr{G}$. The morphism $Z \to Y$ is called the *family of leaves* and $Y$ is called the *space of leaves* of $\mathscr{G}$.

### 2.4 Algebraic and Transcendental Parts

Next, we define the *algebraic* and *transcendental* parts of a foliation (see Araujo and Druel 2014, Definition 2).

Let $\mathscr{G}$ be a foliation on a normal variety $X$. There exist a normal variety $Y$, unique up to birational equivalence, a dominant rational map with connected fibers $\varphi : X \dashrightarrow Y$, and a foliation $\mathscr{E}$ on $Y$ such that the following holds:

1. $\mathscr{E}$ is purely transcendental, i.e., there is no positive-dimensional algebraic subvariety through a general point of $Y$ that is tangent to $\mathscr{E}$; and
2. $\mathscr{G}$ is the pullback of $\mathscr{E}$ via $\varphi$.

The foliation on $X$ induced by $\varphi$ is called the *algebraic part* of $\mathscr{G}$.

The following result due to Campana and Păun will prove to be crucial for the proof of Theorem 1.1. Let $X$ be a normal projective variety, and let $A$ be an ample divisor on $X$. Recall that a $\mathbb{Q}$-divisor $D$ on a normal projective variety is said to be *pseudo-effective* if, for any positive number $\varepsilon \in \mathbb{Q}$, there exists an effective $\mathbb{Q}$-divisor $D_\varepsilon$ such that $D + \varepsilon A \sim_{\mathbb{Q}} D_\varepsilon$.

**Theorem 2.1** *Let $X$ be a normal complex projective variety, and let $\mathscr{G}$ be a foliation on $X$. If $K_{\mathscr{G}}$ is not pseudo-effective, then $\mathscr{G}$ is uniruled.*

***Proof*** This follows easily from Campana and Păun (2015, Theorem 4.7) applied to the pull-back of $\mathscr{G}$ on a resolution of $X$. □

## 3 A Criterion for Regularity

Let $\mathscr{G}$ be a foliation with numerically trivial canonical class on a complex projective manifold, and assume that $\mathscr{G}$ has a compact leaf. Then Theorem 5.6 in Loray et al. (2018) asserts that $\mathscr{G}$ is regular and that there exists a foliation on $X$ transverse to $\mathscr{G}$ at any point in $X$. In this section, we extend this result to our setting (see also Demailly et al. 2001, Proposition 2.7.1) and (Druel 2018, Corollary 7.22).

**Proposition 3.1** *Let $X$ be a compact Kähler manifold of dimension $n$, and let $\mathscr{G}$ be a foliation of rank $r$ on $X$. Suppose that $\mathscr{G}$ has a compact leaf and that $-K_{\mathscr{G}}$ is semi-positive. Then there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of $T_X$ into subbundles. In particular, $\mathscr{G}$ is regular.*

***Proof*** Let $\omega$ be a Kähler form on $X$, and let $v \in H^0\big(X, \wedge^r T_X \otimes \mathscr{O}_X(K_{\mathscr{G}})\big)$ be a $r$-vector defining $\mathscr{G}$. The contraction $v \lrcorner \omega^r$ of $\omega^r$ by $v$ is a $\overline{\partial}$-closed $(0, r)$-form with values in $\mathscr{O}_X(K_{\mathscr{G}})$, and gives a class

$$[v \lrcorner \omega^r] \in H^r\big(X, \mathscr{O}_X(K_{\mathscr{G}})\big).$$

We first show that $[v \lrcorner \omega^r] \neq 0$. Let $F \subseteq X$ be a compact leaf. We have a commutative diagram

$$\begin{array}{ccc} H^r(X, \Omega^r_X) & \xrightarrow{v \lrcorner \bullet} & H^r\big(X, \mathscr{O}_X(K_{\mathscr{G}})\big) \\ \downarrow & & \downarrow \\ H^r(F, \Omega^r_F) & \longrightarrow & H^r\big(F, \mathscr{O}_X(K_{\mathscr{G}})_{|F}\big). \end{array}$$

On the other hand, since $F$ is a leaf of $\mathscr{G}$, we have $\mathscr{O}_F(K_F) \cong \mathscr{O}_X(K_{\mathscr{G}})_{|F}$ and the map

$$H^r(F, \Omega^r_F) \to H^r\big(F, \mathscr{O}_X(K_{\mathscr{G}})_{|F}\big)$$

is an isomorphism. This immediately implies that $[v \lrcorner \omega^r] \neq 0$.

