CODIMENSION 1 FOLIATIONS WITH NUMERICALLY TRIVIAL CANONICAL CLASS ON SINGULAR SPACES

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Abstract

In this article, we describe the structure of codimension 1 *foliations with canonical singularities and numerically trivial canonical class on varieties with terminal singularities, extending a result of Loray, Pereira, and Touzet to this context.*

Contents

1. Introduction

In the last few decades, much progress has been made in the classification of complex projective varieties. The general viewpoint is that complex projective varieties with mild singularities should be classified according to the behavior of their canonical class.

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Similar ideas can be applied in the context of foliations on complex projective varieties. If $\mathscr{G}$ is a foliation on a complex projective variety, then we define its canonical class to be $K_{\mathscr{G}} = -c_1(\mathscr{G})$. In analogy with the case of projective varieties, one expects the numerical properties of $K_{\mathscr{G}}$ to reflect geometric aspects of $\mathscr{G}$ (see, e.g., [4]–[7], [20], [23], [66], [67], [78], [80]). This led, for instance, to the birational classification of foliations by curves on surfaces with quotient singularities (see [20], [67]), generalizing most of the important results of the Enriques–Kodaira classification. The classification exhibits an interesting feature, which is in sharp contrast with the Enriques–Kodaira classification: there exist foliations with pseudoeffective canonical bundle and negative Kodaira dimension. So the abundance conjecture fails already in ambient dimension 2.

The present paper aims at describing one of the most basic classes of codimension 1 foliations, namely, (mildly) singular foliations with numerically trivial canonical class on (mildly) singular spaces.

The Beauville–Bogomolov decomposition theorem asserts that any compact Kähler manifold with numerically trivial canonical bundle admits an étale cover that decomposes into a product of a torus, and irreducible, simply-connected Calabi–Yau and holomorphic symplectic manifolds (see [13]). In [80], Touzet obtained a foliated version of the Beauville–Bogomolov decomposition theorem for codimension 1 regular foliations with numerically trivial canonical bundle on compact Kähler manifolds. The statement below follows from [80, Théorème 1.2] and [31, Lemma 5.9].

THEOREM (Touzet)
Let X be a complex projective manifold, and let $\mathscr{G}$ be a regular codimension 1 foliation on X with $K_{\mathscr{G}} \equiv 0$. Then one of the following holds.

(1) *There exists a $\mathbb{P}^1$-bundle structure $\varphi\colon X \to Y$ onto a complex projective manifold Y with $K_Y \equiv 0$, and $\mathscr{G}$ induces a flat holomorphic connection on φ.*
(2) *There exist an abelian variety A, and a simply connected projective manifold Y with $K_Y \equiv 0$, and a finite étale cover $f : A \times Y \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of a codimension 1 linear foliation on A.*
(3) *There exist a smooth complete curve B of genus at least 2, and a complex projective manifold Y with $K_Y \equiv 0$, and a finite étale cover $f\colon B \times Y \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection morphism $B \times Y \to B$.*

Loray, Pereira, and Touzet [66] recently described the structure of codimension 1 foliations with canonical singularities (we refer to Section 4 for this notion) and numerically trivial canonical class on complex projective manifolds. However, with the development of the minimal model program, it became clear that singularities arise as an inevitable part of higher-dimensional life. In this article, we extend their result to the singular setting. Our first main result is the following.

THEOREM 1.1

Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities. Suppose, furthermore, that $K_{\mathscr{G}} \sim_{\mathbb{Q}} 0$. Then one of the following holds.

(1) *There exist a smooth complete curve C, a complex projective variety Y with canonical singularities and $K_Y \sim_{\mathbb{Z}} 0$, as well as a quasi-étale cover $f : Y \times C \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times C \to C$.*

(2) *There exist complex projective varieties Y and Z with canonical singularities, as well as a quasi-étale cover $f : Y \times Z \to X$ and a foliation $\mathscr{H} \cong \mathscr{O}_Z^{\dim Z - 1}$ on Z such that $f^{-1}\mathscr{G}$ is the pullback of $\mathscr{H}$ via the projection $Y \times Z \to Z$. In addition, we have that $K_Y \sim_{\mathbb{Z}} 0$, that Z is an equivariant compactification of a commutative algebraic group of dimension at least 2, and that $\mathscr{H}$ is induced by a codimension 1 Lie subgroup.*

Remark 1.2

In the setup of Theorem 1.1, suppose that Z is rational, and set $n := \dim Z$. Then Z is an equivariant compactification of $(\mathbb{G}_m)^n$ or $(\mathbb{G}_m)^{n-1} \times \mathbb{G}_a$ (see the proof of Lemma 11.7). In either case, Z is a toric variety by [10].

In Theorem 1.1 above, we assume that the canonical divisor $K_{\mathscr{G}}$ is abundant. In fact, we also show that abundance holds provided that X is terminal. Theorem 1.1 together with Theorem 1.3 then gives the structure of codimension 1 foliations with canonical singularities and numerically trivial canonical class on varieties with terminal singularities.

THEOREM 1.3

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities and $K_{\mathscr{G}} \equiv 0$. Suppose, in addition, that either X has terminal singularities, or that X has canonical singularities and $K_{\mathscr{G}}$ is Cartier. Then $K_{\mathscr{G}}$ is torsion.

If $\dim X = 3$, then Theorem 1.3 is a special case of [23, Theorem 1.7].

As a consequence of Theorem 1.1, we describe the structure of weakly regular (we refer to Section 5 for this notion) codimension 1 foliations with torsion canonical class, extending [80, Théorème 1.2] to this context.

COROLLARY 1.4

Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a weakly regular codimension 1 foliation on X. Suppose, furthermore, that either $K_{\mathscr{G}}$ is Cartier and $K_{\mathscr{G}} \equiv 0$, or $K_{\mathscr{G}} \sim_{\mathbb{Q}} 0$. Then one of the following holds.

(1) *There exist a complex projective manifold Y with $K_Y \equiv 0$, a $\mathbb{P}^1$-bundle $\varphi\colon Z \to Y$, and a quasi-étale cover $f\colon Z \to X$ such that $f^{-1}\mathscr{G}$ induces a flat holomorphic connection on φ.*

(2) *There exists an abelian variety A as well as a simply connected projective manifold Y with $K_Y \equiv 0$, and a finite étale cover $f: Y \times A \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of a codimension 1 linear foliation on A.*

(3) *There exist a smooth complete curve C, a complex projective variety Y with canonical singularities and $K_Y \sim_{\mathbb{Z}} 0$, as well as a quasi-étale cover $f: Y \times C \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times C \to C$.*

However, from the point of view of birational classification of foliations, this class of singularities is inadequate. Indeed, the foliated analogue of the minimal model program aims in particular to reduce the birational study of mildly singular foliations with numerical dimension 0 on complex projective manifolds to the study of associated minimal models, that is, mildly singular foliations with numerically trivial canonical class on klt (Kawamata log terminal) spaces (see, e.g., [23, Theorem 1.7]). Building on the results of the present paper, it has been shown that Theorems 1.1 and 1.3 are valid for codimension 1 foliations with canonical singularities on projective varieties with klt singularities (see [33]).

If $\mathscr{G}$ is a regular foliation on a complex manifold X and L is a compact leaf with finite holonomy group, then the holomorphic version of the local Reeb stability theorem asserts that there exist an invariant open analytic neighborhood U of L and an unramified Galois cover $f_1\colon U_1 \to U$ such that the pullback $f_1^{-1}\mathscr{G}_{|U}$ of $\mathscr{G}_{|U}$ to U_1 is induced by a proper submersion $U_1 \to S$. The proofs of our main results rely on the following global version of the Reeb stability theorem.

THEOREM 1.5

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be an algebraically integrable foliation on X with canonical singularities. Suppose that $K_{\mathscr{G}} \equiv 0$. Then there exist complex projective varieties Y and Z with klt singularities and a quasi-étale cover $f: Y \times Z \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times Z \to Y$.

Outline of the proof

The main steps for the proofs of Theorems 1.1 and 1.3 are as follows. As we will see below, our work owes a great deal to the general strategy introduced in [66].

Let X be a normal complex projective variety with terminal singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities and $K_{\mathscr{G}} \equiv 0$. An analogue of the Bogomolov vanishing theorem says that X has numerical dimension $\nu(X) \leq 1$ (see Lemma 12.5).

Following [66], we show that either $\mathscr{G}$ is closed under pth powers for almost all primes p, or $\mathscr{G}$ is given by a closed rational 1-form (see Proposition 12.3) with values in a flat line bundle, whose zero set has codimension at least 2. In the latter case, one checks that $\nu(X) \le 0$. If $\nu(X) = 0$, then one proves that $\mathscr{G}$ is induced by a linear foliation on an abelian variety (see Proposition 11.10 for a precise statement). If $\nu(X) = -\infty$, then one first reduces to the case where $\mathscr{G}$ is defined by a closed rational 1-form (see Proposition 11.6). One then shows that $\mathscr{G}$ is as in Theorem 1.1(2) (see Theorem 11.3). In particular, abundance holds in this case.

Suppose from now on that $\mathscr{G}$ is closed under pth powers for almost all primes p. We will prove that $\mathscr{G}$ is algebraically integrable, confirming the generalization to foliations by Ekedahl, Shepherd-Barron, and Taylor of the classical Grothendieck–Katz conjecture in this special case. Theorem 1.1 then follows from Theorem 1.5. Moreover, abundance holds for algebraically integrable foliations with canonical singularities and numerically trivial canonical class by Proposition 4.24, as an easy consequence of a theorem of Ambro [1, Theorem 3.5] (see also [36, Theorem 1.2]).

Suppose that $\nu(X) = -\infty$. In this case, we show that it is enough to prove the statement under the additional assumptions that X is $\mathbb{P}^1$-bundle over an abelian variety A and $\mathscr{G}$ is a flat connection on $X \to A$. This follows from the minimal model program together with [40, Theorem I] that says that the fundamental group of the smooth locus of a projective klt variety with numerically trivial canonical class and zero-augmented irregularity does not admit any finite-dimensional linear representation with infinite image. By a result of André [2, Théorème 7.2.2], we see that we can also suppose that $\mathscr{G}$ and $X \to A$ are defined over a number field. The statement then follows from a theorem of Bost [16, Theorem 2.9] who proved the Ekedahl–Shepherd-Barron–Taylor conjecture for flat invariant connections on principal bundles with linear solvable structure groups defined over number fields.

Suppose now that $\nu(X) = 0$. Using Theorem 1.5, we first show that we may assume that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X (see Proposition 8.13). Then, running a minimal model program, one reduces to the case where K_X is torsion. A weak version of the singular analogue of the Beauville–Bogomolov decomposition theorem due to Kawamata [58, Proposition 8.3] implies that, perhaps after passing to a quasi-étale cover, $\mathscr{G}$ is a linear foliation on an abelian variety. Then [16, Theorem 2.3] together with [34, Proposition 3.6] implies that $\dim X = 1$ and $\mathscr{G}$ is the foliation by points.

Suppose finally that $\nu(X) = 1$. The proof in this case is much more involved. We use the assumption that $\mathscr{G}$ is closed under pth powers for almost all primes p to conclude that it is weakly regular. On the other hand, Touzet described in [81] the structure of codimension 1 foliations on complex projective manifolds with pseudoeffective conormal bundle. As a consequence, either the normal sheaf

$\mathscr{N}_{\mathscr{G}}$ satisfies $\kappa(-c_1(\mathscr{N}_{\mathscr{G}})) = \nu(-c_1(\mathscr{N}_{\mathscr{G}})) = 1$ and $\mathscr{G}$ is algebraically integrable, or $\kappa(-c_1(\mathscr{N}_{\mathscr{G}})) = -\infty$ and $\nu(-c_1(\mathscr{N}_{\mathscr{G}})) = 1$. In the latter case, $\mathscr{G}$ is induced by a codimension 1 tautological foliation on a quotient of a polydisk $\mathbb{D}^N$ for some $N \geq 2$ by an arithmetic irreducible lattice $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})^N$. Using the fact that $\mathscr{G}$ is weakly regular, we then show that the image of the map $X \to \mathbb{D}^N / \Gamma$ is a curve, yielding a contradiction (see Theorem 10.4).

These steps are addressed throughout the paper, and are collected together in Section 12.

Structure of the paper

Section 2 gathers notation, known results, and global conventions that will be used throughout the paper. In Section 3, we recall the definitions and basic properties of foliations. In Section 4, we establish a number of properties of foliations with canonical singularities. In particular, we analyze the behavior of foliations with canonical singularities under finite covers and $\mathbb{Q}$-factorial terminalization. We also address foliations with algebraic leaves. In particular, we show that abundance holds for algebraically integrable foliations with canonical singularities and numerically trivial canonical class (see Proposition 4.24). Section 5 is devoted to weakly regular foliations. We first establish basic properties. We then give criteria for a foliation with trivial canonical class to be weakly regular (see Propositions 5.21 and 5.26). We end this section with the local structure of rank 1 weakly regular foliations on surfaces with quotient singularities. Sections 6 and 7 prepare for the proof of Theorem 1.5. It is well known that an algebraically integrable regular foliation on a complex projective manifold is induced by a morphism onto a normal projective variety. In Section 6, we extend this result to weakly regular foliations with canonical singularities on mildly singular varieties (see Theorem 6.1). Section 8 is mostly taken up by the proof of Theorem 1.5. Sections 9 and 10 prepare for the proofs of our main results. In particular, we confirm the Ekedahl–Shepherd-Barron–Taylor conjecture for mildly singular codimension 1 foliations with trivial canonical class first on projective varieties with $\nu(X) = -\infty$ in Section 9, and then on those with $\nu(X) \geq 0$ in Section 10. In Section 11, we describe codimension 1 foliations with numerically trivial canonical class defined by closed (twisted) rational 1-forms. With these preparations at hand, the proofs of Theorems 1.1 and 1.3 and the proof of Corollary 1.4 which we give in Section 12 become reasonably short.

2. Notation, conventions, and used facts

2.1. Global convention

Throughout the paper a *variety* is a reduced and irreducible scheme separated and of finite type over a field.

Given a scheme X, we denote by X_{reg} its smooth locus.

Suppose that $k = \mathbb{C}$. (We will use the notions of terminal, canonical, klt, and lc (log canonical) singularities for pairs without further explanation or comment and simply refer to [63, Section 2.3] for a discussion and for their precise definitions. We refer to [59] and [63] for standard references concerning the minimal model program.)

2.2. $\mathbb{Q}$-Factorializations and $\mathbb{Q}$-factorial terminalizations

Definition 2.1
Let X be a normal complex quasiprojective variety with klt singularities. A $\mathbb{Q}$-*factorialization* is a small birational projective morphism $\beta\colon Z \to X$, where Z is $\mathbb{Q}$-factorial with klt singularities.

FACT 2.2
The existence of $\mathbb{Q}$-factorializations is established in [62, Corollary 1.37]. Note that we must have $K_Z \sim_{\mathbb{Q}} \beta^* K_X$.

Definition 2.3
Let X be a normal complex quasiprojective variety with canonical singularities. A $\mathbb{Q}$-*factorial terminalization* of X is a birational crepant projective morphism $\beta\colon Z \to X$, where Z is $\mathbb{Q}$-factorial with terminal singularities.

FACT 2.4
The existence of $\mathbb{Q}$-factorial terminalizations is established in [14, Corollary 1.4.3].

2.3. Projective space bundle

If $\mathscr{E}$ is a locally free sheaf of finite rank on a variety X, then we denote by $\mathbb{P}(\mathscr{E})$ the variety $\mathrm{Proj}_X(\mathrm{S}^\bullet\mathscr{E})$, and by $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ its tautological line bundle.

2.4. Stability

The word *stable* will always mean *slope-stable with respect to a given movable curve class*. The same goes for *semistable* and *polystable*. We refer to [52, Definition 1.2.12] for their precise definitions.

2.5. Reflexive hull

Given a normal variety X, $m \in \mathbb{N}$, and coherent sheaves $\mathscr{E}$ and $\mathscr{G}$ on X, write $\mathscr{E}^{[m]} := (\mathscr{E}^{\otimes m})^{**}$, $\mathrm{S}^{[m]}\mathscr{E} := (\mathrm{S}^m\mathscr{E})^{**}$, $\det\mathscr{E} := (\Lambda^{\mathrm{rank}\,\mathscr{E}}\mathscr{E})^{**}$, and $\mathscr{E} \boxtimes \mathscr{G} := (\mathscr{E} \otimes \mathscr{G})^{**}$. Given any morphism $f\colon Y \to X$, write $f^{[*]}\mathscr{E} := (f^*\mathscr{E})^{**}$.

2.6. Reflexive Kähler differentials and pullback morphisms

Given a normal variety X, we denote the sheaf of Kähler differentials by Ω^1_X. If $0 \leq p \leq \dim X$ is any integer, then write $\Omega^{[p]}_X := (\Omega^p_X)^{**}$. The tangent sheaf $(\Omega^1_X)^*$ will be denoted by T_X.

If D is a reduced effective divisor on X, then we denote by $(X, D)_{\mathrm{reg}}$ the open set where (X, D) is log smooth. We write $\Omega^{[p]}_X(\log D)$ for the reflexive sheaf on X whose restriction to $U := (X, D)_{\mathrm{reg}}$ is the sheaf of logarithmic differential forms $\Omega^p_U(\log D_{|U})$. We will refer to it as the sheaf of *reflexive logarithmic p-forms*. Suppose that X is smooth, and let t be a defining equation for D on some open set X°. Let α be a rational p-form on X. Then α is a reflexive logarithmic p-form on X° if and only if $t\alpha$ and $t\,d\alpha$ are regular on X° (see [74]).

Suppose that $k = \mathbb{C}$.

If $f: Y \to X$ is any morphism between varieties, then we denote the standard pullback maps of Kähler differentials by

$$df: f^*\Omega^p_X \to \Omega^p_Y \qquad \text{and} \qquad df: H^0(X, \Omega^p_X) \to H^0(Y, \Omega^p_Y).$$

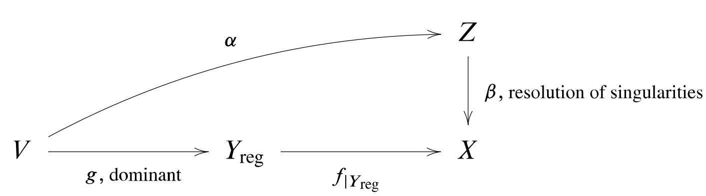
Reflexive differential forms do not generally satisfy the same universal properties as Kähler differentials. However, it has been shown in [42] and [60] that many of the functorial properties do hold if we restrict ourselves to klt spaces.

THEOREM 2.5 ([60, Theorem 1.3])
Let $f: Y \to X$ be a morphism of normal varieties. Suppose that X is klt. Then there exist pullback morphisms

$$d_{\mathrm{refl}} f: f^*\Omega^{[p]}_Y \to \Omega^{[p]}_X \qquad \textit{and} \qquad d_{\mathrm{refl}} f: H^0(X, \Omega^{[p]}_X) \to H^0(Y, \Omega^{[p]}_Y)$$

that agree with the usual pullback morphisms of Kähler differentials wherever this makes sense.

More precisely, the pullback morphism for reflexive forms satisfies the following universal property (see [60, Proposition 6.1]). Let $f: Y \to X$ be any morphism of klt spaces. Given a commutative diagram



where V is smooth, we have

$$dg \circ d_{\mathrm{refl}}(f_{|Y_{\mathrm{reg}}}) = d\alpha \circ d_{\mathrm{refl}}\beta.$$

2.7. Pullback of Weil divisors

Let $\psi\colon X \to Y$ be a dominant equidimensional morphism of normal varieties, and let D be a Weil $\mathbb{Q}$-divisor on Y. The *pullback* ψ^*D *of* D is defined as follows. We define ψ^*D to be the unique $\mathbb{Q}$-divisor on X whose restriction to $\psi^{-1}(Y_{\mathrm{reg}})$ is $(\psi_{|\psi^{-1}(Y_{\mathrm{reg}})})^*(D_{|Y_{\mathrm{reg}}})$. This construction agrees with the usual pullback if D is $\mathbb{Q}$-Cartier.

We will need the following easy observation.

LEMMA 2.6

Let $\psi\colon X \to Y$ be a projective and dominant morphism of normal varieties, and let D be a Weil divisor on Y. Suppose in addition that ψ is equidimensional. If ψ^*D is $\mathbb{Q}$-Cartier, then so is D.

Proof

The statement is local on Y, hence we may shrink Y and assume that Y is affine. By [38, Theorem 6.3], there exists a subvariety $Z \subseteq X$ such that $\psi_{|Z}\colon Z \to Y$ is finite and surjective. Replacing X by the normalization of Z, we may assume without loss of generality that ψ is a finite morphism. Recall that ψ^*D is $\mathbb{Q}$-Cartier by assumption. Shrinking Y again, if necessary, we may also assume that there exist a positive integer m and a rational function t on X such that $\operatorname{div} t = m\psi^*D$. One then readily checks that $\operatorname{div} \mathrm{N}_{X/Y}(t) = m(\deg\psi)D$, where $\mathrm{N}_{X/Y}(t)$ denotes the norm of t. This shows that D is $\mathbb{Q}$-Cartier (see [63, Lemma 5.16] for a somewhat related result). □

2.8. Numerical dimension

Let D be a $\mathbb{Q}$-divisor on a complex projective manifold X, and let A be an ample divisor on X. Following Nakayama (see [69, Definition V.2.5]), we set

$$\sigma(D,A) := \max\Big\{k \in \mathbb{Z}_{\geq 0} \;\Big|\; \varlimsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(\lfloor mD\rfloor + A))}{m^k} > 0\Big\}$$

if $h^0(X, \mathcal{O}_X(\lfloor mD\rfloor + A)) \neq 0$ for some arbitrary large positive integer m, and $\sigma(D,A) := -\infty$ otherwise. The numerical dimension of D is defined as

$$\nu(D) := \max\big\{\sigma(D,A) \;\big|\; A \text{ ample divisor on } X\big\}.$$

Let now D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X, and let $\beta\colon Z \to X$ be a resolution of singularities. The *numerical dimension* of D is defined as $\nu(D) := \nu(\beta^*D)$. Then $\nu(D)$ is independent of the resolution, and it depends only on the numerical class of D by [69, Proposition V.2.7]. We refer to [69] for more detailed properties.

Remark 2.7
Recall that a Weil $\mathbb{Q}$-divisor D on a normal projective variety X is said to be *pseudo-effective* if, for any big $\mathbb{Q}$-divisor B on X and any rational number $\varepsilon > 0$, there exists an effective $\mathbb{Q}$-divisor E on X such that $D + \varepsilon B \sim_{\mathbb{Q}} E$. If D is $\mathbb{Q}$-Cartier, then D is pseudoeffective if and only if $\nu(D) \geq 0$.

2.9. Quasi-étale morphisms
We will need the following definition.

Definition 2.8
A *cover* is a finite and surjective morphism of normal varieties.

A morphism $f : Y \to X$ between normal varieties is called a *quasi-étale morphism* if f is finite and étale in codimension 1.

Remark 2.9
Let $f : Y \to X$ be a quasi-étale cover. By the Nagata–Zariski purity theorem, f branches only on the singular set of X. In particular, we have $f^{-1}(X_{\mathrm{reg}}) \subseteq Y_{\mathrm{reg}}$.

The following elementary fact will be used throughout the paper.

FACT 2.10
Let $f : Y \to X$ be a quasi-étale cover between normal complex varieties. If K_X is Cartier (resp., $\mathbb{Q}$-Cartier), then $K_Y \sim_{\mathbb{Z}} f^* K_X$ is Cartier (resp., $\mathbb{Q}$-Cartier) as well. If X is terminal (resp., canonical, klt), then so is Y by [61, Proposition 3.16].

2.10. Augmented irregularity
The irregularity of normal complex projective varieties is generally not invariant under quasi-étale maps. The notion of augmented irregularity addresses this issue (see [43, Definition 3.1]).

Definition 2.11
Let X be a normal complex projective variety. We denote the irregularity of X by $q(X) := h^1(X, \mathscr{O}_X)$ and define the *augmented irregularity* as

$$\widetilde{q}(X) := \max\{q(Y) \mid Y \to X \text{ a quasi-étale cover}\} \in \mathbb{N} \cup \{\infty\}.$$

Remark 2.12
By a result of Elkik [35, Théorème 1], canonical singularities are rational. It follows that the irregularity is a birational invariant of complex projective varieties with canonical singularities.

Remark 2.13
The augmented irregularity of canonical varieties with numerically trivial canonical class is finite. This follows easily from [58, Proposition 8.3].

The following result often reduces the study of varieties with trivial canonical class to those with $\widetilde{q}(X) = 0$ (see also [58, Proposition 8.3]).

THEOREM 2.14 ([43, Corollary 3.6])
Let X be a normal complex projective variety with canonical singularities. Assume that K_X is numerically trivial. Then there exist an abelian variety A, a normal projective variety Y with $K_Y \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(Y) = 0$, and a quasi-étale cover $A \times Y \to X$.

2.11. Automorphism group

Let X be a complex projective variety, and let $\mathrm{Aut}^\circ(X)$ be the neutral component of the automorphism group $\mathrm{Aut}(X)$ of X; $\mathrm{Aut}^\circ(X)$ is an algebraic group of finite type with $\dim \mathrm{Aut}^\circ(X) = h^0(X, T_X)$. By Chevalley's structure theorem on algebraic groups, $\mathrm{Aut}^\circ(X)$ has a largest connected affine normal subgroup G. Further, the quotient group $\mathrm{Aut}^\circ(X)/G$ is an abelian variety. By [82, Theorem 14.1], if G is nontrivial, then X is uniruled. In particular, if X is canonical and $K_X \equiv 0$, then $\mathrm{Aut}^\circ(X)$ is an abelian variety. Lemma 2.15 below extends this observation to klt spaces.

LEMMA 2.15
Let X be a complex projective variety with klt singularities. Assume that K_X is numerically trivial. Then the neutral component $\mathrm{Aut}^\circ(X)$ of the automorphism group $\mathrm{Aut}(X)$ of X is an abelian variety.

Proof
By [69, Corollary V.4.9], K_X is torsion. Let m be its Cartier index, and let $f\colon Y \to X$ be the index 1 canonical cover, which is quasi-étale (see [63, Definition 2.52]). By Fact 2.10, the variety Y is then klt. Moreover, we have $K_Y \sim_{\mathbb{Z}} 0$ by construction, and therefore Y has canonical singularities. On the other hand,

$$Y \cong \mathrm{Spec}_X \bigoplus_{i=0}^{m-1} \mathscr{O}_X(-iK_X),$$

and hence there is an injective morphism of algebraic groups $\mathrm{Aut}^\circ(X) \subseteq \mathrm{Aut}^\circ(Y)$. The lemma then follows from [82, Theorem 14.1] applied to Y as explained above. $\square$

Remark 2.16
Let X be a complex projective variety with klt singularities. Suppose that $K_X \sim_{\mathbb{Z}} 0$, and set $n := \dim X$. Then

$$\begin{aligned}
&\dim \operatorname{Aut}^\circ(X) \\
&\quad = h^0(X, T_X) \\
&\quad = h^0(X, \Omega_X^{[n-1]}) \quad \text{since } \Omega_X^{[n-1]} \cong T_X \\
&\quad = h^{n-1}(X, \mathscr{O}_X) \quad \text{by Hodge symmetry for klt spaces (see [43, Proposition 6.9])} \\
&\quad = q(X) \quad \text{by Serre duality using the assumption that } K_X \sim_{\mathbb{Z}} 0.
\end{aligned}$$

3. Foliations

In this section, we have gathered a number of results and facts concerning foliations which will later be used in the proofs.

Definition 3.1
A *foliation* on a normal variety X over a field k is a coherent subsheaf $\mathscr{G} \subseteq T_X$ such that

(1) $\mathscr{G}$ is closed under the Lie bracket, and
(2) $\mathscr{G}$ is saturated in T_X; that is, the quotient $T_X/\mathscr{G}$ is torsion-free.

The *rank* r of $\mathscr{G}$ is the generic rank of $\mathscr{G}$. The *codimension* of $\mathscr{G}$ is defined as $q := \dim X - r$.

The *canonical class* $K_{\mathscr{G}}$ of $\mathscr{G}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{G}}) \cong \det \mathscr{G}$.

Suppose that $k = \mathbb{C}$.

Let $X^\circ \subseteq X_{\mathrm{reg}}$ be the open set where $\mathscr{G}_{|X_{\mathrm{reg}}}$ is a subbundle of $T_{X_{\mathrm{reg}}}$. A *leaf* of $\mathscr{G}$ is a maximal connected and immersed holomorphic submanifold $L \subseteq X^\circ$ such that $T_L = \mathscr{G}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure.

The foliation $\mathscr{G}$ is said to be *algebraically integrable* if its leaves are algebraic.

We will use the following notation.

Notation 3.2
Let $\psi\colon X \to Y$ be a dominant equidimensional morphism of normal varieties.

Write $K_{X/Y} := K_X - \psi^* K_Y$. We will refer to it as the *relative canonical divisor of X over Y*.

Set

$$R(\psi) = \sum_D \bigl(\psi^* D - (\psi^* D)_{\mathrm{red}}\bigr),$$

where D runs through all prime divisors on Y. We will refer to it as the *ramification divisor of* ψ.

Example 3.3
Let $\psi: X \to Y$ be a dominant equidimensional morphism of normal varieties, and let $\mathscr{G}$ be the foliation on X induced by ψ. Then $\mathscr{G}$ is the saturation in T_X of the kernel of the tangent map

$$T\psi_{|\psi^{-1}(Y_{\mathrm{reg}})}: T_{\psi^{-1}(Y_{\mathrm{reg}})} \to (\psi_{|\psi^{-1}(Y_{\mathrm{reg}})})^* T_{Y_{\mathrm{reg}}}.$$

A straightforward computation then shows that

$$K_{\mathscr{G}} \sim_{\mathbb{Z}} K_{X/Y} - R(\psi).$$

3.1. Foliations defined by q-forms
Let $\mathscr{G}$ be a codimension q foliation on an n-dimensional normal variety X. The *normal sheaf* of $\mathscr{G}$ is $\mathscr{N}_{\mathscr{G}} := (T_X/\mathscr{G})^{**}$. The qth wedge product of the inclusion $\mathscr{N}_{\mathscr{G}}^* \hookrightarrow \Omega_X^{[1]}$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega_X^q \boxtimes \det \mathscr{N}_{\mathscr{G}})$ whose zero locus has codimension at least 2 in X. Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \dots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{G}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X, and let $\omega \in H^0(X, \Omega_X^q \boxtimes \mathscr{L})$ be a global section whose zero locus has codimension at least 2 in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1} \boxtimes \mathscr{L}$ given by the contraction with ω defines a foliation $\mathscr{G}$ of codimension q on X with $\det \mathscr{N}_{\mathscr{G}} \cong \mathscr{L}$. These constructions are the inverse of each other.

3.2. Foliations described as pullbacks
Let X and Y be normal varieties, and let $\varphi: X \dashrightarrow Y$ be a dominant separable rational map that restricts to a morphism $\varphi^\circ: X^\circ \to Y^\circ$, where $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$ are smooth open subsets.

Let $\mathscr{G}$ be a codimension q foliation on Y. Suppose that the restriction $\mathscr{G}^\circ$ of $\mathscr{G}$ to Y° is defined by a twisted q-form $\alpha_{Y^\circ} \in H^0(Y^\circ, \Omega_{Y^\circ}^q \otimes \det \mathscr{N}_{\mathscr{G}^\circ})$. Then α_{Y° induces a nonzero twisted q-form

$$\alpha_{X^\circ} := d\varphi^\circ(\omega_{Y^\circ}) \in H^0\big(X^\circ, \Omega_{X^\circ}^q \otimes (\varphi^\circ)^*(\det \mathscr{N}_{\mathscr{G}|Y^\circ})\big).$$

Shrinking X° further, we may assume that α_{X° is nowhere vanishing, so that α_{X° defines a codimension q foliation $\mathscr{E}^\circ$ on X°. The *pullback* $\varphi^{-1}\mathscr{G}$ *of* $\mathscr{G}$ *via* φ is the foliation on X whose restriction to X° is $\mathscr{E}^\circ$.

The following observation is rather standard. We include a proof here for the reader's convenience.

LEMMA 3.4

Let $f : Y \to X$ be a quasifinite dominant morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X of rank $1 \le r \le \dim X - 1$. Let also B be a codimension 1 irreducible component of the branch locus of f.

(1) *Suppose that B is $\mathscr{G}$-invariant (see Section 3.5 for this notion). Let D be any irreducible component of $f^{-1}(B)$, and let m denote the ramification index of f along D. Then the natural map $\det f^{[*]}\mathscr{N}^*_{\mathscr{G}} \to \det \mathscr{N}^*_{f^{-1}\mathscr{G}}$ vanishes at order $m-1$ along D. Equivalently, the natural map $f^{[*]}\mathscr{O}_X(K_{\mathscr{G}}) \to \mathscr{O}_Y(K_{f^{-1}\mathscr{G}})$ is an isomorphism at a general point in $f^{-1}(B)$.*

(2) *If B is not $\mathscr{G}$-invariant, then the map $\det f^{[*]}\mathscr{N}^*_{\mathscr{G}} \to \det \mathscr{N}^*_{f^{-1}\mathscr{G}}$ is an isomorphism at a general point in $f^{-1}(B)$. Equivalently, let D be any irreducible component of $f^{-1}(B)$, and let m denote the ramification index of f along D. Then the natural map $f^{[*]}\mathscr{O}_X(K_{\mathscr{G}}) \to \mathscr{O}_Y(K_{f^{-1}\mathscr{G}})$ vanishes at order $m-1$ along D.*

Proof

Set $q := n - r$. Replacing X by a dense open set X° with complement of codimension at least 2 in X and Y by $f^{-1}(X^\circ)$, we may assume without loss of generality that X and Y are smooth, that the branch locus of f is a smooth hypersurface $B \subset X$, and that $\mathscr{G}$ is a regular foliation. We may also assume that the ramification divisor $D := f^{-1}(B)$ is smooth. Given a point $x \in B$, there are analytic coordinates $(x_1, \dots, x_n)$ centered at x such that B is defined by $x_1 = 0$ and such that f is given by $(y_1, \dots, y_n) \mapsto (y_1^m, y_2, \dots, y_n)$ for some integer $m \ge 1$ and some local analytic coordinates $(y_1, \dots, y_n)$ centered at a point y in $f^{-1}(x)$.

Suppose first that B is $\mathscr{G}$-invariant. Then $\mathscr{G}$ is given by a local q-form $dx_1 \wedge \alpha_1 + x_1\alpha_2$, where $\alpha_1 \in \wedge^{q-1}(\mathscr{O}_{X,x}\, dx_2 \oplus \cdots \oplus \mathscr{O}_{X,x}\, dx_n)$ is nowhere vanishing and $\alpha_2 \in \wedge^q(\mathscr{O}_{X,x}\, dx_1 \oplus \cdots \oplus \mathscr{O}_{X,x}\, dx_n)$. It follows that

$$df(dx_1 \wedge \alpha_1 + x_1\alpha_2) = y_1^{m-1}\big(m\, dy_1 \wedge df(\alpha_1) + y_1\, df(\alpha_2)\big)$$

vanishes at order $m-1$ along D.

If B is not $\mathscr{G}$-invariant, then $\mathscr{G}$ is given by a nowhere vanishing q-form $\alpha_3 \in \wedge^q(\mathscr{O}_{X,x}\, dx_2 \oplus \cdots \oplus \mathscr{O}_{X,x}\, dx_n)$. But then $df(\alpha_3)$ is a nowhere vanishing q-form. This completes the proof of the lemma. □

3.3. Ehresmann connection

Let $\pi\colon X \to Y$ be a dominant morphism with Y smooth. A *connection* (also called *Ehresmann connection*) on π is a distribution $\mathscr{E} \subseteq T_X$ such that the restriction of the tangent map $T\pi\colon T_X \to \pi^*T_Y$ to $\mathscr{E}$ induces an isomorphism $\mathscr{E} \cong \pi^*T_Y$. The connection is said to be *flat* if $\mathscr{E}$ is a foliation.

3.4. Projectable foliations

Let $\pi\colon X \to Y$ be a dominant separable morphism between normal varieties, and let $\mathscr{G}$ be a foliation on X. We say that $\mathscr{G}$ is *projectable under* π if there exists a saturated distribution $\mathscr{H} \subseteq T_Y$ such that the restriction of the tangent map

$$T\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}\colon T_{X\,|\pi^{-1}(Y_{\mathrm{reg}})} \to {\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}}^{*}T_{Y_{\mathrm{reg}}}$$

to $\mathscr{G}_{|\pi^{-1}(Y_{\mathrm{reg}})}$ induces an isomorphism

$$\mathscr{G}_{|\pi^{-1}(Y_{\mathrm{reg}})} \cong \left({\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}}^{*}\mathscr{H}_{|\pi^{-1}(Y_{\mathrm{reg}})}\right)^{\mathrm{sat}},$$

where $\left({\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}}^{*}\mathscr{H}_{|\pi^{-1}(Y_{\mathrm{reg}})}\right)^{\mathrm{sat}}$ denotes the saturation of ${\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}}^{*}\mathscr{H}_{|\pi^{-1}(Y_{\mathrm{reg}})}$ in

$$(\pi^*T_Y)_{|\pi^{-1}(Y_{\mathrm{reg}})} \cong {\pi_{|\pi^{-1}(Y_{\mathrm{reg}})}}^{*}T_{Y_{\mathrm{reg}}}.$$

One then checks that $\mathscr{H}$ is a foliation on Y. We refer the reader to [30, Section 2.7] for a more detailed explanation.

3.5. Invariant subvarieties

Let X be a normal variety, let $Y \subseteq X$ be a closed subvariety, and let ∂ be a derivation on X. Say that Y is *invariant under* ∂ if $\partial(\mathscr{I}_Y) \subseteq \mathscr{I}_Y$.

Let $\mathscr{G} \subseteq T_X$ be a foliation on X. Say that Y is *invariant under* $\mathscr{G}$ if for any local section ∂ of $\mathscr{G}$ over some open subset U of X, $Y \cap U$ is invariant under ∂. To prove that Y is invariant under $\mathscr{G}$ it is enough to show that $Y \cap U$ of Y is invariant under $\mathscr{G}_{|U}$ for some open set $U \subseteq X$ such that $Y \cap U$ is dense in Y. If X and Y are smooth and $\mathscr{G} \subseteq T_X$ is a subbundle, then Y is invariant under $\mathscr{G}$ if and only if $\mathscr{G}_{|Y} \subseteq T_Y \subseteq T_{X\,|Y}$.

By [75, Theorem 5], the singular locus of a normal complex variety is invariant under any derivation. Other examples of invariant subsets are provided by the following easy result.

LEMMA 3.5

Let $\mathscr{G} \subseteq T_X$ be a foliation of rank $r \geq 1$ on a complex manifold X. Then any component of the singular locus of $\mathscr{G}$ is invariant under $\mathscr{G}$.

Proof

We argue by induction on $r \geq 1$.

If $r = 1$, then the singular set of $\mathscr{G}$ is obviously invariant under $\mathscr{G}$.

Suppose from now on that $r \geq 2$. The statement is local on X, hence we may shrink X and assume that X is affine. Let S be a component of the singular locus of $\mathscr{G}$, and let also $x \in S$ be a general point. We may also assume that $\partial(\mathscr{I}_{\{x\}}) \nsubseteq \mathscr{I}_{\{x\}}$ for some $\partial \in H^0(X, \mathscr{G}) \subseteq H^0(X, T_X)$. But then $\mathscr{O}_X \partial \subset T_X$ is a regular foliation at x of rank 1. By a theorem of Frobenius, there exist an open neighborhood U of x with respect to the analytic topology and an isomorphism of analytic varieties $U \cong \mathbb{D} \times W$ such that $\mathscr{O}_X \partial \subset T_X$ is induced on U by the projection $\varphi\colon U \cong \mathbb{D} \times W \to W$, where $\mathbb{D}$ is the complex open unit disk and W is a germ of smooth analytic variety. In particular, there is a foliation $\mathscr{H}$ on W such that $\mathscr{G}_{|U} = \varphi^{-1}\mathscr{H}$, and hence $S \cap U = \varphi^{-1}(T)$ for some component T of the singular set of $\mathscr{H}$. By induction, we conclude that T is invariant under $\mathscr{H}$. This immediately implies that S is invariant under $\mathscr{G}$, completing the proof of the lemma. □

3.6. The family of leaves

We refer the reader to [5, Remark 3.12] for a more detailed explanation. Let X be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on X. There is a unique normal complex projective variety Y contained in the normalization of the Chow variety of X whose general point parameterizes the closure of a general leaf of $\mathscr{G}$ (viewed as a reduced and irreducible cycle in X). Let $Z \to Y \times X$ denote the normalization of the universal cycle. It comes with morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ \Big\downarrow{\scriptstyle \psi} & & \\ Y & & \end{array}$$

where $\beta\colon Z \to X$ is birational and, for a general point $y \in Y$, $\beta(\psi^{-1}(y)) \subseteq X$ is the closure of a leaf of $\mathscr{G}$. The morphism $Z \to Y$ is called the *family of leaves* and Y is called the *space of leaves* of $\mathscr{G}$.

Suppose furthermore that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. There is a canonically defined effective Weil $\mathbb{Q}$-divisor B on Z such that

$$K_{\beta^{-1}\mathscr{G}} + B \sim_{\mathbb{Z}} K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}},$$

where $R(\psi)$ denotes the ramification divisor of ψ. Note that B is β-exceptional since $\beta_* K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{\mathscr{G}}$.

The following property holds in addition. Let m be a positive integer, and let $X^\circ \subseteq X$ be a dense open set such that $\mathscr{O}_{X^\circ}(mK_{\mathscr{G}|X^\circ}) \cong \mathscr{O}_{X^\circ}$. Let $f^\circ\colon X_1^\circ \to X^\circ$ be the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]). Finally, let Z_1° be the normalization of the product $Z^\circ \times_{X^\circ} X_1^\circ$, where $Z^\circ := \beta^{-1}(X^\circ)$. If $\beta_1^\circ\colon Z_1^\circ \to X_1^\circ$ and $g^\circ\colon Z_1^\circ \to Z^\circ$ denote the natural morphisms, then there exists an effective β_1°-exceptional divisor B_1° on Z_1° such that

$$K_{(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})} + B_1^\circ \sim_{\mathbb{Z}} (\beta_1^\circ)^* K_{(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})}.$$

Let C_1° denote the non-$(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})$-invariant part of the ramification divisor $R(g^\circ)$ of g°. Using Lemma 3.4 applied to g° and $(\beta^{-1}\mathscr{G})_{|Z^\circ}$, we obtain $(g^\circ)^*(B_{|Z^\circ}) \sim_{\mathbb{Q}} B_1^\circ + C_1^\circ$. By the negativity lemma, we have

$$(g^\circ)^*(B_{|Z^\circ}) = B_1^\circ + C_1^\circ. \tag{3.1}$$

3.7. Bertini-type results

The present subsection is devoted to the following auxiliary result.

PROPOSITION 3.6

Let X be a normal complex projective variety with $\dim X \geq 3$, and let $\mathscr{G} \subseteq T_X$ be a foliation of rank $2 \leq r \leq \dim X - 1$. If $H \in |\mathscr{L}|$ is a general member of a basepoint-free linear system corresponding to $\mathscr{L} \in \operatorname{Pic}(X)$, then $\mathscr{G}_{|H} \subseteq T_{X\,|H}$ and $\mathscr{G}_H := \mathscr{G}_{|H} \cap T_H$ is a foliation on H. In addition, the following hold.

(1) *Suppose that H is transverse to $\mathscr{G}$ at a general point in X. Then $\mathscr{G}_H$ has rank $r-1$, and there exists an effective divisor B on H such that*

$$K_{\mathscr{G}_H} \sim_{\mathbb{Z}} (K_{\mathscr{G}} + H)_{|H} - B.$$

Moreover, if B_1 is a prime divisor on H, then $B_1 \subseteq \operatorname{Supp} B$ if and only if $\mathscr{G}$ is tangent to H at a general point of B_1.

(2) *Suppose that there is a dense open set $X^\circ \subseteq X_{\mathrm{reg}}$ with complement of codimension at least 2 satisfying the following property. For each $x \in X^\circ$, $\mathscr{G}$ is regular at x, and there exist $H_1 \in |\mathscr{L}|$ and $H_2 \in |\mathscr{L}|$ passing through x with $H_1 \neq H_2$ such that any member of $\langle H_1, H_2\rangle$ is transverse to $\mathscr{G}$ at x. Then*

$$K_{\mathscr{G}_H} \sim_{\mathbb{Z}} (K_{\mathscr{G}} + H)_{|H}.$$

(3) *Suppose finally that $\mathscr{L}$ is very ample. Then we have*

$$K_{\mathscr{G}_H} \sim_{\mathbb{Z}} (K_{\mathscr{G}} + H)_{|H}.$$

Proof

If H is sufficiently general, then we have an inclusion $\mathscr{G}_{|H} \subseteq T_{X\,|H}$ and $\mathscr{G}_H = \mathscr{G}_{|H} \cap T_H$ is saturated in T_H by [47, Théorème 12.2.1(i)]. On the other hand, $\mathscr{G}_H$ is closed under the Lie bracket. This proves that $\mathscr{G}_H$ is a foliation.

Set $\mathscr{N} := T_X/\mathscr{G}$ and $q = n - r$, and let $\omega \in H^0(X, \Omega_X^q \boxtimes \det \mathscr{N})$ be a nonzero twisted q-form defining $\mathscr{G}$. Let also $\omega_H \in H^0(H, \Omega_H^q \boxtimes \det \mathscr{N}_{|H})$ be the induced q-form, and let B be the maximal effective divisor on H such that $\omega_H \in H^0(H, \Omega_H^q \boxtimes \det \mathscr{N}_{|H} \boxtimes \mathscr{O}_H(-B))$. Then ω_H is a twisted q-form defining $\mathscr{G}_H$ since the sheaf $\mathscr{G}_H$ is reflexive by [4, Remark 2.3]. A straightforward computation then shows that $K_{\mathscr{G}_H} \sim_{\mathbb{Z}} (K_{\mathscr{G}} + H)_{|H} - B$ using the adjunction formula $K_H \sim_{\mathbb{Z}} (K_X + H)_{|H}$. This shows item (1).

Now, item (2) follows from item (1) and an easy dimension count (see the proof of [8, Lemma 2.9]), while item (3) is an immediate consequence of item (2). □

4. Singularities of foliations

There are several notions of singularities for foliations. The notion of *reduced* foliations has been used in the birational classification of foliations by curves on surfaces (see [20]). More recently, notions of singularities coming from the minimal model program have been shown to be very useful when studying the birational geometry of foliations. We refer the reader to [67, Section I] for an in-depth discussion. Here we only recall the notion of canonical foliation following McQuillan (see [67, Definition I.1.2]).

Definition 4.1
Let $\mathscr{G}$ be a foliation on a normal complex variety X. Suppose that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. Let $\beta\colon Z \to X$ be a projective birational morphism. Then there are uniquely defined rational numbers $a(E, X, \mathscr{G})$ such that

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + \sum_E a(E, X, \mathscr{G}) E,$$

where E runs through all exceptional prime divisors for β. The rational numbers $a(E, X, \mathscr{G})$ do not depend on the birational morphism β, but only on the valuations associated to the E. We say that $\mathscr{G}$ is *canonical* if $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier and, for all E exceptional over X, $a(E, X, \mathscr{G}) \geq 0$.

Note that, in general, Definition 4.1 requires some understanding of the numbers $a(E, X, \mathscr{G})$ for all exceptional divisors of all birational modifications of X.

4.1. *Elementary properties*

In this subsection, we analyze the behavior of canonical singularities with respect to birational maps, finite covers, and projections.

LEMMA 4.2

Let $\beta\colon Z \to X$ be a birational projective morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X. Suppose that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier.

(1) *Suppose that $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + E$ for some effective β-exceptional $\mathbb{Q}$-divisor on Z. If $\beta^{-1}\mathscr{G}$ is canonical, then so is $\mathscr{G}$.*

(2) *If $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$, then $\mathscr{G}$ is canonical if and only if so is $\beta^{-1}\mathscr{G}$.*

Proof

We can write $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + E$ for some β-exceptional $\mathbb{Q}$-divisor on Z.