By Serre duality, there is a $\overline{\partial}$-closed $(n, n-r)$-form $\alpha$ with values in $\mathscr{O}_X(-K_{\mathscr{G}})$ such that

$$\int_X \alpha \wedge (v \lrcorner \omega^r) = 1.$$

Applying Demailly et al. (2001, Theorem 0.1) and using the assumption that $-K_{\mathscr{G}}$ is semi-positive, we see that there exists a $\overline{\partial}$-closed $(r, 0)$-form $\beta$ with values in $\mathscr{O}_X(-K_{\mathscr{G}})$ such that

$$\alpha = \omega^{n-r} \wedge \beta.$$

Now, a straightforward computation shows that

$$\alpha \wedge (v \lrcorner \omega^r) = \omega^{n-r} \wedge \beta \wedge (v \lrcorner \omega^r) = C\beta(v)\omega^n,$$

for some complex number $C \neq 0$. Note that $\beta(v)$ is a holomorphic function, hence constant. We conclude that $\beta(v)$ is non-zero. It follows that the kernel of the morphism $T_X \to \Omega^{r-1}_X \otimes \mathscr{O}_X(-K_{\mathscr{G}})$ given by the contraction with $\beta$ is a vector bundle $\mathscr{E}$ on $X$ such that $T_X \cong \mathscr{G} \oplus \mathscr{E}$. This finishes the proof of the proposition. □

***Remark 3.2*** The conclusion of Proposition 3.1 holds if $-K_{\mathscr{G}}$ is only assumed to be equipped with a (possibly singular) hermitian metric $h$ with curvature $i\Theta_{-K_{\mathscr{G}},h} \geqslant 0$ in the sense of currents whose multiplier ideal sheaf $\mathscr{I}(h)$ is trivial, $\mathscr{I}(h) \cong \mathscr{O}_X$.

## 4 Proofs

The present section is devoted to the proof of Theorem 1.1 and Corollaries 1.2 and 1.3. Theorem 1.1 is an immediate consequence of Theorem 4.2 below together with Lemma 4.1.

**Lemma 4.1** (Druel 2017b, Lemma 6.2) *Let $X$ be a complex projective manifold, and let $\mathscr{G}$ be a foliation on $X$. Suppose that $\mathscr{G}$ is regular, or that it has a compact leaf. Then the algebraic part of $\mathscr{G}$ has a compact leaf.*

**Theorem 4.2** *Let $X$ be a complex projective manifold, and let $\mathscr{G}$ be a foliation on $X$. Suppose that the algebraic part of $\mathscr{G}$ has a compact leaf and that $-K_{\mathscr{G}}$ is semi-positive. Then there exist a smooth morphism $\varphi\colon X \to Y$ onto a complex projective manifold $Y$ and a foliation $\mathscr{E}$ on $Y$ with $K_{\mathscr{E}} \equiv 0$ such that $\mathscr{G} = \varphi^{-1}\mathscr{E}$.*

***Proof*** By assumption, the algebraic part of $\mathscr{G}$ is induced by an almost proper map. Its relative maximal rationally connected fibration then defines a foliation $\mathscr{H}$ on $X$ with a compact leaf. □

***Claim 4.3*** We have $K_{\mathscr{H}} \equiv K_{\mathscr{G}}$.

***Proof*** The proof is similar to that of Druel (2017b, Proposition 6.1) (see also (Druel 2018, Proposition 8.1) for a somewhat related result).

Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism. By construction, $\varphi := \psi \circ \beta^{-1}$ is an almost proper map. Moreover, there is a foliation $\mathscr{E}$ on $Y$ such that $\mathscr{G} = \varphi^{-1}\mathscr{E}$ by Araujo and Druel (2013, Lemma 6.7). By Graber et al. (2003), there is no rational curve tangent to $\mathscr{E}$ passing through a general point of $Y$.

Let $A$ be an ample divisor on $X$, and let $F$ be a general (smooth) fiber of $\varphi$. There exists an open set $U \supset F$ such that $\mathscr{H}_{|U}$ is a subbundle of $\mathscr{G}_{|U}$, $\varphi^*\mathscr{E}_{|U}$ is locally free, and $\big(\mathscr{G}/\mathscr{H}\big)_{|U} \cong \varphi^*\mathscr{E}_{|U}$. In particular, we have

$$K_{\mathscr{G}|F} \sim_{\mathbb{Z}} K_{\mathscr{H}|F} \sim_{\mathbb{Z}} K_F,$$

by the adjunction formula. Let $\varepsilon > 0$ be a rational number. Since $-K_{\mathscr{G}} + \varepsilon A$ is $\mathbb{Q}$-ample, there exists an effective $\mathbb{Q}$-divisor $A_\varepsilon$ such that $A_\varepsilon \sim_{\mathbb{Q}} -K_{\mathscr{G}} + \varepsilon A$. Then $K_F - K_{\mathscr{G}|F} + \varepsilon A_{|F} \sim_{\mathbb{Q}} A_{\varepsilon|F}$ is effective. By (Druel 2017b, Proposition 4.1) applied to $\mathscr{H}$ and $L := A_\varepsilon$, we conclude that $K_{\mathscr{H}} + A_\varepsilon \sim_{\mathbb{Q}} K_{\mathscr{H}} - K_{\mathscr{G}} + \varepsilon A$ is pseudo-effective for any positive rational number $\varepsilon > 0$. It follows that $K_{\mathscr{H}} - K_{\mathscr{G}}$ is pseudo-effective as well.

By Druel (2017b), Section 2.9, there is an effective divisor $R$ on $X$ such that

$$K_{\mathscr{H}} - K_{\mathscr{G}} = -(\varphi^* K_{\mathscr{E}} + R).$$

On the other hand, $K_{\mathscr{E}}$ is pseudo-effective by Theorem 2.1. This easily implies that $\varphi^* K_{\mathscr{E}}$ is pseudo-effective as well. Therefore, we must have $K_{\mathscr{H}} - K_{\mathscr{G}} \equiv 0$, proving our claim. □

By the claim, there is a flat line bundle $\mathscr{L}$ on $X$ such that $\mathscr{O}_X(K_{\mathscr{H}}) \cong \mathscr{O}_X(K_{\mathscr{G}}) \otimes \mathscr{L}$. On the other hand, $\mathscr{L}$ admits a unitary smooth hermitian metric with zero curvature. It follows that $-K_{\mathscr{H}}$ is semi-positive as well. By Proposition 3.1 above, there is a decomposition $T_X \cong \mathscr{H} \oplus \mathscr{H}_1$ of $T_X$ into subbundles. In particular, $\mathscr{H}$ is regular. By construction, its leaves are projective rationally connected manifolds. In particular, they are simply connected. Arguing as in the proof of Druel (2017a, Lemma 4.1), we see that $\beta$ is an isomorphism and that $\psi$ is a smooth morphism onto a complex projective manifold. A straightforward computation then shows that $K_{\mathscr{E}} \equiv 0$ since $K_{\mathscr{H}} \equiv K_{\mathscr{G}}$. This finishes the proof of the theorem. □

***Proof of Corollary 1.2*** The statement follows from Theorem 1.1 together with Touzet (2008, Théorème 1.2) using Druel (2018, Lemma 5.9). □

***Proof of Corollary 1.3*** The claim follows from Theorem 1.1 and Loray et al. (2018, Theorem 5.6) (or Proposition 3.1). □

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