Suppose first that E is effective and that $\beta^{-1}\mathscr{G}$ is canonical. Let $\gamma\colon Z_1 \to X$ be a birational projective morphism of normal varieties, and let F be a prime γ-exceptional divisor on Z_1. We have to show that $a(F, X, \mathscr{G}) \geq 0$. Since $a(F, X, \mathscr{G})$ depends only on the valuation associated to F, we may assume that γ factorizes through β. Let $\beta_1\colon Z_1 \to Z$ be the induced morphism. Write

$$K_{\gamma^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \gamma^* K_{\mathscr{G}} + E_1$$

for some $\mathbb{Q}$-divisor E_1 on Z_1 with support contained in $\operatorname{Exc} \beta \circ \beta_1$. Then

$$K_{\gamma^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta_1^* K_{\beta^{-1}\mathscr{G}} + E_1 - \beta_1^* E.$$

Note that $E_1 - \beta_1^* E$ is supported on $\operatorname{Exc} \beta_1$. Indeed, we have $(\beta_1)_* E_1 - E = (\beta_1)_*(E_1 - \beta_1^* E) \sim_{\mathbb{Q}} 0$ since $(\beta_1)_* K_{(\beta\circ\beta_1)^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{\beta^{-1}\mathscr{G}}$. On the other hand, $(\beta_1)_* E_1 - E$ is β-exceptional, and hence we must have $(\beta_1)_* E_1 - E = 0$. This shows that $E_1 - \beta_1^* E$ is β_1-exceptional. Since $\beta^{-1}\mathscr{G}$ is canonical by assumption, $E_1 - \beta_1^* E$ is effective, and hence so is E_1. This proves item (1).

Suppose now that $E = 0$. If $\beta^{-1}\mathscr{G}$ is canonical, then so is $\mathscr{G}$ by item (1). The above computation also shows that $a(F, Z, \beta^{-1}\mathscr{G}) = a(F, X, \mathscr{G})$ for any prime β_1-exceptional divisor on Z_1. In particular, if $\mathscr{G}$ is canonical, then $\beta^{-1}\mathscr{G}$ is canonical as well. This proves item (2). □

LEMMA 4.3

Let $f\colon X_1 \to X$ be a quasifinite dominant morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X with $K_{\mathscr{G}}$ $\mathbb{Q}$-Cartier. Suppose that any codimension 1 component of the branch locus of f is $\mathscr{G}$-invariant. If $\mathscr{G}$ is canonical, then so is $f^{-1}\mathscr{G}$.

Proof

Set $\mathscr{G}_1 := f^{-1}\mathscr{G}$. By Lemma 3.4, we have $K_{\mathscr{G}_1} \sim_{\mathbb{Z}} f^* K_{\mathscr{G}}$. In particular, $K_{\mathscr{G}_1}$ is $\mathbb{Q}$-Cartier. Let $\beta_1\colon Z_1 \to X_1$ be a birational projective morphism, and let E_1 be a β_1-exceptional prime divisor on Z_1. By [61, Theorem 3.17], there exist a projective birational morphism $\beta\colon Z \to X$ and a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{g} & Z \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta} \\ X_1 & \xrightarrow[f]{} & X \end{array}$$

such that $E := g(E_1)$ is a β-exceptional prime divisor on Z. Let m denote the ramification index of g along E_1. By Lemma 3.4, if E is $\beta^{-1}\mathscr{G}$-invariant, then $a(E_1, X_1, \mathscr{G}_1) = ma(E, X, \mathscr{G})$, and $a(E_1, X_1, \mathscr{G}_1) = ma(E, X, \mathscr{G}) + m - 1$ otherwise. In particular, if $\mathscr{G}$ is canonical, then so is $f^{-1}\mathscr{G}$, proving the lemma. □

The following example shows that the converse is not true in general.

Example 4.4
Let G be a finite subgroup of $\mathrm{GL}(2, \mathbb{C})$ that does not contain any quasireflections, and set $X := \mathbb{A}^2/G$. Suppose that $G \not\subset \mathrm{SL}(2, \mathbb{C})$, so that X is not canonical. Let Y be a normal variety, and consider the foliation $\mathscr{G}$ on $X \times Y$ induced by the projection $X \times Y \to Y$. Let also $f : \mathbb{A}^2 \times Y \to X \times Y$ be the quasi-étale cover induced by the projection morphism $\mathbb{A}^2 \to \mathbb{A}^2/G = X$. Then $\mathscr{G}$ is not canonical while $f^{-1}\mathscr{G}$ is (see Example 4.16 below).

LEMMA 4.5
Let Y and Z be normal complex projective varieties, and let $\mathscr{H}$ be a foliation on Y. Denote by $\psi : Y \times Z \to Y$ the projection, and set $\mathscr{G} := \psi^{-1}\mathscr{H}$. If $\mathscr{G}$ is canonical, then so is $\mathscr{H}$.

Proof
Suppose that $\mathscr{G}$ is canonical. Let $\beta\colon Y_1 \to Y$ be a projective birational morphism with Y_1 normal, and let $F_1 \cong Y_1$ be a fiber of the projection $Y_1 \times Z \to Y \times Z \to Z$. Denote by $\gamma\colon Y_1 \times Z \to Y \times Z$ the natural morphism, and set $F := \gamma(F_1) \cong Y$. One then checks that $K_{\mathscr{G}} \sim_{\mathbb{Z}} K_{Y\times Z/Y} + \psi^* K_{\mathscr{H}}$ and $K_{\gamma^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{Y_1\times Z/Y_1} + \psi_1^* K_{\beta^{-1}\mathscr{H}}$, where $\psi_1\colon Y_1 \times Z \to Y_1$ denotes the projection. It follows that $K_{\mathscr{G}|F} \sim_{\mathbb{Z}} K_{\mathscr{H}}$ and $K_{\gamma^{-1}\mathscr{G}|F_1} \sim_{\mathbb{Z}} K_{\beta^{-1}\mathscr{H}}$. In particular, $K_{\mathscr{H}}$ is $\mathbb{Q}$-Cartier. By assumption, $K_{\gamma^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \gamma^* K_{\mathscr{G}} + E$ for some effective γ-exceptional $\mathbb{Q}$-divisor, and hence $K_{\beta^{-1}\mathscr{H}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{H}} + E_{|F}$. This proves the lemma. □

4.2. $\mathbb{Q}$-Factorial terminalization

In this subsection, we analyze the behavior of canonical singularities with respect to $\mathbb{Q}$-factorial terminalizations.

We will need the following auxiliary result.

LEMMA 4.6

Let X be a normal complex quasiprojective variety with klt singularities, let $1 \leq p \leq \dim X$ be an integer, and let $\mathscr{A} \subseteq \Omega_X^{[p]}$ be a saturated reflexive subsheaf of rank 1. Suppose that $\mathscr{A}^{[m]}$ is a line bundle for some positive integer m. Let $\beta\colon Z \to X$ be a resolution of singularities with exceptional set E, and assume that E is a divisor with simple normal crossings. Let $\mathscr{B} \subseteq \Omega_Z^p$ denote the saturation of $\beta^{[*]}\mathscr{A} \subseteq \Omega_Z^p$, and let E_1 denote the reduced divisor on Z whose support is the union of all irreducible components E' of E such that $\mathscr{B}$ is not saturated in $\Omega_Z^p(\log E)$ at general points of E'. Then there exist an effective β-exceptional $\mathbb{Q}$-divisor E_2 and a rational number $0 \leq \varepsilon < 1$ such that $\beta^* c_1(\mathscr{A}) + E_2 \sim_{\mathbb{Q}} c_1(\mathscr{B}) + \varepsilon E_1$.

Remark 4.7

In the setup of Lemma 4.6, [42, Theorem 4.3] shows that there is an embedding $\beta^{[*]}\mathscr{A} \subseteq \Omega_Z^p$.

Remark 4.8

In the setup of Lemma 4.6, suppose furthermore that $1 \leq p \leq \dim X - 1$, and that $\mathscr{A}$ is the conormal sheaf of a foliation $\mathscr{G}$ on X. Then $\operatorname{Supp} E_1$ is the union of all irreducible components of E that are invariant under $\beta^{-1}\mathscr{G}$.

Proof of Lemma 4.6

The proof is very similar to that of [42, Theorem 7.2], and so we leave some easy details to the reader.

Note that $\mathscr{B}$ is a line bundle by [50, Proposition 1.9]. To prove the statement, it suffices to show that there exists a rational number $0 \leq \varepsilon < 1$ such that, if σ is any section of $\mathscr{A}^{[m]}$ over some open set $U \subseteq X$, then the rational section $(\beta_{|U})^*\sigma$ of $\mathscr{B}^{\otimes m}$ is regular on $\beta^{-1}(U) \setminus \operatorname{Supp} E_1$ and has poles of order at most $m\varepsilon$ along $(\operatorname{Supp} E_1) \cap \beta^{-1}(U)$.

The statement is local on X, hence we may shrink X, and assume that $\mathscr{A}^{[m]} \cong \mathscr{O}_X$. Let $g\colon Y \to X$ be the associated cyclic cover (see [63, Definition 2.52]), and let T denote the normalization of the fiber product $Y \times_X Z$ with natural morphisms $\gamma\colon T \to Y$ and $f\colon T \to Z$. Note that Y is klt by Fact 2.10 and that $g^{[*]}\mathscr{A} \cong \mathscr{O}_Y$.

Let $\sigma \in H^0(X, \mathscr{A}^{[m]})$ be a nowhere vanishing section, and consider the pullback $\beta^*\sigma$, which is a rational section of

$$\mathscr{B}^{\otimes m} \subseteq \mathrm{S}^m \Omega_Z^p,$$

possibly with poles along E. Applying [42, Theorem 4.3] to γ and using the fact that $g^{[*]}\mathscr{A}$ is locally free, we see that there is an embedding

$$\gamma^*(g^{[*]}\mathscr{A}^{[m]}) \cong (\gamma^* g^{[*]}\mathscr{A})^{\otimes m} \subseteq \mathrm{S}^m \Omega_T^{[p]}.$$

This immediately implies that

$$(\beta \circ f)^*\sigma \in H^0(T, \mathrm{S}^m \Omega_T^{[p]}).$$

Set $n := \dim X$. Let $z \in \operatorname{Supp} E$ be a general point, and let $(z_1, \ldots, z_n)$ be local coordinates on some open neighborhood U of z in Z such that $z_1 = 0$ is a local equation of E_1. Let $t \in T$ be such that $f(t) = z$, and let $(t_1, \ldots, t_n)$ be local coordinates on some open neighborhood V of t in T with $f(V) \subseteq U$. We may assume without loss of generality that f is given by $(t_1, \ldots, t_n) \mapsto (t_1^{k_1}, t_2, \ldots, t_n)$ on V.

Suppose first that $\mathscr{B}$ is not saturated in $\Omega_Z^p(\log E)$ at z. Shrinking U, if necessary, we may assume that $\mathscr{B}_{|U}$ is generated by $dz_1 \wedge \alpha_1 + z_1\alpha_2$, where $\alpha_1 \in \wedge^{p-1}(\mathscr{O}_U\, dz_2 \oplus \cdots \oplus \mathscr{O}_U\, dz_n)$ is nowhere vanishing and $\alpha_2 \in \Omega_U^p = \wedge^p(\mathscr{O}_U\, dz_1 \oplus \cdots \oplus \mathscr{O}_U\, dz_n)$. Then $\mathscr{B}^{\otimes m}{}_{|U} \subseteq \mathrm{S}^m \Omega^p_{Z|U}$ is generated by $(dz_1 \wedge \alpha_1 + z_1\alpha_2)^{\otimes m}$ and $f^*\mathscr{B}^{\otimes m}{}_{|V} \subseteq \mathrm{S}^m \Omega_T^{[p]}{}_{|V}$ is generated by $t_1^{m(k_1-1)}(k_1 dt_1 \wedge df(\alpha_1) + t_1 df(\alpha_2))^{\otimes m}$. One then checks that $\beta^*\sigma$ has a pole of order at most $m(1 - \frac{1}{k_1})$ along $z_1 = 0$ since $(\beta \circ f)^*\sigma$ is regular on V.

If $\mathscr{B}$ is saturated in $\Omega_Z^p(\log E)$ at z, then we may assume that $\mathscr{B}_{|U}$ is generated by a nowhere vanishing p-form α_1 with $\alpha_1 \in \wedge^p(\mathscr{O}_U\, dz_2 \oplus \cdots \oplus \mathscr{O}_U\, dz_n)$. Arguing as above, one concludes that $\beta^*\sigma$ is regular in codimension 1 on U.

Write $f^*E_1 = \sum_{i \in I} k_i F_i$, where the F_i's are prime divisors on T and the k_i's are positive integers, and let $0 \le \varepsilon < 1$ be a rational number such that $\varepsilon \ge 1 - \frac{1}{k_i}$ for all indices $i \in I$. We conclude that $\beta^*\sigma$ (viewed as a rational section of the line bundle $\mathscr{B}^{\otimes m}$) is regular on $X \setminus \operatorname{Supp} E_1$ and has poles of order at most $m\varepsilon$ along $\operatorname{Supp} E_1$. This finishes the proof of the lemma. □

The following is an easy consequence of Lemma 4.6.

PROPOSITION 4.9

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X such that $c_1(\mathscr{N}_\mathscr{G})$ is $\mathbb{Q}$-Cartier. Let $\beta\colon Z \to X$ be a resolution of singularities with exceptional set E, and assume that E is a divisor with simple normal crossings. Let also E_1 denote the reduced divisor on Z whose support is the union of all irreducible components of E that are invariant under $\beta^{-1}\mathscr{G}$. There exists a rational number $0 \le \varepsilon < 1$ such that

$$\kappa\big(-c_1(\mathscr{N}_{\mathscr{G}})\big) = \kappa\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big) \qquad \textit{and}$$
$$\nu\big(-c_1(\mathscr{N}_{\mathscr{G}})\big) = \nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big).$$

In particular, $-c_1(\mathscr{N}_{\mathscr{G}})$ *is pseudoeffective if and only if so is* $-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1$*.*

Proof
Applying Lemma 4.6 and using Remark 4.8, we see that there exist an effective β-exceptional $\mathbb{Q}$-divisor E_2 and a rational number $0 \le \varepsilon < 1$ such that $-\beta^* c_1(\mathscr{N}_{\mathscr{G}}) + E_2 \sim_{\mathbb{Q}} -c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1$. By [69, Lemma II.3.11], we have $\kappa(-\beta^* c_1(\mathscr{N}_{\mathscr{G}}) + E_2) = \kappa(-c_1(\mathscr{N}_{\mathscr{G}}))$ since E_2 is effective and β-exceptional. We also have $\nu(-\beta^* c_1(\mathscr{N}_{\mathscr{G}}) + E_2) = \nu(-c_1(\mathscr{N}_{\mathscr{G}}))$ by [69, Proposition V.2.7]. This proves the proposition. □

The following results often reduce the study of mildly singular foliations with numerically trivial canonical class on varieties with canonical singularities to those on varieties with terminal singularities.

PROPOSITION 4.10
Let X *be a normal complex projective variety with canonical singularities, and let* $\mathscr{G}$ *be a foliation on* X*. Suppose that* $\mathscr{G}$ *is canonical and that* $K_{\mathscr{G}}$ *is Cartier. Let* $\beta\colon Z \to X$ *be a* $\mathbb{Q}$*-factorial terminalization of* X*. Then* $\beta^{-1}\mathscr{G}$ *is canonical with* $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$*.*

Proof
Note that $c_1(\mathscr{N}_{\mathscr{G}})$ is $\mathbb{Q}$-Cartier since $c_1(\mathscr{N}_{\mathscr{G}}) \sim_{\mathbb{Z}} K_{\mathscr{G}} - K_X$. Recall also that $K_Z \sim_{\mathbb{Q}} \beta^* K_X$. Applying Lemma 4.6 to $\det \mathscr{N}_{\mathscr{G}}^*$ on a resolution of Z, we see that there exist effective β-exceptional $\mathbb{Q}$-divisors E_1 and E_2 with E_1 reduced, and a rational number $0 \le \varepsilon < 1$ such that $c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + E_2 \sim_{\mathbb{Q}} \beta^* c_1(\mathscr{N}_{\mathscr{G}}) + \varepsilon E_1$. On the other hand, we have $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}} + F$ for some effective integral β-exceptional Weil divisor F since $\mathscr{G}$ is canonical and $K_{\mathscr{G}}$ is Cartier. Since $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} K_Z + c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}})$ and $K_{\mathscr{G}} \sim_{\mathbb{Z}} K_X + c_1(\mathscr{N}_{\mathscr{G}})$, we must have $\varepsilon E_1 = F + E_2$ by the negativity lemma. It follows that $F = 0$, and hence $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$. By Lemma 4.2, we see that $\beta^{-1}\mathscr{G}$ is canonical, completing the proof of the proposition. □

Example 4.11 below shows that Proposition 4.10 is wrong if one drops the assumption that $K_{\mathscr{G}}$ is Cartier.

Example 4.11
Let E be an elliptic curve, and set $X_1 := E \times \mathbb{P}^1$. Let G be a cyclic group of order

2 acting on E by $x \mapsto -x$ and acting on $\mathbb{P}^1$ by $(x:y) \mapsto (y:x)$, where $(x:y)$ are homogeneous coordinates on $\mathbb{P}^1$. Set $X := X_1/G$, and denote by $f\colon X_1 \to X$ the projection map, which is two-to-one quasi-étale cover. We obtain a rational surface containing eight rational double points. Consider the foliation $\mathscr{G}$ on X given by the morphism $X \to \mathbb{P}^1/G \cong \mathbb{P}^1$. The canonical divisor $K_{\mathscr{G}}$ is not Cartier, but $2K_{\mathscr{G}}$ is. Let $\beta\colon Z \to X$ be the blowup of the eight singular points, and denote by E_i the β-exceptional divisors. We have $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + \frac{1}{2}\sum_i E_i$. Now, we claim that $\mathscr{G}$ and $\beta^{-1}\mathscr{G}$ are canonical. By Lemma 4.2, it suffices to show that $\beta^{-1}\mathscr{G}$ is canonical. Any of the four singular fibers of $Z \to \mathbb{P}^1$ is a chain of three smooth rational curves: the union of two reduced disjoint (-2)-curves and a (-1)-curve with multiplicity 2. Moreover, it has simple normal crossings. Thus, $\beta^{-1}\mathscr{G}$ is locally given by the 1-form $2u\,dv + v\,du$, and hence canonical (see [67, Fact I.2.4(d)] or Proposition 4.15 below).

LEMMA 4.12

Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a foliation on X. Suppose that $\mathscr{G}$ is canonical and that $\det \mathscr{N}_{\mathscr{G}}$ is Cartier. Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorial terminalization of X. Then $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$.

Proof

Recall that $K_Z \sim_{\mathbb{Q}} \beta^* K_X$. By [42, Theorem 4.3], there is an effective β-exceptional Weil divisor E on Z such that $c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) \sim_{\mathbb{Z}} \beta^* c_1(\mathscr{N}_{\mathscr{G}}) - E$. Since $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} K_Z + c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}})$ and $K_{\mathscr{G}} \sim_{\mathbb{Q}} K_X + c_1(\mathscr{N}_{\mathscr{G}})$, we must have $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} - E$. It follows that $E = 0$ since $\mathscr{G}$ is canonical by assumption. Applying Lemma 4.2, we see that $\beta^{-1}\mathscr{G}$ is canonical. This finishes the proof of the lemma. □

4.3. Algebraically integrable foliations

In this subsection, we address algebraically integrable foliations with canonical singularities.

LEMMA 4.13

Let X be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on X. Suppose that $\mathscr{G}$ is canonical. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6). Then the following hold.

(1) *The foliation $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$.*

(2) *There exists a dense open set $Y^\circ \subseteq Y$ such that Z has canonical singularities over Y°. In particular, a general fiber of ψ has canonical singularities.*

(3) *If P is a prime divisor on Y and $C := (\psi^* P)_{\mathrm{red}}$, then the pair (Z, C) is lc over an open neighborhood of the generic point of P.*

Proof
Recall from Section 3.6 that there is an effective Weil $\mathbb{Q}$-divisor B on Z such that

$$K_{\beta^{-1}\mathscr{G}} + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}. \tag{4.1}$$

On the other hand, since $\mathscr{G}$ is canonical by assumption, there exists an effective Weil $\mathbb{Q}$-divisor E on Z such that

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + E. \tag{4.2}$$

From equations (4.1) and (4.2), we obtain

$$B + E \sim_{\mathbb{Q}} 0.$$

This immediately implies that $B = 0$ and $E = 0$, and shows that $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. By Lemma 4.2, $\beta^{-1}\mathscr{G}$ is then canonical, proving item (1).

Let $\beta_1 \colon Z_1 \to Z$ be a resolution of singularities, and set $\psi_1 := \psi \circ \beta_1$. By Example 3.3, we have $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{Z/Y} - R(\psi)$ and $K_{\beta_1^{-1}\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{Z_1/Y} - R(\psi_1) + F_1$, where $R(\psi)$ and $R(\psi_1)$ denote the ramification divisors of ψ and ψ_1, respectively, and F_1 is a β_1-exceptional $\mathbb{Q}$-divisor on Z_1 such that $\psi_1(\operatorname{Supp} F_1)$ has codimension at least 2 in Y. In particular, $K_{Z/Y} - R(\psi)$ is $\mathbb{Q}$-Cartier. Since $\beta^{-1}\mathscr{G}$ is canonical, there exists an effective β_1-exceptional $\mathbb{Q}$-divisor E_1 on Z_1 such that

$$K_{\beta_1^{-1}\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta_1^* K_{\beta^{-1}\mathscr{G}} + E_1. \tag{4.3}$$

Set also $Y^\circ := Y \setminus \psi_1(\operatorname{Supp} R(\psi_1) \cup \operatorname{Supp} F_1)$, $Z^\circ := \psi^{-1}(Y^\circ)$, and $Z_1^\circ := \psi_1^{-1}(Y^\circ)$. Equation (4.3) then gives

$$K_{Z_1^\circ} \sim_{\mathbb{Q}} \beta_1^* K_{Z^\circ} + E_{1|Z_1^\circ}.$$

This shows that Z° has canonical singularities. Item (2) follows easily.

To prove item (3), we may assume that $\beta \colon Z_1 \to Z$ is a log resolution of (Z, C). Shrinking Y, if necessary, we may assume that Y is smooth, and that ψ_1 is also equidimensional. Then $K_{\beta_1^{-1}\beta^{-1}\mathscr{G}} = K_{Z_1/Y} - R(\psi_1)$. We may also assume without loss of generality that either $R(\psi_1) = 0$ or $\psi_1(\operatorname{Supp} R(\psi_1)) = P$. It follows that $C = \psi^* P - R(\psi)$ and that $C_1 := (\psi_1^* P)_{\mathrm{red}} = \psi_1^* P - R(\psi_1)$. Equation (4.3) now yields

$$K_{Z_1} + C_1 \sim_{\mathbb{Q}} \beta^*(K_Z + C) + E_1.$$

Since C_1 is reduced, we conclude that (Z, C) is lc, completing the proof of the lemma. □

Remark 4.14
In the setup of Lemma 4.13(1), suppose in addition that $K_{\mathscr{G}}$ is Cartier. Then $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$.

The converse is also true if $\dim Y = 1$ by Proposition 4.15 below.

PROPOSITION 4.15
Let X be a normal complex projective variety, and let $\psi\colon X \to Y$ be a surjective morphism onto a smooth projective curve. Denote by $\mathscr{G}$ the foliation induced by ψ. Let $\beta\colon X_1 \to X$ be a resolution of singularities, and denote by $R(\psi_1)$ the ramification divisor of $\psi_1 := \psi \circ \beta$. If X has canonical singularities over $Y \setminus \psi_1(\operatorname{Supp} R(\psi_1))$ and, for any point P in $\psi_1(\operatorname{Supp} R(\psi_1))$, the pair $(X, (\psi^* P)_{\mathrm{red}})$ is lc over an open neighborhood of P, then $\mathscr{G}$ is canonical.

Proof
Let $R(\psi)$ denote the ramification divisor of ψ, and denote by $B(\psi)$ the reduced divisor on Y with support $\psi(\operatorname{Supp} R(\psi))$. By Example 3.3, we have $K_{\mathscr{G}} = K_{X/Y} - R(\psi)$. Note that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier since $R(\psi) = \psi^* B(\psi) - (\psi^* B(\psi))_{\mathrm{red}}$ and $K_X + (\psi^* B(\psi))_{\mathrm{red}}$ is $\mathbb{Q}$-Cartier.

Let $\beta\colon X_1 \to X$ be a resolution of singularities, and let E be the β-exceptional divisor on X_1 such that

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} + E. \tag{4.4}$$

Denote by $B(\psi_1)$ the reduced divisor on Y with support $\psi_1(\operatorname{Supp} R(\psi_1))$. Set $C := (\psi^* B(\psi_1))_{\mathrm{red}}$ and $C_1 := (\psi_1^* B(\psi_1))_{\mathrm{red}}$. Then (4.4) above yields

$$K_{X_1} + C_1 = \beta^*(K_X + C) + E.$$

Since X has canonical singularities over $Y^\circ := Y \setminus \psi_1(\operatorname{Supp} R(\psi_1))$, we see that E is effective over Y°. On the other hand, since the pair (X, C) is lc over some open neighborhood of $\operatorname{Supp} B(\psi_1)$, we conclude that E is effective over some open neighborhood of $\operatorname{Supp} B(\psi_1)$. This proves that E is effective, completing the proof of the proposition. □

Example 4.16
Let Y and Z be normal projective varieties, and let $\mathscr{G}$ be the foliation on $X := Y \times Z$ induced by the projection $Y \times Z \to Y$. Then $\mathscr{G}$ is canonical if and only if Z has canonical singularities. Indeed, if $\mathscr{G}$ is canonical, then Z has canonical singularities by Lemma 4.13 above. Suppose that Z has canonical singularities, and let $\beta_1\colon Z_1 \to Z$ be a resolution of singularities. Let also $Y_1 \to Y$ be a resolution of Y_1. Let $\mathscr{G}_1$ be

the foliation on $Y_1 \times Z$ induced by the projection $Y_1 \times Z \to Y_1$, and denote by $\gamma_1 \colon X_1 := Y_1 \times Z \to Y \times Z = X$ the natural morphism. Notice that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier and that $K_{\mathscr{G}_1} \sim_{\mathbb{Q}} \gamma_1^* K_{\mathscr{G}}$. Thus, by Lemma 4.2, it suffices to show that $\mathscr{G}_1$ is canonical. Let $\mathscr{G}_2$ be the foliation on $X_2 := Y_1 \times Z_1$ induced by the projection $Y_1 \times Z_1 \to Y_1$, and denote by $\gamma_2 \colon X_2 = Y_1 \times Z_1 \to Y_1 \times Z = X_1$ the natural morphism. Since Z has canonical singularities, $K_{\mathscr{G}_2} \sim_{\mathbb{Q}} \gamma_2^* K_{\mathscr{G}_1} + E_2$ for some effective and γ_2-exceptional Weil $\mathbb{Q}$-divisor. Now, $\mathscr{G}_2$ is canonical since it is a regular foliation (see Lemma 5.9). In particular, if $\gamma_3 \colon X_3 \to X_2$ is any projective birational morphism with X_3 normal, then $K_{\gamma_3^{-1}\mathscr{G}_2} \sim_{\mathbb{Q}} \gamma_3^* K_{\mathscr{G}_2} + E_3$ for some effective and γ_3-exceptional Weil $\mathbb{Q}$-divisor. It follows that $K_{\gamma_3^{-1}\mathscr{G}_2} \sim_{\mathbb{Q}} (\gamma_2 \circ \gamma_3)^* K_{\mathscr{G}_1} + \gamma_3^* E_2 + E_3$. This shows that $\mathscr{G}$ is canonical.

The following result, which will be crucial for the proof of Theorem 6.1, extends [8, Lemma 2.12] to the singular setting.

PROPOSITION 4.17
Let $\mathscr{G}$ be a foliation of rank $r \geq 1$ on a normal complex projective variety X. Suppose that $\mathscr{G}$ is algebraically integrable and that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. Let $\psi \colon Z \to Y$ be the family of leaves, and let $\beta \colon Z \to X$ be the natural morphism (see Section 3.6). Let also B be the effective β-exceptional $\mathbb{Q}$-divisor on Z such that $K_{\beta^{-1}\mathscr{G}} + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. If E is a β-exceptional prime divisor on Z such that $\psi(E) = Y$, then $E \subseteq \operatorname{Supp} B$.

Remark 4.18
Proposition 4.17 says that $\mathscr{G}$ is not canonical along $\beta(E)$.

Proof of Proposition 4.17
We argue by induction on $r \geq 1$. Let E be a β-exceptional prime divisor on Z, and assume that $\psi(E) = Y$.

Suppose first that $r = 1$. Let m be a positive integer, and let $X^\circ \subseteq X$ be a nonempty open set such that $\mathscr{O}_{X^\circ}(mK_{\mathscr{G}|X^\circ}) \cong \mathscr{O}_{X^\circ}$. Suppose in addition that $\beta(E) \cap X^\circ \neq \emptyset$. Let $f^\circ \colon X_1^\circ \to X^\circ$ be the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]). Finally, let Z_1° be the normalization of the product $Z^\circ \times_{X^\circ} X_1^\circ$, where $Z^\circ := \beta^{-1}(X^\circ)$, and let $\beta_1^\circ \colon Z_1^\circ \to X_1^\circ$ and $g^\circ \colon Z_1^\circ \to Z^\circ$ denote the natural morphisms. Recall from Section 3.6 that there exists an effective β_1°-exceptional divisor B_1° on Z_1° such that $K_{(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})} + B_1^\circ \sim_{\mathbb{Z}} 0$. Moreover, $(g^\circ)^*(B_{|Z^\circ}) - B_1^\circ$ is effective (see (3.1)). Since $\psi(E) = Y$ by assumption, we see that E is not invariant under $\beta^{-1}\mathscr{G}$. Let E_1° be a prime divisor on Z_1° such that $g^\circ(E_1^\circ) = E \cap Z^\circ$. Notice that E_1° is not invariant under $(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})$ and that E_1° is β_1°-exceptional. By

[75, Theorem 5], the singular locus of X_1° is invariant under any derivation on X_1°. Thus, if $\beta_1^\circ(E_1^\circ)$ is contained in $X_1^\circ \setminus (X_1^\circ)_{\mathrm{reg}}$, then $E_1^\circ \subseteq \operatorname{Supp} B_1^\circ$ by Lemma 4.19 below. Suppose that $\beta_1^\circ(E_1^\circ) \cap (X_1^\circ)_{\mathrm{reg}} \neq \emptyset$. By construction, the image in X_1° of any irreducible component of a general fiber of the composed map $Z_1^\circ \to Z^\circ \to Y$ is the closure of a leaf of $(f^\circ)^{-1}(\mathscr{G}_{|X_1^\circ})$. Since E_1° is β_1°-exceptional, we see that there are infinitely many such subvarieties through a general point of $\beta_1^\circ(E_1^\circ)$. Therefore, $\beta_1^\circ(E_1^\circ)$ is contained in the singular locus of $(f^\circ)^{-1}(\mathscr{G}_{|X_1^\circ})$, and hence $E_1^\circ \subseteq \operatorname{Supp} B_1^\circ$ by Lemma 4.19 again and Lemma 3.5. In either case, since $(g^\circ)^*(B_{|Z^\circ}) - B_1^\circ$ is effective, we see that $E \subseteq \operatorname{Supp} B$. This proves the proposition when $r = 1$.

Suppose from now on that $r \geq 2$. We may assume without loss of generality that $X \subseteq \mathbb{P}^N$ for some positive integer N.

Let $H \subset X$ be a general hyperplane section. We may assume that H and $G := \beta^{-1}(H)$ are normal varieties (see [47, Théorème 12.2.4]). Set $\gamma := \beta_{|G} \colon G \to H$ and $\mathscr{E} := \mathscr{G}_{|H} \cap T_H$, so that $\gamma^{-1}\mathscr{E} = \beta^{-1}\mathscr{G}_{|G} \cap T_G$. Applying [47, Théorème 12.2.4] again, we see that general fibers of $\psi_{|G} \colon G \to Y$ are integral by general choice of H. This implies that $\psi_{|G} \colon G \to Y$ is the family of leaves of $\mathscr{E}$. Since the restriction of β to any fiber of ψ is finite, we have $\dim \beta(E) \geq r - 1 \geq 1$. In particular, $E_{|G}$ is a nonzero divisor on G.

Using Proposition 3.6 and the formula $K_{\beta^{-1}\mathscr{G}} + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$, we obtain

$$K_{\mathscr{E}} \sim_{\mathbb{Z}} K_{\mathscr{G}|H} + H_{|H} \tag{4.5}$$

and

$$K_{\gamma^{-1}\mathscr{E}} \sim_{\mathbb{Z}} K_{\beta^{-1}\mathscr{G}|G} + G_{|G} - B_G \sim_{\mathbb{Q}} \gamma^* K_{\mathscr{E}} - B_G - B_{|G}, \tag{4.6}$$

for some effective γ-exceptional divisor B_G on G. By induction, we must have

$$\operatorname{Supp} E_{|G} \subseteq \operatorname{Supp}(B_G + B_{|G}).$$

Given a general fiber F of ψ, we may assume that $H \cap F_{\mathrm{reg}}$ is smooth by Bertini's theorem. This immediately implies that $\psi_{|G}(\operatorname{Supp} B_G) \subsetneq Y$ (see Proposition 3.6(1)). On the other hand, by general choice of H, any irreducible component of $E \cap G$ is mapped onto Y by $\psi_{|G}$. Therefore, we have $\operatorname{Supp} E_{|G} \subseteq \operatorname{Supp} B_{|G}$, and hence

$$E \subseteq \operatorname{Supp} B.$$

This completes the proof of the proposition. □

LEMMA 4.19
Let Z and X be normal complex varieties, and let $\beta \colon Z \to X$ be a birational projective morphism. Let $\mathscr{G}$ be a foliation of rank 1 on X. Suppose that $K_{\mathscr{G}}$ is Cartier,

and write $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}} + B$ for some β-exceptional divisor B on Z. Let $E \subset Z$ be a prime divisor not contained in Supp B. *If $\beta(E)$ is contained in a proper closed $\mathscr{G}$-invariant subvariety $Y \subsetneq X$, then E is invariant under $\beta^{-1}\mathscr{G}$.*

Proof
The statement is local on X, hence we may shrink X and assume that $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$ and that X is affine, $X \subseteq \mathbb{A}^N$ for some integer $N \geq 1$.

Let $\partial_Z \in H^0(Z, T_Z \boxtimes \mathscr{O}_Z(B))$ be such that $\beta^{-1}\mathscr{G} \boxtimes \mathscr{O}_Z(B) = \mathscr{O}_Z \partial_Z$, and denote by $\partial_X \in H^0(X, T_X)$ the derivation on X induced by ∂_Z, so that $\mathscr{G} = \mathscr{O}_X \partial_X$. By construction, ∂_Z is a regular derivation on $Z \setminus \operatorname{Supp} B$.

Let f be a nonzero regular function on X, vanishing on Y, such that $m := v_E(f)$ is minimal, where v_E denotes the divisorial valuation on the field of rational functions on X induced by E. Note that m is positive since f vanishes on Y. Let also g be a local equation of E on some open subset $U \subseteq Z \setminus \operatorname{Supp}(B)$. There exists a function u on U such that $u_{|E\cap U} \not\equiv 0$ and $f \circ \beta_{|U} = ug^m$. It follows that

$$\partial_Z(f \circ \beta)_{|U} = g^m \partial_{Z|U}(u) + mug^{m-1}\partial_{Z|U}(g).$$

On the other hand, we have

$$\partial_Z(f \circ \beta) = \beta \circ \partial_X(f)$$

and thus

$$v_E\big(\partial_Z(f \circ \beta)\big) = v_E\big(\partial_X(f)\big) \geq v_E(f) = m$$

by choice of f, using the fact that $\partial_X(f)$ vanishes on Y since f does and Y is $\mathscr{G}$-invariant.

Suppose that $v_E(\partial_Z(g)) = 0$. We have $v_E(\partial_{Z|U}(u)) \geq 0$ since $\partial_{Z|U}$ is regular on U, and thus

$$v_E\big(g^m \partial_{Z|U}(u)\big) \geq m.$$

But then

$$v_E\big(g^m \partial_{Z|U}(u) + mug^{m-1}\partial_{Z|U}(g)\big) = v_E\big(mug^{m-1}\partial_{Z|U}(g)\big) = m - 1$$

since $v_E(u) = 0$, yielding a contradiction. This shows that $v_E(\partial_Z(g)) \geq 1$, proving the lemma. ☐

The following easy consequences of Lemma 4.19 might be of independent interest.

PROPOSITION 4.20

Let Z and X be normal complex varieties, and let $\beta\colon Z \to X$ be a proper birational morphism. Let $\partial_Z \in H^0(Z, T_Z)$, and let $\partial_X \in H^0(X, T_X)$ be the induced derivation on X. Suppose that $\partial_X \neq 0$ in codimension 1. Let also $E \subset Z$ be a prime divisor. If $\beta(E)$ is contained in the singular locus of X, then E is invariant under ∂_Z.

Proof

By [75, Theorem 5], the singular locus of X is invariant under ∂_X. Let B be the maximal effective divisor on Z such that $\partial_Z \in H^0(Z, T_Z \boxtimes \mathscr{O}_Z(-B))$. Observe that B is β-exceptional since $\partial_X \neq 0$ in codimension 1 by assumption. If $E \subseteq \operatorname{Supp} B$, then $\partial_{Z|E} \equiv 0$ and E is invariant under ∂_Z. If E is not contained in $\operatorname{Supp} B$, then the claim follows from Lemma 4.19 above applied to the foliation $\mathscr{G} = \mathscr{O}_X \partial \subseteq T_X$. □

Recall that an *equivariant resolution* of a normal variety X is a projective birational morphism $\beta\colon Z \to X$ with Z smooth such that β restricts to an isomorphism over the smooth locus of X and such that $\beta_* T_Z = T_X$. The following consequence of Lemma 4.19 is a special case of [41, Corollary 4.7].

COROLLARY 4.21

Let X be a normal complex variety, and let $\beta\colon Z \to X$ be an equivariant resolution of X. Then $\beta_* T_Z(-\log E) = T_X$, where E denotes the union of all prime β-exceptional divisors.

4.4. Singularities of foliations with numerically trivial canonical class

In general, Definition 4.1 requires some understanding of the numbers $a(E, X, \mathscr{G})$ for all exceptional divisors of all birational modifications of X. However, if $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier and $K_{\mathscr{G}} \equiv 0$, then we have the following characterization of canonical singularities, due to Loray, Pereira, and Touzet when X is smooth.

PROPOSITION 4.22

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation on X with $K_{\mathscr{G}}$ $\mathbb{Q}$-Cartier and $K_{\mathscr{G}} \equiv 0$. Then $\mathscr{G}$ has canonical singularities if and only if $\mathscr{G}$ is not uniruled.

Proof

The same argument used in the proof of [66, Corollary 3.8] shows that the conclusion of Proposition 4.22 holds. One only needs to replace the use of [66, Theorem 3.7] by Theorem 4.23 below. □

THEOREM 4.23

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation on X with canonical singularities. Then $\mathscr{G}$ is uniruled if and only if $K_{\mathscr{G}}$ is not pseudoeffective.

Proof

Let $\beta\colon Z \to X$ be a resolution of singularities.

Suppose first that $K_{\mathscr{G}}$ is not pseudoeffective. Then $K_{\beta^{-1}\mathscr{G}}$ is not pseudoeffective as well. Applying [22, Theorem 4.7] to $\beta^{-1}\mathscr{G}$, we see that $\mathscr{G}$ is uniruled.

Suppose now that $\mathscr{G}$ is uniruled. The same argument used in the proof of [66, Theorem 3.7] applied to $\beta^{-1}\mathscr{G}$ shows that $K_{\beta^{-1}\mathscr{G}}$ is not pseudoeffective. This in turn implies that $K_{\mathscr{G}}$ is not pseudoeffective since $\mathscr{G}$ is canonical, finishing the proof of the theorem. □

4.5. Abundance for algebraically integrable foliations with numerically trivial canonical class

Let X be a projective klt variety with numerically trivial canonical class. By a theorem of Nakayama, K_X is torsion (see [69, Corollary V.4.9]). Proposition 4.24 below extends Nakayama's result to mildly singular algebraically integrable foliations.

PROPOSITION 4.24

Let X be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}} \equiv 0$. Then $K_{\mathscr{G}}$ is torsion.

Proof

Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6). Let also F be a general fiber of ψ. By Lemma 4.13, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$, and moreover, F has canonical singularities. Recall from Example 3.3 that $K_{\beta^{-1}\mathscr{G}} = K_{Z/Y} - R(\psi)$, where $R(\psi)$ denotes the ramification divisor of ψ. From the adjunction formula, we conclude that $K_F \sim_{\mathbb{Z}} K_{Z|F} \equiv 0$, and thus K_F is torsion by [69, Corollary V 4.9]. It follows that there exists a $\mathbb{Q}$-divisor B on Z with $\psi(\operatorname{Supp} B) \subsetneq Y$ such that

$$K_{Z/Y} - R(\psi) \sim_{\mathbb{Q}} B.$$

On the other hand, since $K_{Z/Y} - R(\psi) \equiv 0$ by assumption, there exists a $\mathbb{Q}$-divisor on Y such that $B = \psi^* D$. This follows easily from [73, Theorem A.7]. Therefore, we have

$$K_Z + \psi^{-1}(C) \sim_{\mathbb{Q}} \psi^*(K_Y + C + D),$$

where C is the reduced divisor on Y with support $\psi(\operatorname{Supp} R(\psi))$. By Lemma 2.6, D is $\mathbb{Q}$-Cartier. Moreover, we have $D \equiv 0$ by the projection formula. By Lemma 4.13

applied to $\beta^{-1}\mathscr{G}$, the discriminant of the lc-trivial fibration $\psi\colon (Z, \psi^{-1}(C)) \to Y$ is C (we refer the reader to [1] for the definitions of lc-trivial fibration and discriminant). From [1, Theorem 3.5] (see also [36, Theorem 1.2]), we conclude that D is torsion, proving the proposition. □

5. Weakly regular foliations on singular spaces

5.1. Definitions and examples

There are several notions of regularity for foliations on singular spaces. We first recall the notion of strongly regular foliation following [51, Definition 1.9].

Definition 5.1

Let $\mathscr{G}$ be a foliation of rank $r \geq 1$ on a normal complex variety X. Say that $\mathscr{G}$ is *strongly regular at* $x \in X$ if x has an open analytic neighborhood U that is biholomorphic to $\mathbb{D}^r \times M$, where $\mathbb{D}$ is the complex open unit disk and M is a germ of normal complex analytic variety, such that $\mathscr{G}_{|U}$ is induced by the projection $\mathbb{D}^r \times M \to M$. Say that $\mathscr{G}$ is *strongly regular* if $\mathscr{G}$ is strongly regular at any point $x \in X$.

Remark 5.2

By [15, Lemma 1.3.2], $\mathscr{G}$ is strongly regular at $x \in X$ if and only if $\mathscr{G}$ is locally free in a neighborhood of x and the natural map $\Omega^r_X \to \mathscr{O}_X(K_{\mathscr{G}})$ induced by the rth wedge product of the inclusion $\mathscr{G} \hookrightarrow T_X$ is surjective at x.

Example 5.3

If Y and Z are normal varieties and $\mathscr{G}$ is the foliation on $X := Y \times Z$ induced by the projection $Y \times Z \to Y$, then $\mathscr{G}$ is strongly regular if and only if Z is smooth.

The notion of strong regularity is, however, not flexible enough to allow for applications. The following notion of regularity for foliations addresses this issue (see [5, Definition 3.5]).

Definition 5.4

Let $\mathscr{G}$ be a foliation of rank $r \geq 1$ on a normal complex variety X. The rth wedge product of the inclusion $\mathscr{G} \hookrightarrow T_X$ gives rise to a nonzero map $\mathscr{O}_X(-K_{\mathscr{G}}) \to (\wedge^r T_X)^{**}$. We will refer to the dual map $\Omega^{[r]}_X \to \mathscr{O}_X(K_{\mathscr{G}})$ as the *Pfaff field* associated to $\mathscr{G}$.

The *singular locus* S of $\mathscr{G}$ is the closed subscheme of X whose ideal sheaf is the image of the induced map $\Omega^{[r]}_X \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$, which we will refer to as the *twisted Pfaff field* associated to $\mathscr{G}$.

We say that $\mathscr{G}$ is *weakly regular at* $x \in X$ if $x \notin S$. We say that $\mathscr{G}$ is *weakly regular* if $S = \emptyset$.

Example 5.5
Let X be a normal variety, and consider $\mathscr{G} = T_X$. Then $\mathscr{G}$ is weakly regular.

A strongly regular foliation is weakly regular in the sense of Definition 5.4 above. The converse is true if X is smooth by Frobenius's theorem. Other examples of weakly regular foliations are provided by the following results.

PROPOSITION 5.6
Let X be a normal complex variety, and let $\psi\colon X \to Y$ be a dominant morphism onto a variety Y. Let $\mathscr{G}$ be the foliation on X induced by ψ. Then $\mathscr{G}$ is weakly regular over the generic point of Y.

Proof
Note that we may assume without loss of generality that Y is smooth. By [49, Proposition III.10.6], the tangent map $T\psi\colon T_X \to \psi^* T_Y$ is surjective along a general fiber F of ψ. The claim then follows from Lemma 5.7 below. □

LEMMA 5.7
Let X be a normal variety of dimension $n \geq 2$, and let $\mathscr{G} \subsetneq T_X$ be a foliation of rank $r \geq 1$ on X. Set $q := n - r$. Then $\mathscr{G}$ is weakly regular if and only if the map $\eta\colon (\wedge^q T_X)^{**} \boxtimes \det \mathscr{N}_{\mathscr{G}}^* \to \mathscr{O}_X$ induced by the qth wedge product of the quotient map $T_X \to \mathscr{N}_{\mathscr{G}}$ is surjective.

Proof
The wedge product of differential forms on X_{reg} induces an isomorphism of reflexive sheaves

$$\Omega_X^{[r]} \cong (\wedge^q T_X)^{**} \boxtimes \mathscr{O}_X(K_X).$$

The canonical isomorphism $\mathscr{O}_X(K_{\mathscr{G}}) \cong \mathscr{O}_X(K_X) \boxtimes \det \mathscr{N}_{\mathscr{G}}$ then yields

$$\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \cong (\wedge^q T_X)^{**} \boxtimes \det \mathscr{N}_{\mathscr{G}}^*.$$

One readily checks that the map $\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$ induced by η is the twisted Pfaff field associated with $\mathscr{G}$. This shows the lemma. □

LEMMA 5.8
Let X be a normal variety, and let $\mathscr{G}$ be a foliation on X. Suppose that there exists a distribution $\mathscr{E}$ on X such that $T_X = \mathscr{G} \oplus \mathscr{E}$. Then $\mathscr{G}$ is weakly regular.

Proof

Set $r := \operatorname{rank} \mathscr{G}$. Observe that $\det \mathscr{G}^* \cong \mathscr{O}_X(K_{\mathscr{G}})$ is a direct summand of $\Omega_X^{[r]}$ and that the twisted Pfaff field $\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$ associated to $\mathscr{G}$ is induced by the projection $\Omega_X^{[r]} \to \det \mathscr{G}^*$. This immediately implies that $\mathscr{G}$ is weakly regular. □

5.2. Elementary properties

The following lemma says that a weakly regular foliation $\mathscr{G}$ has mild singularities if $K_{\mathscr{G}}$ is Cartier.

LEMMA 5.9

Let X be a normal complex variety with klt singularities, and let $\mathscr{G}$ be a foliation on X. Suppose that $K_{\mathscr{G}}$ is Cartier. If $\mathscr{G}$ is weakly regular, then it has canonical singularities.

Proof

Let Z be a normal variety, and let $\beta \colon Z \to X$ be a birational projective morphism. Let $\eta_X : \Omega_X^{[r]} \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{G}})$ and $\eta_Z : \Omega_Z^{[r]} \to \mathscr{O}_Z(K_{\beta^{-1}\mathscr{G}})$ be the Pfaff fields associated to $\mathscr{G}$ and $\beta^{-1}\mathscr{G}$, respectively. Recall from Section 2.6 that there exists a morphism of sheaves $d_{\text{refl}}\beta \colon \beta^*\Omega_X^{[r]} \to \Omega_Z^{[r]}$ that agrees with the usual pullback morphism of Kähler differentials wherever this makes sense. Next, we show that there exist a morphism $\beta^*\mathscr{O}_X(K_{\mathscr{G}}) \to \mathscr{O}_Z(K_{\beta^{-1}\mathscr{G}})$ and a commutative diagram as follows:

$$\begin{array}{ccc} \beta^*\Omega_X^{[r]} & \xrightarrow{d_{\text{refl}}\beta} & \Omega_Z^{[r]} \\ \downarrow{\scriptstyle \beta^*\eta_X} & & \downarrow{\scriptstyle \eta_Z} \\ \beta^*\mathscr{O}_X(K_{\mathscr{G}}) & \longrightarrow & \mathscr{O}_Z(K_{\beta^{-1}\mathscr{G}}) \end{array}$$

Let $\mathscr{K}$ denote the kernel $\beta^*\eta_X$. Since β is birational, the image of $\mathscr{K}$ by $\eta_Z \circ d_{\text{refl}}\beta$ is a torsion subsheaf of $\mathscr{O}_Z(K_{\beta^{-1}\mathscr{G}})$. But the latter is torsion-free by construction, proving our claim. In particular, there is a β-exceptional effective divisor E on Z such that $K_{\beta^{-1}\mathscr{G}} = \beta^*K_{\mathscr{G}} + E$, proving the lemma. □

Remark 5.10

We will show that the converse is also true if $\mathscr{G}$ is algebraically integrable with $K_{\mathscr{G}} \equiv 0$ (see Corollary 5.23).

Example 5.11 below shows that Lemma 5.9 is wrong if one drops the assumption that $K_{\mathscr{G}}$ is Cartier.

Example 5.11
Let Y and Z be normal varieties, and let $\mathscr{G}$ be the foliation on $X := Y \times Z$ induced by the projection $Y \times Z \to Y$. Then $\mathscr{G}$ is weakly regular by Lemma 5.8. But $\mathscr{G}$ has canonical singularities if and only if Z has canonical singularities by Example 4.16.

Next, we analyze the behavior of weakly regular foliations with respect to smooth morphisms, quasifinite maps, and birational modifications.

LEMMA 5.12
Let $\pi: Y \to X$ be a surjective étale morphism of normal varieties, and let $\mathscr{G}$ be a foliation on X. Then $\mathscr{G}$ is weakly regular if and only if so is $\pi^{-1}\mathscr{G}$.

Proof
Recall that the pullback of a reflexive sheaf by a flat morphism is reflexive as well by [50, Proposition 1.8]. Note also that $\pi^*(\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}})) \cong \Omega_Y^{[r]} \boxtimes \mathscr{O}_Y(-\pi^* K_{\mathscr{G}})$ since both are reflexive sheaves and agree on $\pi^{-1}(X_{\mathrm{reg}}) = Y_{\mathrm{reg}}$. Let $\eta\colon \Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$ be the twisted Pfaff field associated to $\mathscr{G}$. One readily checks that the induced map $\pi^*\eta\colon \Omega_Y^{[r]} \boxtimes \mathscr{O}_Y(-\pi^* K_{\mathscr{G}}) \to \mathscr{O}_Y$ is the twisted Pfaff field associated to $\pi^{-1}\mathscr{G}$. The lemma follows since π is a faithfully flat morphism. $\square$

PROPOSITION 5.13
Let X be a normal complex variety, let $\mathscr{G}$ be a foliation on X, and let $\pi: Y \to X$ be a quasifinite dominant morphism. Suppose that any codimension 1 irreducible component of the branch locus of π is $\mathscr{G}$-invariant. Then the following hold.

(1) *If $\mathscr{G}$ is weakly regular, then so is $\pi^{-1}\mathscr{G}$.*

(2) *Suppose in addition that π is finite and surjective. If $\pi^{-1}\mathscr{G}$ is weakly regular, then so is $\mathscr{G}$.*

Proof
Set $r := \operatorname{rank}\mathscr{G}$, and let $\eta_X: \Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$ and $\eta_Y: \Omega_Y^{[r]} \boxtimes \mathscr{O}_Y(-K_{\pi^{-1}\mathscr{G}}) \to \mathscr{O}_Y$ be the twisted Pfaff fields associated to $\mathscr{G}$ and $\pi^{-1}\mathscr{G}$, respectively.

Suppose first that $\mathscr{G}$ is weakly regular. By Lemma 3.4, we have $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^* K_{\mathscr{G}}$. This implies that

$$\pi^{[*]}\big(\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}})\big) \cong \pi^{[*]}\Omega_X^{[r]} \boxtimes \mathscr{O}_Y(-K_{\pi^{-1}\mathscr{G}}).$$

One then checks that we have a commutative diagram

$$\begin{array}{ccc} \pi^*(\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}})) & \longrightarrow & \Omega_Y^{[r]} \boxtimes \mathscr{O}_Y(-K_{\pi^{-1}\mathscr{G}}) \\ {\scriptstyle \pi^*\eta_X} \downarrow & & \downarrow {\scriptstyle \eta_Y} \\ \mathscr{O}_Y & = & \mathscr{O}_Y \end{array}$$

This shows that $\pi^{-1}\mathscr{G}$ is a weakly regular foliation, proving item (1).

Suppose from now on that π is a finite cover and that $\pi^{-1}\mathscr{G}$ is weakly regular. Let $\gamma\colon Y_1 \to Y$ be a finite cover such that the induced cover $\pi_1\colon Y_1 \to X$ is Galois with Galois group G. We may also assume that π_1 is quasi-étale away from the branch locus of π. By item (1) applied to γ, $\pi_1^{-1}\mathscr{G}$ is weakly regular as well. Thus, we may assume without loss of generality that π is Galois with Galois group G. By Lemma 3.4, the tangent map $T\pi$ induces an isomorphism $\pi^{-1}\mathscr{G} \cong (\pi^*\mathscr{G})^{**}$. It follows that $\mathscr{O}_Y(K_{\pi^{-1}\mathscr{G}}) \cong \mathscr{O}_Y(\pi^*K_{\mathscr{G}})$ as G-sheaves. Note that η_Y is G-equivariant. By [42, Lemma A.4] and [18, Theorem 2], we have

$$\big(\pi_*\big(\Omega_Y^{[r]} \boxtimes \mathscr{O}_Y(-K_{\pi^{-1}\mathscr{G}})\big)\big)^G \cong \Omega_X^{[r]} \boxtimes \mathscr{O}_Y(-K_{\mathscr{G}}).$$

It follows that the map $\eta_Y^G\colon \Omega_X^{[r]} \boxtimes \mathscr{O}_Y(-K_{\mathscr{G}}) \to \mathscr{O}_X$ induced by η_Y is the twisted Pfaff field associated to $\mathscr{G}$. From [42, Lemma A.3], we see that $\eta_X = \eta_Y^G$ is surjective. This shows that $\mathscr{G}$ is weakly regular, completing the proof of the proposition. □

The following is an immediate consequence of Proposition 5.13.

COROLLARY 5.14
Let X be a normal complex variety, let $\mathscr{G}$ be a foliation on X, and let $\pi: Y \to X$ be a quasi-étale cover. Then $\mathscr{G}$ is weakly regular if and only if so is $\pi^{-1}\mathscr{G}$.

LEMMA 5.15
Let $\pi: Y \to X$ be a projective birational morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X. Suppose that $K_{\mathscr{G}}$ is Cartier and that $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^*K_{\mathscr{G}}$. Suppose furthermore that X has klt singularities. If $\mathscr{G}$ is weakly regular, then so is $\pi^{-1}\mathscr{G}$.

Proof
Let $\eta_X: \Omega_X^{[r]} \twoheadrightarrow \mathscr{O}_X(K_{\mathscr{G}})$ and $\eta_Y: \Omega_Y^{[r]} \to \mathscr{O}_Y(K_{\pi^{-1}\mathscr{G}})$ be the Pfaff fields associated to $\mathscr{G}$ and $\pi^{-1}\mathscr{G}$, respectively. Recall from the proof of Lemma 5.9 that there is a commutative diagram

$$\begin{array}{ccc} \pi^*\Omega_X^{[r]} & \xrightarrow{d_{\mathrm{refl}}\pi} & \Omega_Y^{[r]} \\ \Big\downarrow{\scriptstyle \pi^*\eta_X} & & \Big\downarrow{\scriptstyle \eta_Y} \\ \pi^*\mathscr{O}_X(K_{\mathscr{G}}) & \xrightarrow{\sim} & \mathscr{O}_Y(K_{\pi^{-1}\mathscr{G}}) \end{array}$$

This immediately implies that $\pi^{-1}\mathscr{G}$ is weakly regular, proving the lemma. □

LEMMA 5.16

Let $\pi : Y \to X$ be a dominant morphism of normal complex varieties, and let $\mathscr{G}$ be a foliation on X.

(1) *Suppose that $Y = X \times Z$ and that π is the projection onto X. Then $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^*K_{\mathscr{G}} + K_{Y/X}$, and $\mathscr{G}$ is weakly regular if and only if so is $\pi^{-1}\mathscr{G}$.*

(2) *If π is a smooth morphism, then $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^*K_{\mathscr{G}} + K_{Y/X}$. Moreover, $\mathscr{G}$ is weakly regular if and only if $\pi^{-1}\mathscr{G}$ is weakly regular.*

(3) *Suppose that X has klt singularities and that π is a small projective birational map. Suppose in addition that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. Then $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \pi^*K_{\mathscr{G}}$. Moreover, if $\mathscr{G}$ is weakly regular, then so is $\pi^{-1}\mathscr{G}$.*

Proof

Set $r := \operatorname{rank}\mathscr{G}$, and let $\eta_X : \Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}}) \to \mathscr{O}_X$ be the twisted Pfaff field associated to $\mathscr{G}$.

Suppose first that $Y = X \times Z$ and that π is the projection onto X. Denote by p the projection onto Z, and set $m := \dim Z$. Recall that the pullback of a reflexive sheaf by a flat morphism is reflexive as well by [50, Proposition 1.8]. Then

$$\pi^{-1}\mathscr{G} \cong T_{Y/X} \oplus \pi^*\mathscr{G} \cong p^*T_Z \oplus \pi^*\mathscr{G},$$

and hence $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^*K_{\mathscr{G}} + K_{Y/X} \sim_{\mathbb{Z}} \pi^*K_{\mathscr{G}} + p^*K_Z$.

We have

$$\Omega_Y^{[r+m]} \cong \bigoplus_{i+j=r+m} \pi^*\Omega_X^{[i]} \boxtimes p^*\Omega_Z^{[j]}$$

and the twisted Pfaff field associated to $\pi^{-1}\mathscr{G}$ is the composed map

$$\begin{aligned} &\Omega_Y^{[r+m]} \boxtimes \mathscr{O}_X(-K_{\pi^{-1}\mathscr{G}}) \\ &\quad \twoheadrightarrow (\pi^*\Omega_X^{[r]} \boxtimes p^*\Omega_Z^{[m]}) \boxtimes \mathscr{O}_X(-K_{\pi^{-1}\mathscr{G}}) \cong \pi^*\big(\Omega_X^{[r]} \boxtimes \mathscr{O}_X(-K_{\mathscr{G}})\big) \\ &\quad \xrightarrow{\pi^*\eta_X} \pi^*\mathscr{O}_X \cong \mathscr{O}_Y. \end{aligned}$$

It follows that $\mathscr{G}$ is weakly regular if and only if so is $\pi^{-1}\mathscr{G}$ since π is a faithfully flat morphism.

Suppose now that π is smooth. Set $m := \dim Y - \dim X$, $X^\circ := X_{\mathrm{reg}}$, and $Y^\circ := \pi^{-1}(X^\circ) \subseteq Y_{\mathrm{reg}}$. We have an exact sequence of vector bundles

$$0 \to T_{Y^\circ/X^\circ} \to (\pi^{-1}\mathscr{G})_{|Y^\circ} \to (\pi_{|Y^\circ})^*\mathscr{G}_{|X^\circ} \to 0$$

and thus $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^* K_{\mathscr{G}} + K_{Y/X}$ since Y° has complement of codimension at least 2 in Y. We proceed to show that $\mathscr{G}$ is weakly regular if and only if so is $\pi^{-1}\mathscr{G}$. The statement is local on X for the étale topology by Lemma 5.12. Thus, we may assume that $Y = X \times \mathbb{A}^m$ and that π is given by the projection $Y = X \times \mathbb{A}^m \to X$. The claim then follows from the previous case.

Suppose finally that X has klt singularities and that π is a small projective birational map. Suppose in addition that $K_{\mathscr{G}}$ is $\mathbb{Q}$-Cartier. We clearly have $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \pi^* K_{\mathscr{G}}$. Replacing X by an open subset, if necessary, we may assume that $K_{\mathscr{G}}$ is torsion. It follows that $K_{\pi^{-1}\mathscr{G}}$ is torsion as well. By Corollary 5.14, replacing Y and X by the associated cyclic quasi-étale covers (see [63, Definition 2.52]), we may also assume that $K_{\mathscr{G}}$ is Cartier and that $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^* K_{\mathscr{G}}$. The statement then follows from Lemma 5.15. □

Remark 5.17
In the setup of Lemma 5.16(3), suppose in addition that $K_{\mathscr{G}}$ is Cartier. Then $K_{\pi^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \pi^* K_{\mathscr{G}}$.

LEMMA 5.18
Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be an algebraically integrable foliation on X with canonical singularities. Let $\psi: Z \to Y$ be the family of leaves, and let $\beta: Z \to X$ be the natural morphism (see Section 3.6). If $\mathscr{G}$ is weakly regular, then so is $\beta^{-1}\mathscr{G}$.

Proof
By Lemma 4.13, we have $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. Note that the statement is local on X. Let m be a positive integer, and let $X^\circ \subseteq X$ be a dense open subset such that $\mathscr{O}_{X^\circ}(mK_{\mathscr{G}|X^\circ}) \cong \mathscr{O}_{X^\circ}$. Let $f^\circ: X_1^\circ \to X^\circ$ be the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]), and let Z_1° be the normalization of the product $Z^\circ \times_{X^\circ} X_1^\circ$, where $Z^\circ := \beta^{-1}(X^\circ)$. Let also $\beta_1^\circ: Z_1^\circ \to X_1^\circ$ and $g^\circ: Z_1^\circ \to Z^\circ$ denote the natural morphisms. Recall from Section 3.6 that $K_{(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})} \sim_{\mathbb{Z}} (\beta_1^\circ)^* K_{(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})}$ and that the support of the ramification divisor $R(g^\circ)$ of g° must be $(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})$-invariant. By construction, $K_{(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})}$ is Cartier. Moreover, $(f^\circ)^{-1}(\mathscr{G}_{|X^\circ})$ is weakly regular by Corollary 5.14. Applying Lemma 5.15, we

see that $(\beta_1^\circ)^{-1}(f^\circ)^{-1}(\mathscr{G}_{|X^\circ}) = (g^\circ)^{-1}((\beta^{-1}\mathscr{G})_{|Z^\circ})$ is weakly regular. The statement then follows from Proposition 5.13. □

5.3. Criteria for weak regularity

Let $\mathscr{G}$ be a foliation with numerically trivial canonical class on a complex projective manifold. Suppose that $\mathscr{G}$ has a compact leaf. Then Theorem 5.6 in [66] asserts that $\mathscr{G}$ is regular and that there exists a foliation on X transverse to $\mathscr{G}$ at any point in X. In this subsection, we extend this result to mildly singular varieties (see Corollary 5.22). We also show that algebraically integrable foliations with mild singularities and numerically trivial canonical class are weakly regular (see Corollary 5.23). Finally, we provide another criterion for regularity of foliations (see Proposition 5.26).

We will need the following easy observations.

LEMMA 5.19

Let X be a normal complex variety, and let $\mathscr{G}$ be a foliation of rank r on X. Suppose that $K_{\mathscr{G}}$ is Cartier, and let $\eta\colon \Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{G}})$ be the Pfaff field associated to $\mathscr{G}$. Let $L \subset X$ be a subvariety which is not entirely contained in the union of the singular loci of X and $\mathscr{G}$. Suppose in addition that $\dim L = r$. Then the following hold.

(1) *The variety $L \cap X_{\mathrm{reg}}$ is a leaf of $\mathscr{G}_{|X_{\mathrm{reg}}}$ if and only if the composed map $\Omega^r_{X\,|L} \to \Omega^{[r]}_{X\,|L} \to \mathscr{O}_X(K_{\mathscr{G}})_{|L}$ factors through the natural map $\Omega^r_{X\,|L} \twoheadrightarrow \Omega^r_L$.*

(2) *Suppose that $L \cap X_{\mathrm{reg}}$ is a leaf of $\mathscr{G}_{|X_{\mathrm{reg}}}$, and let F be the normalization of L. Denote by $n\colon F \to X$ the natural morphism. Then there is a commutative diagram*

$$\begin{array}{ccccc}
n^*\Omega^r_X & \longrightarrow & n^*\Omega^{[r]}_X & \xrightarrow{n^*\eta} & n^*\mathscr{O}_X(K_{\mathscr{G}}) \\
\Big\downarrow{\scriptstyle dn} & & & & \Big\| \\
\Omega^r_F & \longrightarrow & \Omega^{[r]}_F & \longrightarrow & n^*\mathscr{O}_X(K_{\mathscr{G}})
\end{array}$$

Proof

Item (1) follows from [4, Lemma 2.7] using the fact that $\mathscr{O}_X(K_{\mathscr{G}})_{|L}$ is torsion-free. Item (2) follows from item (1) and [9, Proposition 4.5]. □

LEMMA 5.20

Let X be a normal complex projective variety, and let H be an ample divisor on X. For any integer $1 \le r \le \dim X$, the image of $c_1(H)^r \in H^r(X, \Omega^r_X)$ under the natural map $H^r(X, \Omega^r_X) \to H^r(X, \Omega^{[r]}_X)$ is nonzero.

Proof

In order to prove the lemma, it suffices to consider the case when $r = \dim X$. Let $\beta\colon Z \to X$ be a resolution of X. The image of $c_1(H)^{\dim X} \in H^{\dim X}(X, \Omega_X^{\dim X})$ under the map $H^{\dim X}(X, \Omega_X^{\dim X}) \to H^{\dim X}(Z, \Omega_Z^{\dim X})$ is nonzero, and hence $c_1(H)^{\dim X}$ is nonzero as well. On the other hand, the kernel and the cokernel of the natural map $\Omega_X^{\dim X} \to \Omega_X^{[\dim X]}$ are supported on closed subsets of codimension at least 2. It follows that the natural map $H^{\dim X}(X, \Omega_X^{\dim X}) \to H^{\dim X}(X, \Omega_X^{[\dim X]})$ is an isomorphism, completing the proof of the lemma. □

PROPOSITION 5.21

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation of rank r on X with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$. Let $L \subset X$ be a proper subvariety which is not entirely contained in the union of the singular loci of X and $\mathscr{G}$. Suppose that $L \cap X_{\mathrm{reg}}$ is a leaf of $\mathscr{G}_{|X_{\mathrm{reg}}}$. Let F be the normalization of L, and denote by $n\colon F \to X$ the natural morphism. Suppose furthermore that the map $\eta_F\colon \Omega_F^{[r]} \to n^*\mathscr{O}_X(K_{\mathscr{G}})$ given by Lemma 5.19 is an isomorphism, and that F has rational singularities. Then $\mathscr{G}$ is weakly regular, and there exists a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.

Proof

Let $\beta\colon Z \to X$ be an embedded resolution of L, and let T be the strict transform of L in Z. Observe that $\beta_{|T}\colon T \to L$ factors through $F \to L$, and denote by $\gamma\colon T \to F$ the induced map.

Let H be an ample divisor on X. Let also $\eta\colon \Omega_X^{[r]} \to \mathscr{O}_X(K_{\mathscr{G}})$ be the Pfaff field associated to $\mathscr{G}$. Consider the image c of $c_1(H)^r \in H^r(X, \Omega_X^r)$ under the composed map

$$H^r(X, \Omega_X^r) \to H^r(X, \Omega_X^{[r]}) \to H^r\big(X, \mathscr{O}_X(K_{\mathscr{G}})\big).$$

We will show that $c \neq 0$ and that $H^r(n^*)(c) \neq 0$, where $H^r(n^*)\colon H^r(X, \mathscr{O}_X(K_{\mathscr{G}})) \to H^r(F, n^*\mathscr{O}_X(K_{\mathscr{G}}))$ denotes the map induced by n^*. By Lemma 5.19, we have a commutative diagram

$$\begin{array}{ccccc}
H^r(X, \Omega_X^r) & \longrightarrow & H^r(X, \Omega_X^{[r]}) & \xrightarrow{H^r(\eta)} & H^r(X, \mathscr{O}_X(K_{\mathscr{G}})) \\
\downarrow{\scriptstyle H^r(dn)} & & & & \downarrow{\scriptstyle H^r(n^*)} \\
H^r(F, \Omega_F^r) & \longrightarrow & H^r(F, \Omega_F^{[r]}) & \longrightarrow & H^r(F, n^*\mathscr{O}_X(K_{\mathscr{G}}))
\end{array}$$

By assumption, the map $H^r(F, \Omega_F^{[r]}) \to H^r(F, n^*\mathscr{O}_X(K_\mathscr{G}))$ is an isomorphism. On the other hand, by Lemma 5.20 above, the image of $c_1(H_{|F})^r \in H^r(F, \Omega_F^r)$ under the map $H^r(F, \Omega_F^r) \to H^r(F, \Omega_F^{[r]})$ is nonzero. This immediately implies that $c \neq 0$ and that $H^r(n^*)(c) \neq 0$.

Consider the commutative diagram

$$\begin{array}{ccc} H^r(Z, \beta^*\mathscr{O}_X(K_\mathscr{G})) & \longrightarrow & H^r(T, (n\circ\gamma)^*\mathscr{O}_X(K_\mathscr{G})) \\ \uparrow & & \uparrow \\ H^r(X, \mathscr{O}_X(K_\mathscr{G})) & \xrightarrow{H^r(n^*)} & H^r(F, n^*\mathscr{O}_X(K_\mathscr{G})) \end{array}$$

Since F has rational singularities, the morphism $H^r(F, n^*\mathscr{O}_X(K_\mathscr{G})) \to H^r(T, (n\circ\gamma)^*\mathscr{O}_X(K_\mathscr{G}))$ is an isomorphism. This implies that the image c_Z of c under the map $H^r(X, \mathscr{O}_X(K_\mathscr{G})) \to H^r(Z, \beta^*\mathscr{O}_X(K_\mathscr{G}))$ is nonzero.

On the other hand, by Hodge symmetry with coefficients in local systems, there are natural isomorphisms

$$H^r\big(Z, \beta^*\mathscr{O}_X(K_\mathscr{G})\big) \cong \overline{H^0\big(Z, \Omega_Z^r \otimes \beta^*\mathscr{O}_X(-K_\mathscr{G})\big)} \qquad \text{and}$$
$$H^r\big(T, (n\circ\gamma)^*\mathscr{O}_X(K_\mathscr{G})\big) \cong \overline{H^0\big(T, \Omega_T^r \otimes (n\circ\gamma)^*\mathscr{O}_X(-K_\mathscr{G})\big)}.$$

It follows that $\alpha_Z := \overline{c_Z} \in H^0(Z, \Omega_Z^r \otimes \beta^*\mathscr{O}_X(-K_\mathscr{G}))$ is a twisted r-form that restricts to a nonzero twisted r-form $\alpha_T \in H^0(T, \Omega_T^r \otimes (n\circ\gamma)^*\mathscr{O}_X(-K_\mathscr{G}))$. Let $\alpha \in H^0(X, \Omega_X^{[r]} \otimes \mathscr{O}_X(-K_\mathscr{G}))$ be the twisted reflexive r-form on X induced by α_Z, and let $\alpha_F \in H^0(F, \Omega_F^{[r]} \otimes n^*\mathscr{O}_X(-K_\mathscr{G}))$ be the nonzero twisted reflexive r-form on F induced by α_T. By construction, $\eta(\alpha)$ is a regular function that restricts to the nonzero regular function $\eta_F(\alpha_F)$ on F. It follows that $\eta(\alpha)$ is constant. The contraction with α then gives a morphism $(\wedge^{r-1}\mathscr{G})^{**} \to \Omega_X^{[1]} \otimes \mathscr{O}_X(-K_\mathscr{G})$ such that the composed map $(\wedge^{r-1}\mathscr{G})^{**} \to \Omega_X^{[1]} \otimes \mathscr{O}_X(-K_\mathscr{G}) \to \mathscr{G}^* \otimes \mathscr{O}_X(-K_\mathscr{G})$ is an isomorphism. This shows that there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$. Note that $\mathscr{E}$ is given by the twisted r-form α which extends to the twisted r-form α_Z on Z with values in the flat line bundle $\beta^*\mathscr{O}_X(-K_\mathscr{G})$, which is automatically closed. This implies that $\mathscr{E}$ is involutive. Finally, $\mathscr{G}$ is weakly regular by Lemma 5.8, completing the proof of the proposition. □

COROLLARY 5.22

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a foliation on X. Suppose that $K_\mathscr{G}$ is Cartier and that $K_\mathscr{G} \equiv 0$. Let $L \subset X$ be a proper subvariety disjoint from the singular locus of $\mathscr{G}$ and not contained in the singular locus of X. Suppose that $L \cap X_{\text{reg}}$ is a leaf of $\mathscr{G}_{|X_{\text{reg}}}$. Suppose furthermore that the *normalization F of L has rational singularities. Then $\mathscr{G}$ is weakly regular, and there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.*

Proof
Denote by $n\colon F \to X$ the natural morphism, and let $\eta_F : \Omega_F^{[r]} \to n^*\mathscr{O}_X(K_{\mathscr{G}})$ be the map given by Lemma 5.19. By Lemma 5.19 and Section 2.6, we have a commutative diagram

$$\begin{array}{ccc} n^*\Omega_X^{[r]} & \xrightarrow{n^*\eta} & n^*\mathscr{O}_X(K_{\mathscr{G}}) \\ \downarrow{\scriptstyle d_{\mathrm{refl}} n} & & \| \\ \Omega_F^{[r]} & \xrightarrow[\eta_F]{} & n^*\mathscr{O}_X(K_{\mathscr{G}}) \end{array}$$

On the other hand, the map $n^*\eta$ is surjective by assumption. This immediately implies that η_F is an isomorphism, so that Proposition 5.21 applies. □

COROLLARY 5.23
Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be an algebraically integrable foliation on X with canonical singularities. Suppose that $K_{\mathscr{G}}$ is Cartier and that $K_{\mathscr{G}} \equiv 0$. Then $\mathscr{G}$ is weakly regular, and there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.

Proof
Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6). Let also F be a general fiber of ψ. Note that $L := \beta(F)$ is the closure of a leaf of $\mathscr{G}$. By Lemma 4.13, F has canonical singularities and $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^*K_{\mathscr{G}}$. In particular, F is the normalization of L, and it has rational singularities by [35, Théorème 1]. Moreover, $K_F \sim_{\mathbb{Q}} n^*\mathscr{O}_X(K_{\mathscr{G}})$ by Example 3.3 and the adjunction formula, where $n\colon F \to X$ denotes the restriction of β to F. It follows that the map $\eta_F : \Omega_F^{[r]} \cong \mathscr{O}_F(K_F) \to n^*\mathscr{O}_X(K_{\mathscr{G}})$ given by Lemma 5.19 is an isomorphism, where r denotes the rank of $\mathscr{G}$. The conclusion then follows from Proposition 5.21. □

Example 5.24 shows that Corollary 5.23 above is wrong if one drops the assumption that $K_{\mathscr{G}} \equiv 0$.

Example 5.24
Let B and C be smooth projective curves, and let X be the blowup of $B \times C$ at some

point. Let also $\mathscr{G}$ be the foliation on X induced by the natural morphism $X \to B$. Then $\mathscr{G}$ has canonical singularities, but it is not regular.

The proof of Proposition 5.26 below makes use of the following result, which might be of independent interest.

LEMMA 5.25
Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation of rank r on X with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$. Let H be a very ample Cartier divisor on X, and let $\Omega^r_X \to \mathscr{O}_X(K_{\mathscr{G}})$ be the map induced by the Pfaff field associated to $\mathscr{G}$. Let $\beta\colon Z \to X$ be a resolution of singularities. Suppose that the image c of $c_1(H)^r \in H^r(X, \Omega^r_X)$ under the composed map

$$H^r(X, \Omega^r_X) \to H^r\big(X, \mathscr{O}_X(K_{\mathscr{G}})\big) \to H^r\big(Z, \beta^*\mathscr{O}_X(K_{\mathscr{G}})\big)$$

is nonzero. Then $\mathscr{G}$ is weakly regular, and there exists a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.

Proof
The proof is similar to that of [27, Proposition 2.7.1].

Set $n := \dim X$. The linear system $|H|$ embeds X into $\mathbb{P}^N$ for some positive integer N. Denote by $\gamma\colon Z \to \mathbb{P}^N$ the natural map. Let $(U_i)_{i\in I}$ be a finite covering of $\mathbb{P}^N$ by open sets such that $\mathscr{O}_X(K_{\mathscr{G}})_{|X_i} \cong \mathscr{O}_{X_i}$, where $X_i := U_i \cap X$. Denote by $v_i \in H^0(X_i, \wedge^{[r]}T_{X_i})$ an r-field defining $\mathscr{G}_{|X_i}$, and let $u_i \in H^0(U_i, \wedge^r T_{U_i})$ be such that $u_{i\,|X_i} = v_i \in H^0(X_i, \wedge^{[r]}T_{X_i}) \subset H^0(X_i, \wedge^r T_{U_i\,|X_i})$. Let ω_{FS} be the Fubini–Study form on $\mathbb{P}^N$, and denote by ω_i its restriction to U_i. The pullback η_i on $\gamma^{-1}(U_i)$ of the contraction $u_i \lrcorner \omega_i^r$ of ω_i^r by u_i is a $\overline{\partial}$-closed $(0,r)$-form. Moreover, the η_i glue to give a $\overline{\partial}$-closed $(0,r)$-form η with coefficients in the unitary flat line bundle $\beta^*\mathscr{O}_X(K_{\mathscr{G}})$. By construction, η represents c if $H^r(Z, \beta^*\mathscr{O}_X(K_{\mathscr{G}}))$ is identified with the corresponding Dolbeault cohomology group.

By Hodge symmetry with coefficients in local systems, there exists a holomorphic r-form α with values in $\beta^*\mathscr{O}_X(-K_{\mathscr{G}})$ such that $\{\overline{\alpha}\} = c \in H^r(Z, \beta^*\mathscr{O}_X(K_{\mathscr{G}}))$. In particular, there exists a $(0, r-1)$-form ξ with values in $\beta^*\mathscr{O}_X(K_{\mathscr{G}})$ such that $\overline{\alpha} = \eta + \overline{\partial}\xi$. Note that α is harmonic with respect to any Kähler form. In particular, α is closed.

Set $X^\circ := X \setminus \beta(\operatorname{Exc}\beta)$, and let $v^\circ \in H^0(X^\circ, \wedge^r T_{X^\circ} \otimes \mathscr{O}_{X^\circ}(K_{\mathscr{G}_{|X^\circ}}))$ be a twisted r-field defining $\mathscr{G}_{|X^\circ}$. Let also $\alpha^\circ \in H^0(X^\circ, \Omega^r_{X^\circ} \otimes \mathscr{O}_{X^\circ}(-K_{\mathscr{G}_{|X^\circ}}))$ be the twisted r-form induced by α on X°. Notice that the contraction $\alpha^\circ(v^\circ)$ is a regular function, and hence constant since X° has complement of codimension at least 2 in X. To prove the statement, it suffices to show that $\alpha^\circ(v^\circ)$ is nonzero (see Lemma 5.8).

Let ω be the smooth closed semipositive $(1,1)$-form on Z induced by ω_{FS}. Since α and ω are closed, we have

$$d(\alpha \wedge \xi \wedge \omega^{n-r}) = (-1)^{r-1}\alpha \wedge d\xi \wedge \omega^{n-r} = (-1)^{r-1}\alpha \wedge \overline{\partial}\xi \wedge \omega^{n-r},$$

and hence

$$\int_Z \alpha \wedge \overline{\alpha} \wedge \omega^{n-r} = \int_Z \alpha \wedge \eta \wedge \omega^{n-r}$$

by Stokes's theorem. By the Hodge–Riemann bilinear relations, there exists a complex number $C_1 \neq 0$ such that the smooth (n,n)-form

$$C_1 \alpha \wedge \overline{\alpha} \wedge \omega^{n-r}{}_{|\beta^{-1}(X^\circ)} = C_1 \alpha^\circ \wedge \overline{\alpha^\circ} \wedge \omega_{\mathrm{FS}}^{n-r}$$

is semipositive and not identically zero since $\alpha \neq 0$ by assumption. It follows that $\int_Z \alpha \wedge \eta \wedge \omega^{n-r} \neq 0$. On the other hand, a straightforward computation shows that

$$\int_Z \alpha \wedge \eta \wedge \omega^{n-r} = \int_{X^\circ} \alpha^\circ \wedge (v^\circ \lrcorner \omega_{\mathrm{FS}}{}^r_{|X^\circ}) \wedge \omega_{\mathrm{FS}}{}^{n-r}_{|X^\circ} = C_2 \alpha^\circ(v^\circ),$$

for some complex number $C_2 \neq 0$. This immediately implies that $\alpha^\circ(v^\circ)$ is nonzero. Arguing as in the proof of Proposition 5.21, one shows that there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves and that $\mathscr{G}$ is weakly regular. This finishes the proof of the lemma. □

PROPOSITION 5.26

Let X be a normal complex $\mathbb{Q}$-Gorenstein projective variety, and let $\mathscr{G}$ be a codimension 1 foliation on X with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$. Suppose that X is smooth in codimension 2 with rational singularities and that $K_X \cdot H^{\dim X - 1} \neq 0$ for some ample Cartier divisor H on X. Suppose furthermore that there exists an open set $X^\circ \subseteq X_{\mathrm{reg}}$ with complement of codimension at least 3 such that $\mathscr{G}_{|X^\circ}$ is defined by closed holomorphic 1-forms with zero set of codimension at least 2 locally for the analytic topology. Then $\mathscr{G}$ is weakly regular, and there exists a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.

Proof

Set $n := \dim X$, and let $\eta\colon \Omega_X^{n-1} \to \mathscr{O}_X(K_{\mathscr{G}})$ be the map induced by the Pfaff field associated to $\mathscr{G}$. Let $\beta\colon Z \to X$ be a resolution of singularities. The natural map $H^{n-1}(X, \mathscr{O}_X(K_{\mathscr{G}})) \to H^{n-1}(Z, \beta^*\mathscr{O}_X(K_{\mathscr{G}}))$ is an isomorphism since X has rational singularities. Thus, by Lemma 5.25, it suffices to show that the image of $c_1(H)^{n-1} \in H^{n-1}(X, \Omega_X^{n-1})$ under the map $H^{n-1}(\eta)\colon H^{n-1}(X, \Omega_X^{n-1}) \to H^{n-1}(X, \mathscr{O}_X(K_{\mathscr{G}}))$ is nonzero.

Set $\mathscr{L} := \mathscr{N}_{\mathscr{G}}^{*}$. By Lemma 5.27 below, the cohomology class $c_1(\mathscr{L}_{|X^\circ}) \in H^1(X^\circ, \Omega^1_{X^\circ})$ lies in the image of the natural map $H^1(X^\circ, \mathscr{L}_{|X^\circ}) \to H^1(X^\circ, \Omega^1_{X^\circ})$. Recall that rational singularities are Cohen–Macaulay. It follows that $\mathscr{O}_X(K_X)$ and $\mathscr{O}_X(K_X - K_{\mathscr{G}}) \cong \mathscr{L}$ are Cohen–Macaulay sheaves. Then the restriction map $H^1(X, \mathscr{L}) \to H^1(X^\circ, \mathscr{L}_{|X^\circ})$ is an isomorphism by [77, Theorem 1.14]. Applying [77, Theorem 1.14] again together with [50, Proposition 1.6], we see that the restriction map $H^1(X, \Omega^{[1]}_X) \to H^1(X^\circ, \Omega^1_{X^\circ})$ is injective. It follows that the image of $c_1(\mathscr{L}) \in H^1(X, \Omega^1_X)$ under the natural map $H^1(X, \Omega^1_X) \to H^1(X, \Omega^{[1]}_X)$ is the image of a class $c \in H^1(X, \mathscr{L})$.

The same argument used in the proof of Lemma 5.20 shows that $c_1(H)^{n-1} \otimes c_1(\mathscr{L})$ maps to a nonzero class $c_1(H)^{n-1} \cup c_1(\mathscr{L}) \in H^n(X, \Omega^{[n]}_X)$ under the composed map

$$H^{n-1}(X, \Omega^{n-1}_X) \otimes H^1(X, \Omega^1_X) \xrightarrow{\bullet \cup \bullet} H^n(X, \Omega^n_X) \longrightarrow H^n(X, \Omega^{[n]}_X)$$

using the assumptions that $K_X \cdot H^{\dim X - 1} \not\equiv 0$ and $K_{\mathscr{G}} \equiv 0$. On the other hand, one checks that $c_1(H)^{n-1} \cup c_1(\mathscr{L})$ is the image of $c_1(H)^{n-1} \otimes c$ under the composed map

$$H^{n-1}(X, \Omega^{n-1}_X) \otimes H^1(X, \mathscr{L}) \xrightarrow{H^{n-1}(\eta) \otimes \mathrm{Id}} H^{n-1}(X, \mathscr{O}_X(K_{\mathscr{G}})) \otimes H^1(X, \mathscr{L}) \xrightarrow{\bullet \cup \bullet} H^n(X, \mathscr{O}_X(K_{\mathscr{G}}) \otimes \mathscr{L}) \cong H^n(X, \Omega^{[n]}_X)$$

This immediately implies that $H^{n-1}(\eta)(c_1(H)^{n-1}) \neq 0$, completing the proof of the proposition. □

The following result generalizes [12, Corollary 3.4].

LEMMA 5.27
Let X be a complex manifold, let $\mathscr{G} \subset T_X$ be a codimension 1 foliation, and set $\mathscr{L} := \mathscr{N}_{\mathscr{G}}^{*}$. Suppose that $\mathscr{G}$ is defined by closed holomorphic 1-forms with zero set of codimension at least 2 locally for the analytic topology. Then the cohomology class $c_1(\mathscr{L}) \in H^1(X, \Omega^1_X)$ lies in the image of the natural map $H^1(X, \mathscr{L}) \to H^1(X, \Omega^1_X)$.

Proof
Note that $\mathscr{L}$ is a line bundle by [50, Proposition 1.9]. Let $(U_i)_{i \in I}$ be a covering of X by analytically open sets such that $\mathscr{G}_{|U_i}$ is defined by a closed 1-form ω_i with zero set of codimension at least 2. Then we can write $\omega_i = g_{ij}\omega_j$ on $U_{ij} := U_i \cap$

U_j, where g_{ij} is a nowhere vanishing holomorphic function on U_{ij}. The cocycle $[(g_{ij})] \in H^1(X, \mathscr{O}_X^\times)$ then satisfies $[(g_{ij})] = [\mathscr{L}]$ since both classes agree away from the singular set of $\mathscr{G}$ which has codimension at least 2 in X. Now, since ω_i and ω_j are closed, we must have $0 = dg_{ij} \wedge \omega_j$ on U_{ij}. It follows that

$$dg_{ij} \in H^0(U_{ij}, \mathscr{L}_{|U_{ij}}) \subset H^0(U_{ij}, \Omega^1_{U_{ij}})$$

since $\mathscr{L}$ is saturated in Ω^1_X by [4, Lemma 9.7]. This implies that the cohomology class $c_1(\mathscr{L}) = [(d \log g_{ij})] \in H^1(X, \Omega^1_X)$ lies in the image of the natural map $H^1(X, \mathscr{L}) \to H^1(X, \Omega^1_X)$, proving the lemma. □

5.4. Local structure in codimension 2 *of weakly regular rank* 1 *foliations*

In the present section, we exemplify the notion of weakly regular foliation by describing weakly regular rank 1 foliations on surfaces with klt singularities. The results are not used elsewhere so that this section can be safely skipped.

We first show that a weakly regular foliation given by a derivation with zero set of codimension at least 2 is strongly regular in codimension 2.

PROPOSITION 5.28

Let X be a normal complex variety with klt singularities, and let $\mathscr{G} \subset T_X$ be a weakly regular foliation of rank 1*. Suppose in addition that $K_{\mathscr{G}}$ is Cartier. Then there exists a closed subset $Z \subseteq X$ with* $\operatorname{codim} Z \geq 3$ *such that $\mathscr{G}_{|X \setminus Z}$ is strongly regular.*

Proof

Set $n := \dim X$. Recall from [42, Proposition 9.3] that klt spaces have quotient singularities in codimension 2. Thus, we may assume without loss of generality that X has quotient singularities. Given $x \in X$, we have $(X, x) \cong (\mathbb{C}^n/G, 0)$ for some finite subgroup G of $\mathrm{GL}(n, \mathbb{C})$ that does not contain any quasireflections. In particular, the quotient map $\pi \colon \mathbb{C}^n \to \mathbb{C}^n/G$ is étale outside of the singular set. The statement is local on X, hence we may shrink X and assume that there exists $\partial \in H^0(X, T_X)$ such that $\mathscr{G} = \mathscr{O}_X \partial$. By Proposition 5.13, ∂ induces a nowhere vanishing vector field $\partial_U \in H^0(U, T_U)$ on some open G-stable neighborhood U of $0 \in \mathbb{C}^n$. Since $\mathscr{G}$ is weakly regular, there exists a G-invariant holomorphic 1-form α_U on U such that $\alpha_U(\partial_U) = 1$. Let α_U^0 be the 0th jet of α_U at 0. Then α_U^0 is G-invariant as well, and $\alpha_U^0(\partial_U)(0) = 1$. In particular, we have $\alpha_U^0 \neq 0$. On the other hand, $\alpha_U^0 = df$ for some holomorphic function f at 0 such that $f(0) = 0$. Observe that f must be G-invariant, so that there exists a holomorphic function t on some open analytic neighborhood of x in X such that $\partial(t)(x) \neq 0$. By a result of Zariski [83, Lemma 4], this implies that $R = R_1[[t]]$, where R is the formal completion of the local ring $(\mathscr{O}_{X,x}, \mathfrak{m}_x)$

and $R_1 \subsetneq R$ is a Noetherian normal ring with $\dim R_1 = \dim R - 1$. Moreover, the extension of $\frac{1}{\partial(t)}\partial$ to R coincides with ∂_t. The proposition then follows from [15, Lemma 1.3.2] (see Remark 5.2). □

The following is an immediate consequence of Proposition 5.28.

COROLLARY 5.29
Let X be a normal surface, and let $\mathscr{G} \subsetneq T_X$ be a weakly regular foliation of rank 1. Suppose that X has klt singularities and that $K_{\mathscr{G}}$ is Cartier. Then X is smooth.

We finally describe the structure of weakly regular foliations on klt surfaces.

PROPOSITION 5.30
Let (X,x) be a germ of normal surface with klt singularities, and let $\mathscr{G} \subset T_X$ be a foliation of rank 1. Then $\mathscr{G}$ is weakly regular if and only if there exists a positive integer m, as well as nonnegative integers a and b with $(a,m)=1$ and $(b,m)=1$ such that $(X,x) \cong (\mathbb{C}^2/G, 0)$ and $\pi^{-1}\mathscr{G} = \mathscr{O}_{\mathbb{C}^2}\partial_{y_1}$, where (y_1,y_2) are coordinates on $\mathbb{C}^2$, $G = \langle\zeta\rangle$ is a cyclic group of order m acting on $\mathbb{C}^2$ by $\zeta\cdot(y_1,y_2) = (\zeta^a y_1, \zeta^b y_2)$, and $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2/G$ denotes the quotient map.

Proof
Suppose first that $\mathscr{G}$ is weakly regular. Recall that X is $\mathbb{Q}$-factorial by [63, Proposition 4.11]. Let m be the smallest positive integer such that $mK_{\mathscr{G}}$ is Cartier at x, and let $\pi\colon Y \to X$ be the associated local cyclic cover, which is a quasi-étale cover of degree m (see [63, Definition 5.19]). By Proposition 5.13, $\pi^{-1}\mathscr{G}$ is a weakly regular foliation on Y with $K_{\pi^{-1}\mathscr{G}}$ Cartier. It follows from Corollary 5.29 that Y is smooth. The same argument used in the proof of [67, Corollary I.2.2] then shows our claim.

Conversely, let m be a positive integer, let a and b be nonnegative integers such that $(a,m)=1$ and $(b,m)=1$, and let $G=\langle\zeta\rangle$ be a cyclic group of order m acting on $\mathbb{C}^2$ with coordinates (y_1,y_2) by $\zeta\cdot(y_1,y_2) = (\zeta^a y_1, \zeta^b y_2)$. Set $X := \mathbb{C}^2/G$, and denote by $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2/G$ the natural morphism. Let $\mathscr{L}$ be the line bundle $\mathscr{O}_{\mathbb{A}^2}$ equipped with the G-linearization given by the character $\zeta \mapsto \zeta^a$ of G. Note that there exists a reflexive rank 1 sheaf $\mathscr{M}$ on X such that $\mathscr{L} \cong \pi^{[*]}\mathscr{M}$. Then ∂_{y_1} (resp., dy_1) is a global G-invariant section of $T_{\mathbb{C}^2} \otimes \mathscr{L}$ (resp., $\Omega^1_{\mathbb{C}^2} \otimes \mathscr{L}^*$), and thus yields a global section τ (resp., α) of $T_X \boxtimes \mathscr{M}$ (resp., $\Omega^{[1]}_X \boxtimes \mathscr{M}^*$). Since $dy_1(\partial_{y_1}) = 1$, we must have $\alpha(\tau)=1$. This shows that the foliation induced by τ on X is weakly regular, completing the proof of the proposition. □

6. Weakly regular foliations with algebraic leaves

It is well known that an algebraically integrable regular foliation on a complex projective manifold is induced by a morphism onto a normal projective variety (see [53, Proposition 2.5]). In the present section, we extend this result to weakly regular foliations with canonical singularities on mildly singular varieties.

THEOREM 6.1

Let X be a normal complex projective variety with $\mathbb{Q}$-factorial klt singularities, and let $\mathscr{G}$ be a weakly regular algebraically integrable foliation on X. Suppose in addition that $\mathscr{G}$ has canonical singularities. Then $\mathscr{G}$ is induced by a surjective equidimensional morphism $\psi\colon X \to Y$ onto a normal projective variety Y. Moreover, there exists an open subset Y° with complement of codimension at least 2 in Y such that $\psi^{-1}(y)$ is irreducible for any $y \in Y^\circ$.

Before we give the proof of Theorem 6.1, we need the following auxiliary lemma.

LEMMA 6.2

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a weakly regular algebraically integrable foliation on X. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6). Suppose in addition that $\mathscr{G}$ has canonical singularities. Then there exists an open subset Y° with complement of codimension at least 2 in Y such that $\psi^{-1}(y)$ is irreducible for any $y \in Y^\circ$.

Proof

By [29, Lemma 4.2], there exists a finite surjective morphism $g\colon Y_1 \to Y$ with Y_1 normal and connected such that the following holds. If Z_1 denotes the normalization of $Y_1 \times_Y Z$, then the induced morphism $\psi_1\colon Z_1 \to Y_1$ has reduced fibers over codimension 1 points in Y_1. We obtain a commutative diagram

$$
\begin{array}{ccccc}
Z_1 & \xrightarrow{f} & Z & \xrightarrow{\beta} & X \\
{\scriptstyle\psi_1}\downarrow & & \downarrow{\scriptstyle\psi} & & \\
Y_1 & \xrightarrow[g]{} & Y & &
\end{array}
$$

Let $Y^\circ \subseteq Y_{\mathrm{reg}}$ be an open subset with complement of codimension at least 2 in Y such that $Y_1^\circ := g^{-1}(Y^\circ)$ is smooth and $R(\psi_{1|Z_1^\circ}) = 0$, where $Z_1^\circ := \psi_1^{-1}(Y_1^\circ)$ and $R(\psi_{1|Z_1^\circ})$ denotes the ramification divisor of the restriction $\psi_{1|Z_1^\circ}$ of ψ_1 to Z_1°. Set $Z^\circ := \psi^{-1}(Y^\circ)$, and denote by $R(\psi_{|Z^\circ})$ the ramification divisor of the restriction

$\psi_{|Z^\circ}$ of ψ to Z°. By [29, Lemma 5.4], the pair $(Z^\circ, -R(\psi_{|Z^\circ}))$ has canonical singularities. On the other hand, we have

$$K_{Z_1^\circ/Y_1^\circ} = K_{Z_1^\circ/Y_1^\circ} - R(\psi_{1|Z_1^\circ}) \sim_{\mathbb{Z}} (f_{|Z_1^\circ})^*\big(K_{Z^\circ/Y^\circ} - R(\psi_{|Z^\circ})\big)$$

by Lemma 3.4. Applying [61, Proposition 3.16], we see that Z_1° has canonical singularities.

To prove the statement, it suffices to show that $\psi_1^{-1}(y)$ is irreducible for any $y \in Y_1$ away from a codimension 2 closed subset. We argue by contradiction and assume that there exists a prime divisor $D \subset Y_1$ such that $\psi_1^{-1}(y)$ is reducible for a general point $y \in D$.

Let $S \subseteq \psi_1^{-1}(D)$ be a subvariety of codimension 2 in Z_1 such that for a general point $z \in S$ there are at least two irreducible components of $\psi_1^{-1}(\psi_1(z))$ passing through z. Let $z \in S$ be a general point. Recall from [42, Proposition 9.3] that z has an analytic neighborhood U that is biholomorphic lytic neighborhood of the origin in a variety of the form $\mathbb{C}^{\dim X}/G$, where G is a finite subgroup of $\mathrm{GL}(\dim X, \mathbb{C})$ that does not contain any quasireflections. In particular, if W denotes the inverse image of U in the affine space $\mathbb{C}^{\dim X}$, then the quotient map $\pi\colon W \to W/G \cong U$ is étale outside of the singular set.

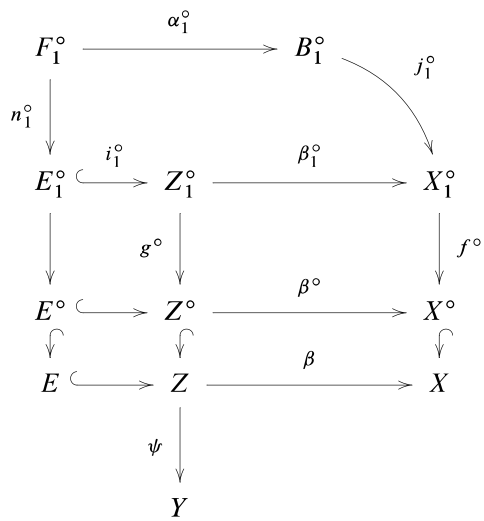
From Lemma 5.18 together with Proposition 5.13, we see that $\mathscr{G}$ induces a regular foliation on W. Let F_1 and F_2 be irreducible components of $\psi_1^{-1}(\psi_1(z))$ passing through z with $F_1 \neq F_2$. Note that $\pi^{-1}(F_1 \cap U) \cap \pi^{-1}(F_2 \cap U) \neq \emptyset$. By general choice of z, F_1 and F_2 are not contained in the singular locus of $f^{-1}\beta^{-1}\mathscr{G}$, and hence both $\pi^{-1}(F_1 \cap U)$ and $\pi^{-1}(F_2 \cap U)$ are a disjoint union of leaves. But then any leaf passing through some point of $\pi^{-1}(F_1 \cap U) \cap \pi^{-1}(F_2 \cap U)$ is a connected component of both $\pi^{-1}(F_1 \cap U)$ and $\pi^{-1}(F_2 \cap U)$. This in turn implies that $F_1 = F_2$, yielding a contradiction. This finishes the proof of the lemma. □

Proof of Theorem 6.1

Set $r := \operatorname{rank} \mathscr{G}$. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6). Set also $\mathscr{G}_Z := \beta^{-1}\mathscr{G}$. To prove Theorem 6.1, we have to show that the β-exceptional set $\operatorname{Exc}\beta$ is empty. We argue by contradiction and assume that $\operatorname{Exc}\beta \neq \emptyset$. Let E be an irreducible component of $\operatorname{Exc}\beta$. Note that E has codimension 1 since X is $\mathbb{Q}$-factorial by assumption. By Lemma 4.13, we have $K_{\mathscr{G}_Z} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. Applying Proposition 4.17, we see that $\psi(E) \subsetneq Y$. On the other hand, there exists an open subset Y° with complement of codimension at least 2 in Y such that $\psi^{-1}(y)$ is irreducible for any $y \in Y^\circ$ by Lemma 6.2. This implies that $E = \psi^{-1}(\psi(E))$. In particular, E is invariant under $\mathscr{G}_Z$.

Let m be a positive integer, and let $X^\circ \subseteq X$ be a dense open subset such that $\mathscr{O}_{X^\circ}(mK_{\mathscr{G}|X^\circ}) \cong \mathscr{O}_{X^\circ}$. Suppose in addition that $\beta(E) \cap X^\circ \neq \emptyset$. Let $f^\circ\colon X_1^\circ \to X^\circ$

be the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]), and let Z_1° be the normalization of the product $Z^\circ \times_{X^\circ} X_1^\circ$, where $Z^\circ := \beta^{-1}(X^\circ)$. Let also $\beta_1^\circ\colon Z_1^\circ \to X_1^\circ$ and $g^\circ\colon Z_1^\circ \to Z^\circ$ denote the natural morphisms. Recall from the proof of Lemma 5.18 that the foliations $\mathscr{G}_{X_1^\circ}$ and $\mathscr{G}_{Z_1^\circ}$ induced by $\mathscr{G}$ on X_1° and Z_1°, respectively, are weakly regular foliations, and that $K_{\mathscr{G}_{Z_1^\circ}} \sim_{\mathbb{Z}} (\beta_1^\circ)^* K_{\mathscr{G}_{X_1^\circ}}$. Let E_1° be a prime divisor on Z_1° such that $g^\circ(E_1^\circ) = E \cap Z^\circ =: E^\circ$. Notice that E_1° is invariant under $\mathscr{G}_{Z_1^\circ}$ since E is $\mathscr{G}_Z$-invariant. Moreover, E_1° is β_1°-exceptional. Let F_1° denote the normalization of E_1°, and let $\alpha_1^\circ\colon F_1^\circ \to B_1^\circ$ be the Stein factorization of the map $F_1^\circ \to X_1^\circ$. Shrinking X°, if necessary, we may assume without loss of generality that B_1° is smooth. We obtain a commutative diagram



CLAIM 6.3
The foliation on F_1° induced by $\mathscr{G}_{Z_1^\circ}$ is projectable under the map $\alpha_1^\circ\colon F_1^\circ \to B_1^\circ$.

Proof
Let $\eta_{Z_1^\circ}\colon \Omega^{[r]}_{Z_1^\circ} \twoheadrightarrow \mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})$ be the Pfaff field associated to $\mathscr{G}_{Z_1^\circ}$. Since E_1° is invariant under $\mathscr{G}_{Z_1^\circ}$, the composed map of sheaves $\Omega^r_{Z_1^\circ}{}_{|E_1^\circ} \to \Omega^{[r]}_{Z_1^\circ}{}_{|E_1^\circ} \to \mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})_{|E_1^\circ}$ factors through the natural morphism $\Omega^r_{Z_1^\circ}{}_{|E_1^\circ} \twoheadrightarrow \Omega^r_{E_1^\circ}$ and gives a map $\Omega^r_{E_1^\circ} \to \mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})_{|E_1^\circ}$ (see [4, Lemma 2.7]). By [9, Proposition 4.5], the latter extends to a morphism

$$\eta_{F_1^\circ}\colon \Omega^r_{F_1^\circ} \to (n_1^\circ)^*\big(\mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})_{|E_1^\circ}\big).$$

By construction, $\eta_{F_1^\circ}$ is the Pfaff field associated to the foliation induced by $\mathscr{G}_{Z_1^\circ}$ on F_1° on a dense open set.

Notice that

$$(n_1^\circ)^*\big(\mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})_{|E_1^\circ}\big) \cong (\alpha_1^\circ)^*\big((j_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}})\big)$$

since $\mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}}) \cong (\beta_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}})$ by construction. Therefore, there exists a morphism of sheaves

$$\eta_{B_1^\circ}\colon \Omega^r_{B_1^\circ} \to (j_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}})$$

whose pullback under α_1° is the composition

$$(\alpha_1^\circ)^*\Omega^r_{B_1^\circ} \xrightarrow{d\alpha_1^\circ} \Omega^r_{F_1^\circ} \xrightarrow{\eta_{F_1^\circ}} (n_1^\circ)^*\big(\mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})_{|E_1^\circ}\big) \cong (\alpha_1^\circ)^*\big((j_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}})\big).$$

Now, to show that the foliation on F_1° induced by $\mathscr{G}_{Z_1^\circ}$ is projectable under the map $\alpha_1^\circ\colon F_1^\circ \to B_1^\circ$, it suffices to prove that $\eta_{B_1^\circ}$ is nonzero.

Let $\eta_{X_1^\circ}\colon \Omega^{[r]}_{X_1^\circ} \twoheadrightarrow \mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}})$ be the Pfaff field associated to $\mathscr{G}_{X_1^\circ}$. Let $E_2^\circ \subseteq E_1^\circ$ denote a smooth dense open set contained in the smooth locus of Z_1°, and let $F_2^\circ \subseteq F_1^\circ$ be its inverse image in F_1°. Notice that $E_2^\circ \cong F_2^\circ$. Set $i_2^\circ := i^\circ_{1|E_2^\circ}\colon E_2^\circ \hookrightarrow Z_1^\circ$ and $\alpha_2^\circ := \alpha^\circ_{1|F_2^\circ}\colon F_2^\circ \to B_1^\circ$. By [60, Proposition 6.1] (see also Section 2.6), the composition

$$\big((\beta_1^\circ)^*\Omega^{[r]}_{X_1^\circ}\big)_{|E_2^\circ} \xrightarrow{d_{\mathrm{refl}}\beta^\circ_{1|E_2^\circ}} \Omega^{[r]}_{Z^\circ_{1|E_2^\circ}} \cong \Omega^r_{Z^\circ_{1|E_2^\circ}} \overset{di_2^\circ}{\twoheadrightarrow} \Omega^r_{E_2^\circ}$$

agrees with

$$\begin{aligned}&\big((\beta_1^\circ)^*\Omega^{[r]}_{X_1^\circ}\big)_{|E_2^\circ}\\ &\quad\cong \big((\alpha_1^\circ)^*\big((j_1^\circ)^*\Omega^{[r]}_{X_1^\circ}\big)\big)_{|F_2^\circ} \xrightarrow{((\alpha_1^\circ)^*d_{\mathrm{refl}}j_1^\circ)_{|F_2^\circ}} \big((\alpha_1^\circ)^*\Omega^r_{B_1^\circ}\big)_{|F_2^\circ} \xrightarrow{d\alpha_2^\circ} \Omega^r_{F_2^\circ}.\end{aligned}$$

On the other hand, recall from the proof of Lemma 5.15 that there is a commutative diagram

$$\begin{array}{ccc}(\beta_1^\circ)^*\Omega^{[r]}_{X_1^\circ} & \xrightarrow{d_{\mathrm{refl}}\beta_1^\circ} & \Omega^{[r]}_{Z_1^\circ}\\ \Big\downarrow{\scriptstyle (\beta_1^\circ)^*\eta_{X_1^\circ}} & & \Big\downarrow{\scriptstyle \eta_{Z_1^\circ}}\\ (\beta_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}}) & \xrightarrow{\sim} & \mathscr{O}_{Z_1^\circ}(K_{\mathscr{G}_{Z_1^\circ}})\end{array}$$

One then checks that the diagram

$$\begin{array}{ccc}
(j_1^\circ)^*\Omega^{[r]}_{X_1^\circ} & \xrightarrow{\;d_{\mathrm{refl}}\, j_1^\circ\;} & \Omega^r_{B_1^\circ} \\
{\scriptstyle (j_1^\circ)^*\eta_{X_1^\circ}}\big\downarrow & \swarrow {\scriptstyle \eta_{B_1^\circ}} & \\
(j_1^\circ)^*\mathscr{O}_{X_1^\circ}(K_{\mathscr{G}_{X_1^\circ}}) & &
\end{array}$$

is commutative as well. This immediately implies that $\eta_{B_1^\circ}$ is nonzero, completing the proof of the claim. □

Since the foliation induced by $\mathscr{G}_{Z_1^\circ}$ on F_1° is projectable under α_1° and $\dim E_1^\circ > \dim B_1^\circ$, there are infinitely many leaves of $\mathscr{G}_{Z_1^\circ}$ contained in E_1° that map to the same leaf of the foliation induced by $\eta_{B_1^\circ}$ on B_1°. On the other hand, recall that $\psi^{-1}(y)$ is irreducible for any point $y \in \psi(E) \cap Y^\circ$. Hence, there exists a positive integer m such that the cycle-theoretic fiber $\psi^{[-1]}(y)$ is $m[\psi^{-1}(y)]$ for a general point y in $\psi(E)$. It follows that the restriction of the map $Y \to \mathrm{Chow}(X)$ (see Section 3.6) to $\psi(E)$ has positive-dimensional fibers, yielding a contradiction. This finishes the proof of the theorem. □

Remark 6.4
In the setup of Theorem 6.1, let $\psi : Z \to Y$ be the family of leaves, and let $\beta : Z \to X$ be the natural morphism (see Section 3.6). If X is only assumed to have klt singularities, then the same argument used in the proof of the theorem shows that β is a small birational map.

COROLLARY 6.5
Let X be a normal complex projective surface with klt singularities, and let $\mathscr{G}$ be an algebraically integrable foliation by curves on X with canonical singularities. If $\mathscr{G}$ is weakly regular, then it is induced by a surjective equidimensional morphism $\psi : X \to Y$ onto a smooth complete curve Y.

Proof
This is an immediate consequence of Theorem 6.1 since X is automatically $\mathbb{Q}$-factorial by [63, Proposition 4.11]. □

The following is an easy consequence of Theorem 6.1 above together with Lemma 5.9.

COROLLARY 6.6

Let X be a normal complex projective variety with $\mathbb{Q}$-factorial klt singularities, and let $\mathscr{G}$ be a weakly regular algebraically integrable foliation on X. Suppose in addition that $K_{\mathscr{G}}$ is Cartier. Then $\mathscr{G}$ is induced by a surjective equidimensional morphism $\psi\colon X \to Y$ onto a normal projective variety Y. Moreover, there exists an open subset Y° with complement of codimension at least 2 in Y such that $\psi^{-1}(y)$ is irreducible for any $y \in Y^\circ$

7. Quasi-étale trivializable reflexive sheaves

In this section, we provide another technical tool for the proof of the main results.

PROPOSITION 7.1

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a coherent reflexive sheaf of rank r. Suppose that there exists a finite cover $f : Y \to X$ such that $f^{[*]}\mathscr{G} \cong \mathscr{O}_Y^{\oplus r}$. Then there exists a quasi-étale cover $g\colon Z \to X$ such that $g^{[*]}\mathscr{G} \cong \mathscr{O}_Z^{\oplus r}$.

7.1. *The holonomy group of a stable reflexive sheaf*

We briefly recall the definition of algebraic holonomy groups following Balaji and Kollár [11].

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a coherent reflexive sheaf on X. Suppose that $\mathscr{G}$ is stable with respect to an ample Cartier divisor H and that $\mu_H(\mathscr{G}) = 0$. For a sufficiently large positive integer m, let $C \subset X$ be a general complete intersection curve of elements in $|mH|$, and let $x \in C$. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{G}_{|C}$ is stable with $\deg \mathscr{G}_{|C} = 0$, and hence it corresponds to a unique unitary representation $\rho\colon \pi_1(C,x) \to \mathbb{U}(\mathscr{G}_x)$ by a result of Narasimhan and Seshadri [71, Theorem 2].

Definition 7.2

The holonomy group $\mathrm{Hol}_x(\mathscr{G})$ of $\mathscr{G}$ is the Zariski closure of $\rho(\pi_1(C,x))$ in $\mathrm{GL}(\mathscr{G}_x)$.

Remark 7.3

The holonomy group $\mathrm{Hol}_x(\mathscr{G})$ does not depend on $C \ni x$ provided that m is large enough (see [11]).

7.2. *Strong stability*

The following notion is used in the formulation of Lemmas 7.5 and 7.6.

Definition 7.4

Let X be a normal projective variety, let H be a nef and big Cartier divisor on X, and let $\mathscr{G}$ be a coherent reflexive sheaf. We say that $\mathscr{G}$ is *strongly stable with respect to H* if, for any normal projective variety Y and any generically finite surjective morphism $f: Y \to X$, the reflexive pullback sheaf $f^{[*]}\mathscr{G}$ is f^*H-stable.

LEMMA 7.5 ([30, Lemma 6.3])
Let X be a normal complex projective variety, let $x \in X$ be a general point, and let $\mathscr{G}$ be a coherent reflexive sheaf. Suppose that $\mathscr{G}$ is stable with respect to an ample divisor H and that $\mu_H(\mathscr{G}) = 0$. Suppose furthermore that the holonomy group $\mathrm{Hol}_x(\mathscr{G})$ is connected. Then $\mathscr{G}$ is strongly stable with respect to H.

The following observation is needed for the main result of this section.

LEMMA 7.6
Let X be a normal complex projective variety, and let $\mathscr{G}$ be a coherent reflexive sheaf. Suppose that $\mathscr{G}$ is polystable with respect to an ample divisor H and that $\mu_H(\mathscr{G}) = 0$. Then there exists a quasi-étale cover $g: Z \to X$ as well as coherent reflexive sheaves $(\mathscr{G}_i)_{i\in I}$ on Z such that the following hold.
(1) *There is a decomposition $g^{[*]}\mathscr{G} \cong \bigoplus_{i\in I} \mathscr{G}_i$.*
(2) *The sheaves $\mathscr{G}_i$ are strongly stable with respect to g^*H with $\mu_{g^*H}(\mathscr{G}_i) = 0$.*

Proof
Suppose that there exists a quasi-étale cover $g_1: Z_1 \to X$ such that the reflexive pullback $g_1^{[*]}\mathscr{G}$ is not stable with respect to g_1^*H. Applying [52, Lemma 3.2.3], we see that $g_1^{[*]}\mathscr{G}$ is polystable, and hence, there exist nonzero reflexive sheaves $(\mathscr{G}_i)_{i\in I}$, g_1^*H-stable with slopes $\mu_{g_1^*H}(\mathscr{G}_i) = \mu_{g_1^*H}(g_1^{[*]}\mathscr{G}) = 0$ such that

$$g_1^{[*]}\mathscr{G} \cong \bigoplus_{i\in I} \mathscr{G}_i.$$

Suppose in addition that the number of direct summands is maximal. Then, for any quasi-étale cover $g_2: Z_2 \to Z_1$, the reflexive pullback $g_2^{[*]}\mathscr{G}_i$ is stable with respect to $(g_1 \circ g_2)^*H$.

By [11, Lemma 40] (see also [31, Lemma 6.19]), there exists a quasi-étale cover $g_i: Z_i \to Z_1$ such that $\mathrm{Hol}_{z_i}((g_1 \circ g_i)^{[*]}\mathscr{G}_i)$ is connected, where z_i is a general point on Z_i. Let Z be the normalization of Z_1 in the compositum of the function fields $\mathbb{C}(Z_i)$. Observe that the natural morphism $g: Z \to Z_1$ is a quasi-étale cover, and factors through each $Z_i \to Z_1$. The proposition then follows from Lemma 7.5 above. □

Remark 7.7
In the setup of Lemma 7.6, suppose furthermore that $\mathscr{G}$ is polystable with respect to

any polarization H on X and that $\mu_H(\mathscr{G}) = 0$. Let H_1 be any ample divisor on X. Then the sheaves $\mathscr{G}_i$ are strongly stable with respect to g^*H_1 and $\mu_{g^*H_1}(\mathscr{G}_i) = 0$.

Proof of Proposition 7.1

Let H be an ample Cartier divisor on X. Applying [52, Lemma 3.2.3], we see that $\mathscr{G}$ is polystable with respect to H with $\mu_H(\mathscr{G}) = 0$. By Lemma 7.6, there exists a quasi-étale cover $g_1\colon Z_1 \to X$ as well as coherent reflexive sheaves $(\mathscr{G}_i)_{i\in I}$ on Z_1 such that the following hold.

(1) There is a decomposition $g_1^{[*]}\mathscr{G} \cong \bigoplus_{i\in I} \mathscr{G}_i$.

(2) The sheaves $\mathscr{G}_i$ are strongly stable with respect to g_1^*H with $\mu_{g_1^*H}(\mathscr{G}_i) = 0$.

Let Y_1 be the normalization of the product $Y \times_X Z_1$, with natural morphism $f_1\colon Y_1 \to Z_1$. By construction, we have $(g_1 \circ f_1)^{[*]}\mathscr{G} \cong \mathscr{O}_{Y_1}^{\oplus r}$. It follows that $(g_1 \circ f_1)^{[*]}\mathscr{G}_i \cong \mathscr{O}_{Y_1}$ for all $i \in I$, and hence $\mathscr{G}_i^{[N]} \cong \mathscr{O}_{Z_1}$, where $N := \deg g_1 \circ f_1$. Replacing Z_1 by a further quasi-étale cover, we may assume that $\mathscr{G}_i \cong \mathscr{O}_{Z_1}$, proving the proposition. □

Remark 7.8

In the setup of Proposition 7.1, suppose that X is a (smooth complete) curve. Then there is an alternative argument. Indeed, by [52, Lemma 3.2.3], $\mathscr{G}$ is a polystable vector bundle. We may assume without loss of generality that $\mathscr{G}$ is stable. By [71], $\mathscr{G}$ corresponds to a unique unitary representation

$$\rho\colon \pi_1(X,x) \to \mathbb{U}(\mathscr{G}_x).$$

It follows that $f^*\mathscr{G}$ corresponds to the induced representation

$$\rho \circ \pi_1(f)\colon \pi_1(Y,y) \to \mathbb{U}(\mathscr{G}_x),$$

where y is a point on Y such that $f(y) = x$. Applying [70, Proposition 4.2] to $\rho \circ \pi_1(f)$ and the trivial representation of $\pi_1(Y,y)$ in $\mathbb{U}(\mathscr{G}_x)$, we see that $\rho \circ \pi_1(f)$ must be the trivial representation. The statement then follows since the image of $\pi_1(Y,y)$ has finite index in $\pi_1(X,x)$.

8. A global Reeb stability theorem

8.1. Global Reeb stability theorem

Let X be a complex manifold, and let $\mathscr{G}$ be a regular foliation on X. Let L be a compact leaf with finite holonomy group G, and let $x \in L$. The holomorphic version of the local Reeb stability theorem (see [53, Theorem 2.4]) asserts that there exist an invariant open analytic neighborhood U of L, a (local) transversal section S at x

with a G-action, an unramified Galois cover $U_1 \to U$ with group G, a smooth proper G-equivariant morphism $U_1 \to S$, and a commutative diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{\text{unramified}} & U \\ {\scriptstyle\text{proper submersion}}\downarrow & & \downarrow \\ S & \longrightarrow & S/G \end{array}$$

such that the pullback of $\mathscr{G}_{|U}$ to U_1 is induced by the map $U_1 \to S$. In this section, we prove a global version of the Reeb stability theorem for weakly regular algebraically integrable foliations with trivial canonical class on mildly singular spaces. The following is the main result of this section (see [32, Proposition 4.2] for a somewhat related result).

THEOREM 8.1
Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a weakly regular algebraically integrable foliation on X. Suppose that $K_{\mathscr{G}} \sim_{\mathbb{Q}} 0$. Then there exist complex projective varieties Y and Z with klt singularities and a quasi-étale cover $f: Y \times Z \to X$ such that $f^{-1}\mathscr{G}$ is induced by the projection $Y \times Z \to Y$.

The proof of Theorem 8.1 makes use of the following result, which might be of independent interest.

PROPOSITION 8.2
Let X be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on X. Suppose that $\mathscr{G}$ is canonical and that $\mathscr{G} \cong \mathscr{O}_X^{\oplus \operatorname{rank} \mathscr{G}}$. Then there exist an abelian variety A, a normal projective variety X_1, and an étale cover $f: A \times X_1 \to X$ such that $f^{-1}\mathscr{G} = T_{A \times X_1/X_1}$.

Proof
Let $\psi: Z \to Y$ be the family of leaves, and let $\beta: Z \to X$ be the natural morphism (see Section 3.6). Let also F be a general fiber of ψ. Then $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} 0$ by Lemma 4.13 and Remark 4.14. Moreover, F has canonical singularities. By Example 3.3 and the adjunction formula, we have $K_F \sim_{\mathbb{Z}} K_{\beta^{-1}\mathscr{G}|F} \sim_{\mathbb{Z}} 0$.

The same argument used in the proof of [4, Lemma 3.2] (see also [4, Remark 3.8]) shows that the dual map $\Omega_X^{[1]} \to \mathscr{G}^*$ gives a generically surjective morphism $\Omega_{Z/Y}^{[1]} \to \beta^*\mathscr{G}^*$ and a commutative diagram

$$\begin{array}{ccccc}
\beta^*\Omega^1_X & \xrightarrow{d\beta} & \Omega^1_Z & \twoheadrightarrow & \Omega^1_{Z/Y} \\
\downarrow & & & & \downarrow \\
\beta^*\mathscr{G}^* & & \longleftarrow & & \Omega^{[1]}_{Z/Y}
\end{array}$$

This immediately implies that $T_F \cong \mathscr{O}_F^{\oplus \dim F}$ since $K_F \sim_{\mathbb{Z}} 0$. By Lemma 2.15, we see that the neutral component $\mathrm{Aut}^\circ(F)$ of the automorphism group $\mathrm{Aut}(F)$ of F is an abelian variety.

Let $\mathrm{Aut}^\circ(X)$ denote the neutral component of $\mathrm{Aut}(X)$, and let $H \subseteq \mathrm{Aut}^\circ(X)$ be the connected complex Lie subgroup with Lie algebra $H^0(X, \mathscr{G}) \subseteq H^0(X, T_X)$. Note that $\beta(F)$ is invariant under the action of H, and thus H also acts on the normalization F of $\beta(F)$, so that $H \subseteq \mathrm{Aut}^\circ(F)$. Since both $\mathscr{G}$ and T_F are trivial vector bundles, the tangent map

$$\mathrm{Lie}\, H = H^0(X, \mathscr{G}) \to H^0(F, T_F) = \mathrm{Lie}\,\mathrm{Aut}^\circ(F)$$

is surjective, and hence $H \cong \mathrm{Aut}^\circ(F)$. In particular, H is a closed (projective) algebraic subgroup of $\mathrm{Aut}^\circ(X)$.

By the proof of [19, Theorem 1.2, p. 10], there exist a normal projective variety X_1 and an H-equivariant finite étale cover $f\colon H \times X_1 \to X$, where H acts trivially on X_1 and diagonally on $H \times X_1$. In particular, we have $f^{-1}\mathscr{G} = T_{H \times X_1 / X_1}$, completing the proof of the proposition. □

Next, we consider the special case where the foliation is induced by a morphism equipped with a flat connection.

LEMMA 8.3

Let X be a normal complex quasiprojective variety, and let $\varphi\colon X \to Y$ be a projective equidimensional morphism with connected fibers onto a smooth quasiprojective variety. Suppose that X has klt singularities over the generic point of Y, and suppose that φ has reduced fibers over codimension 1 points in Y. Suppose furthermore that $K_{X/Y}$ is relatively numerically trivial and that there exists a foliation $\mathscr{E}$ on X such that $T_X = T_{X/Y} \oplus \mathscr{E}$.

(1) *Then there exist complex varieties B and G, as well as a quasi-étale cover $f\colon B \times G \to X$ such that $T_{B\times G/B} = f^{-1}T_{X/Y}$. Moreover, B is smooth and quasiprojective, G is projective with canonical singularities, and $K_G \sim_{\mathbb{Z}} 0$.*

(2) *Let F be a general fiber of φ. Suppose in addition that $h^0(F, T_F) = 0$. Then there exists a smooth quasiprojective variety Y_1 as well as a finite étale cover $Y_1 \to Y$ such that $Y_1 \times_Y X \cong Y_1 \times F$ as varieties over Y_1.*

Proof
Let $X^\circ \subseteq X$ be the open set where $\varphi_{|X^\circ}$ is smooth. Notice that X° has complement of codimension at least 2 since φ is equidimensional with reduced fibers over codimension 1 points in Y by assumption. The restriction of the tangent map

$$T\varphi_{|X^\circ}\colon T_{X^\circ} \to (\varphi_{|X^\circ})^* T_Y$$

to $\mathscr{E}_{|X^\circ} \subseteq T_{X^\circ}$ then induces an isomorphism $\mathscr{E}_{|X^\circ} \cong (\varphi_{|X^\circ})^* T_Y$. Since $\mathscr{E}_{|X^\circ}$ and $(\varphi_{|X^\circ})^* T_Y$ are both reflexive sheaves, we must have $\mathscr{E} \cong \varphi^* T_Y$. Thus, $\mathscr{E}$ yields a flat connection on φ. Now, a classical result of complex analysis says that complex flows of vector fields on analytic spaces exist (see [56]). It follows that φ is a locally trivial analytic fibration for the analytic topology.

Now, the same argument used in the proof of [30, Lemma 6.4] shows that the conclusion of Lemma 8.3 holds. □

Remark 8.4
In the setup of Lemma 8.3 above, suppose in addition that X is projective with klt singularities and that K_X is Cartier. Then the existence of $\mathscr{E}$ follows from the assumption $K_{X/Y} \equiv 0$ by Corollary 5.22 together with Proposition 5.6.

The proof of Theorem 8.1 relies in part on the following descent result for foliations.

LEMMA 8.5
Let X be a normal complex variety, and let $\mathscr{G}$ be a foliation on X. Let also Y be a normal variety, let A be an abelian variety, and let B be a projective variety with canonical singularities, $K_B \sim_{\mathbb{Z}} 0$, and $\widetilde{q}(B) = 0$. Suppose that there is a finite cover $f\colon Y \times A \times B \to X$ such that $f^{-1}\mathscr{G} = T_{Y\times A\times B/Y}$. Suppose in addition that any codimension 1 irreducible component of the branch locus of f is $\mathscr{G}$-invariant. Then there exist foliations $\mathscr{G}_1 \subseteq \mathscr{G}$ and $\mathscr{G}_2 \subseteq \mathscr{G}$ such that $f^{-1}\mathscr{G}_1 = T_{Y\times A\times B/Y\times B}$ and $f^{-1}\mathscr{G}_2 = T_{Y\times A\times B/Y\times A}$.

Proof
Set $Z := Y \times A \times B$. Let $X^\circ \subseteq X$ be the open set such that $f_{|f^{-1}(X^\circ)}\colon f^{-1}(X^\circ) =: Z^\circ \to X^\circ$ is étale. By a result of Serre (see [68, Lemma 2.10]), there exists a finite étale cover $g^\circ\colon Z_1^\circ \to Z^\circ$ such that the induced cover $Z_1^\circ \to X^\circ$ is Galois. Let Z_1 be the normalization of X in the function field of Z_1°, and let $f_1\colon Z_1 \to X$ be the natural morphism. Note that there exists a finite morphism $g\colon Z_1 \to Z$ such that $f_1 = f \circ g$. By construction, g is étale away from the branch locus of f, so that any codimension 1 irreducible component of the branch locus of f_1 is $\mathscr{G}$-invariant as well. By Lemma 3.4, we must have $f_1^{-1}\mathscr{G} \cong f_1^{[*]}\mathscr{G}$.

Set $\mathscr{E}_1 := g^{-1}T_{Y\times A\times B/Y\times B}$ and $\mathscr{E}_2 := g^{-1}T_{Y\times A\times B/Y\times A}$, and let γ be any automorphism of the covering $f_1\colon Z_1 \to X$. In order to prove the statement, it suffices to show that $\mathscr{E}_i \subseteq T_{Z_1}$ is γ-invariant (see [25, Lemme 2.13]).

Notice that $\mathscr{E}_1 \oplus \mathscr{E}_2 = f_1^{-1}\mathscr{G} \cong f_1^{[*]}\mathscr{G}$ is γ-invariant by construction. Moreover, the natural maps

$$\mathscr{E}_1 = g^{-1}T_{Y\times A\times B/Y\times B} \to g^{[*]}T_{Y\times A\times B/Y\times B} \qquad \text{and}$$
$$\mathscr{E}_2 = g^{-1}T_{Y\times A\times B/Y\times A} \to g^{[*]}T_{Y\times A\times B/Y\times A}$$

are isomorphisms since we observed that the induced map

$$f_1^{-1}\mathscr{G} = \mathscr{E}_1 \oplus \mathscr{E}_2 \to g^{[*]}(f^{[*]}\mathscr{G}) \cong f_1^{[*]}\mathscr{G}$$

is an isomorphism. In particular, $\mathscr{E}_1 \cong \mathscr{O}_{Z_1}^{\oplus \dim A}$. Let $F \cong A \times B$ be a general fiber of the projection $Y \times A \times B \to Y$. By general choice of F, any irreducible component G of $g^{-1}(F)$ is normal (see [47, Théorème 12.2.4]) with $T_G \cong \mathscr{E}_{1|G} \oplus \mathscr{E}_{2|G}$. In particular, we must have $K_G \sim_{\mathbb{Z}} 0$. Applying Theorem 2.14, one checks that there exist an abelian variety A_1, a projective variety B_1 with canonical singularities, $K_{B_1} \sim_{\mathbb{Z}} 0$, and $\widetilde{q}(B_1) = 0$, and quasi-étale covers $A_1 \to A$ and $B_1 \to B$ such that $G \cong A_1 \times B_1$ and such that the restriction of g to G identifies with $G \cong A_1 \times B_1 \to A \times B \cong F$. Moreover, $\mathscr{E}_{1|G} \cong T_{A_1\times B_1/B_1}$ and $\mathscr{E}_{2|G} \cong T_{A_1\times B_1/A_1}$. By Remark 2.16, we have $h^0(B_1, T_{B_1}) = 0$ and $h^1(B_1, \Omega^{[1]}_{B_1}) = 0$. This immediately implies that any map $\gamma^*\mathscr{E}_1 \to \mathscr{E}_2$ or $\gamma^*\mathscr{E}_2 \to \mathscr{E}_1$ vanishes identically, completing the proof of the lemma. ☐

Before we give the proof of Theorem 8.1, we need the following auxiliary statements.

LEMMA 8.6

Let X be a normal complex projective variety, and let $\psi\colon X \to Y$ be a surjective equidimensional morphism with connected fibers onto a normal projective variety. Suppose that X is $\mathbb{Q}$-factorial. Let F denote a general fiber of ψ, and assume that $q(F) = 0$. If $\psi^{-1}(Y^\circ) \cong Y^\circ \times F$ for some open set $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least 2, then $X \cong Y \times F$.

Proof

Set $X^\circ := \psi^{-1}(Y^\circ)$. Let H be an ample divisor on X. We have $\mathrm{Pic}(Y^\circ \times F) \cong \mathrm{Pic}(Y^\circ) \times \mathrm{Pic}(F)$ since $q(F) = 0$ by assumption. Thus, there exist Cartier divisors H_{Y° and H_F on Y° and F, respectively, such that $\mathscr{O}_{X^\circ}(H_{|X^\circ}) \cong \mathscr{O}_{Y^\circ}(H_{Y^\circ}) \boxtimes \mathscr{O}_F(H_F)$. Let H_Y be the Weil divisor on Y that restricts to H_{Y° on Y°. By

Lemma 2.6, H_Y is $\mathbb{Q}$-Cartier. Notice that $m_0(H - \psi^* H_Y)$ is relatively ample over Y for any positive integer m_0 such that $m_0(H - \psi^* H_Y)$ is Cartier. It follows that

$$X \cong \operatorname{Proj}_Y \bigoplus_{m \geq 0} \psi_* \mathscr{O}_X\big(m m_0(H - \psi^* H_Y)\big).$$

On the other hand, by [50, Corollary 1.7], the coherent sheaves $\psi_* \mathscr{O}_X(m m_0(H - \psi^* H_Y))$ are reflexive, and restrict to $H^0(F, \mathscr{O}_F(m m_0 H_F)) \otimes \mathscr{O}_{Y^\circ}$ by assumption. Therefore, there is an isomorphism of sheaves of graded $\mathscr{O}_Y$-algebras

$$\bigoplus_{m \geq 0} \psi_* \mathscr{O}_X\big(m m_0(H - \psi^* H_Y)\big) \cong \Big(\bigoplus_{m \geq 0} H^0\big(F, \mathscr{O}_F(m m_0 H_F)\big)\Big) \otimes \mathscr{O}_Y.$$

This implies that $X \cong Y \times F$, finishing the proof of the lemma. □

LEMMA 8.7 ([31, Lemma 4.6])
Let X_1, X_2, and Y be complex normal projective varieties. Suppose that there exists a surjective morphism with connected fibers $\beta\colon X_1 \times X_2 \to Y$. Suppose furthermore that $q(X_1) = 0$. Then Y decomposes as a product $Y \cong Y_1 \times Y_2$ of normal projective varieties, and there exist surjective morphisms with connected fibers $\beta_1\colon X_1 \to Y_1$ and $\beta_2\colon X_2 \to Y_2$ such that $\beta = \beta_1 \times \beta_2$.

We are now in position to prove Theorem 8.1.

Proof of Theorem 8.1
We maintain the notation and assumptions of Theorem 8.1. For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. Reduction to the case where $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$. By [63, Lemma 2.53] and Fact 2.10, there exists a quasi-étale cover $f\colon X_1 \to X$ with X_1 klt such that $f^* K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$. Moreover, $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} f^* K_{\mathscr{G}}$ and $f^{-1}\mathscr{G}$ is weakly regular by Proposition 5.13. To prove Theorem 8.1, we can therefore assume without loss of generality that the following holds.

Assumption 8.8
The canonical class $K_{\mathscr{G}}$ is trivial, $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$.

This implies that $\mathscr{G}$ is canonical by Lemma 5.9.

Step 2. Reduction to the case where $\mathscr{G}$ is given by a morphism. Suppose that the conclusion of Theorem 8.1 holds under the additional assumption that the foliation is given by an equidimensional morphism. Then we show that it holds in general.

Let $\beta\colon X_1 \to X$ be a $\mathbb{Q}$-factorialization (see Section 2.2), and set $\mathscr{G}_1 := \beta^{-1}\mathscr{G}$. By Lemma 5.16 and Remark 5.17, $\mathscr{G}_1$ is a weakly regular foliation with $K_{\mathscr{G}_1} \sim_{\mathbb{Z}} 0$. By Lemma 5.9 again, we see that $\mathscr{G}_1$ is canonical. It then follows from Theorem 6.1 that $\mathscr{G}_1$ is induced by a surjective equidimensional morphism onto a normal projective variety. By assumption, there exist complex projective varieties Y_1 and Z_1 with klt singularities and a quasi-étale cover $f_1\colon Y_1 \times Z_1 \to X_1$ such that $f_1^{-1}\mathscr{G}_1$ is induced by the projection $Y_1 \times Z_1 \to Y_1$. In particular, we have $K_{Z_1} \sim_{\mathbb{Z}} 0$. By Theorem 2.14, replacing Z_1 by a further quasi-étale cover, if necessary, we may assume without loss of generality that $Z_1 \cong A_1 \times B_1$, where A_1 is an abelian variety and B_1 is normal projective variety with $\widetilde{q}(B_1) = 0$. Let X_2 be the normalization of X in the function field of $Y_1 \times Z_1$. We have a commutative diagram

$$\begin{array}{ccc}
Y_1 \times Z_1 & \xrightarrow{\ \beta_2,\ \text{small and birational}\ } & X_2 \\
{\scriptstyle f_1,\ \text{quasi-étale}}\downarrow & & \downarrow{\scriptstyle f,\ \text{quasi-étale}} \\
X_1 & \xrightarrow[\ \beta,\ \text{small and birational}\ ]{} & X
\end{array}$$

Note that $T_{Y_1\times A_1\times B_1} \cong T_{Y_1} \boxplus T_{A_1} \boxplus T_{B_1}$. The direct summand T_{A_1} of $T_{Y_1\times A_1\times B_1}$ induces an algebraically integrable foliation $\mathscr{E}_2 \subseteq f^{-1}\mathscr{G}$ with $\mathscr{E}_2 \cong \mathscr{O}_{X_2}^{\oplus \dim A_1}$. By Lemma 5.8, $\mathscr{E}_2$ is a weakly regular foliation, and it has canonical singularities by Lemma 5.9. Applying Proposition 8.2 to $\mathscr{E}_2$, we see that there exist an abelian variety A_3, a normal projective variety Z_3, and a finite étale cover $f_2\colon X_3 := A_3 \times Z_3 \to X_2$ such that the foliation $\mathscr{E}_3 := f_2^{-1}\mathscr{E}_2$ is induced by the projection $A_3 \times Z_3 \to Z_3$. Observe that the direct summand T_{B_1} of $T_{Y_1\times A_1\times B_1}$ induces an algebraically integrable foliation $\mathscr{E}_3^{\perp}$ on X_3 such that $\mathscr{E}_3 \oplus \mathscr{E}_3^{\perp} = f_2^{-1} f^{-1}\mathscr{G}$. One then checks that there exists an algebraically integrable foliation $\mathscr{G}_3$ on Z_3 such that $\mathscr{E}_3^{\perp}$ is the pullback of $\mathscr{G}_3$ under the projection $A_3 \times Z_3 \to Z_3$. Note that $\mathscr{G}_3$ is weakly regular with $K_{\mathscr{G}_3} \sim_{\mathbb{Z}} 0$ by Lemma 5.16. By replacing X by Z_3 and $\mathscr{G}$ by $\mathscr{G}_3$, and repeating the process finitely many times, we may therefore assume that $\dim A_1 = 0$. But then the conclusion of Theorem 8.1 follows from Lemma 8.7 since $q(B_1) = 0$ by construction. To prove Theorem 8.1, we can therefore assume without loss of generality that the following holds.

Assumption 8.9
The foliation $\mathscr{G}$ is induced by a surjective equidimensional morphism with connected fibers $\psi\colon X \to Y$ onto a normal projective variety Y.

Step 3. Let F be a general fiber of ψ. Note that F has klt singularities by [61, Corollary 4.9]. Applying Corollary 5.22, we see that there is a decomposition $T_X \cong \mathscr{G} \oplus \mathscr{E}$ of T_X into involutive subsheaves.

CLAIM 8.10
There exist an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least 2 in Y, and a finite Galois cover $g^\circ \colon Y_1^\circ \to Y^\circ$ such that the following holds. Let X_1° be the normalization of $Y_1^\circ \times_{Y^\circ} X$, and denote by $\psi_1^\circ \colon X_1^\circ \to Y_1^\circ$ the natural morphism. Then ψ_1° is a locally trivial analytic fibration for the analytic topology. In particular, ψ_1° has integral fibers. Moreover, there exists a decomposition $T_{X_1^\circ} = T_{X_1^\circ/Y_1^\circ} \oplus \mathscr{E}_1^\circ$ of $T_{X_1^\circ}$ into involutive subsheaves.

Proof
By [29, Lemma 4.2], there exists a finite surjective morphism $g \colon Y_1 \to Y$ with Y_1 normal and connected such that the following holds. If X_1 denotes the normalization of $Y_1 \times_Y X$, then the induced morphism $\psi_1 \colon X_1 \to Y_1$ has reduced fibers over codimension 1 points in Y_1. By replacing Y_1 with a further finite cover, if necessary, we may assume without loss of generality that the finite cover $Y_1 \to Y$ is Galois. We have a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f,\ \text{finite}} & X \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi} \\ Y_1 & \xrightarrow[g,\ \text{finite}]{} & Y \end{array}$$

Next, we show that the tangent sheaf T_{X_1} decomposes as a direct sum $T_{X_1} = f^{-1}\mathscr{G} \oplus f^{-1}\mathscr{E}$ and that $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} 0$. Set $q := \operatorname{rank} \mathscr{G}$, and let $\omega \in H^0(X, \Omega_X^{[q]})$ be a reflexive q-form defining $\mathscr{E}$. The reflexive pullback $\omega_1 \in H^0(X_1, \Omega_{X_1}^{[q]})$ of ω then defines $f^{-1}\mathscr{E}$ on a nonempty open set. In particular, ω_1 induces an $\mathscr{O}_{X_1}$-linear map $(\wedge^q T_{X_1})^{**} \to \mathscr{O}_{X_1}$ such that the composed morphism of reflexive sheaves of rank 1

$$\tau \colon \det f^{-1}\mathscr{G} \to (\wedge^q T_{X_1})^{**} \to \mathscr{O}_{X_1}$$

is generically nonzero. On the other hand, by Lemma 3.4, we have $\det f^{-1}\mathscr{G} \cong f^*\mathscr{O}_X(-K_{\mathscr{G}}) \cong \mathscr{O}_{X_1}$ and hence τ must be an isomorphism. This easily implies that $T_{X_1} = f^{-1}\mathscr{G} \oplus f^{-1}\mathscr{E}$.

Let Y_1° be the smooth locus of Y_1, and set $X_1^\circ := \psi_1^{-1}(Y_1^\circ)$. Let $Z_1^\circ \subseteq X_1^\circ$ be the open set where $\psi_{1|X_1^\circ}$ is smooth. Notice that Z_1° has complement of codimension at least 2 in X_1° since ψ_1 has reduced fibers over codimension 1 points in Y_1°. The restriction of the tangent map

$$T\psi_{1|X_1^\circ}\colon T_{X_1^\circ} \to (\psi_{1|X_1^\circ})^* T_{Y_1^\circ}$$

to $f^{-1}\mathscr{E}_{|Z_1^\circ} \subseteq T_{Z_1^\circ}$ then induces an isomorphism $f^{-1}\mathscr{E}_{|Z_1^\circ} \cong (\psi_{1|Z_1^\circ})^* T_{Y_1^\circ}$. Since $f^{-1}\mathscr{E}_{|X_1^\circ}$ and $(\psi_{1|X_1^\circ})^* T_{Y_1^\circ}$ are both reflexive sheaves, we finally obtain an isomorphism

$$f^{-1}\mathscr{E}_{|X_1^\circ} \cong (\psi_{1|X_1^\circ})^* T_{Y_1^\circ}.$$

Now, a classical result of complex analysis says that complex flows of vector fields on analytic spaces exist (see [56]). It follows that $\psi_{1|X_1^\circ}$ is a locally trivial analytic fibration for the analytic topology. Set $Y^\circ := Y_{\mathrm{reg}} \setminus g(Y_1 \setminus Y_1^\circ)$. Then $g_{|g^{-1}(Y^\circ)}\colon g^{-1}(Y^\circ) \to Y^\circ$ satisfies the conclusions of Claim 8.10. □

Step 4. Reduction to the case where $\widetilde{q}(F) = 0$. We maintain the notation of Claim 8.10. Set $f^\circ := f_{|X_1^\circ}\colon X_1^\circ \to X^\circ := \psi^{-1}(Y^\circ)$.

By Lemma 8.3(1) and Theorem 2.14, there exist a quasiprojective variety Y_2°, an abelian variety A and a projective variety B with canonical singularities, $K_B \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(B) = 0$, and a quasi-étale cover $f_1^\circ\colon X_2^\circ := Y_2^\circ \times A \times B \to X_1^\circ$ such that $(f_1^\circ)^{-1}(\mathscr{G}_{|X_1^\circ})$ is the foliation given by the projection $Y_2^\circ \times A \times B \to Y_2^\circ$. Notice that any codimension 1 irreducible component of the branch locus of $f^\circ \circ f_1^\circ$ is invariant under $\mathscr{G}$ so that Lemma 8.5 applies. There exist algebraically integrable foliations $\mathscr{G}_1 \subseteq \mathscr{G}$ and $\mathscr{G}_2 \subseteq \mathscr{G}$ with torsion canonical class such that $(f^\circ \circ f_1^\circ)^{-1}(\mathscr{G}_{1|X^\circ}) = T_{Y_2^\circ \times A \times B / Y_2^\circ \times B}$ and $(f^\circ \circ f_1^\circ)^{-1}(\mathscr{G}_{2|X^\circ}) = T_{Y_2^\circ \times A \times B / Y_2^\circ \times A}$. Note that $\mathscr{G}_1$ and $\mathscr{G}_2$ are weakly regular foliations by Lemma 5.8 since $\mathscr{G}_1 \oplus \mathscr{G}_2 \oplus \mathscr{E} = T_X$ by construction.

Let X_2 be the normalization of X in the function field of X_2°. Applying Proposition 7.1 to the cover $X_2 \to X$ and to $\mathscr{G}_1$, we see that, replacing X by a further quasi-étale cover, if necessary, we may assume without loss of generality that $\mathscr{G}_1 \cong \mathscr{O}_X^{\dim A}$. By Proposition 8.2 together with Lemma 5.9, there exist an abelian variety A_1, a normal projective variety X_1, and an étale cover $f\colon A_1 \times X_1 \to X$ such that $f^{-1}\mathscr{G}_1 = T_{A_1 \times X_1 / X_1}$. Then $f^{-1}\mathscr{G}_2 \subseteq T_{A_1 \times X_1 / A_1}$ and it is induced by a weakly regular algebraically integrable foliation $\mathscr{H}_1$ on X_1 with $K_{\mathscr{H}_1} \sim_{\mathbb{Z}} 0$ (see Lemma 5.16). Moreover, by the rigidity lemma, $\mathscr{H}_1$ is given by a surjective equidimensional morphism $\psi_1\colon X_1 \to Y_1$ onto a normal variety Y_1 whose general fiber F_1 satisfies $\widetilde{q}(F_1) = 0$ since irregularity increases in covers by [65, Lemma 4.1.14]. To prove Theorem 8.1, we can therefore assume without loss of generality that the following holds.

Assumption 8.11
The augmented irregularity of a general fiber F of ψ is zero, $\widetilde{q}(F) = 0$.

Step 5. Next, we show the following.

CLAIM 8.12
There exist a finite cover $g\colon Y_1 \to Y$ and an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least 2 in Y such that the following holds. Let X_1 be the normalization of $Y_1 \times_Y X$, and denote by $f\colon X_1 \to X$ and $\psi_1\colon X_1 \to Y_1$ the natural morphisms. Then f is a quasi-étale cover, and $\psi_1^{-1}(Y_1^\circ) \cong Y_1^\circ \times F$ as varieties over Y_1°, where $Y_1^\circ := g^{-1}(Y^\circ)$.

Proof
We maintain the notation of Claim 8.10. Notice that $h^0(F, T_F) = 0$ since $\widetilde{q}(F) = 0$ (see Remark 2.16), so that Lemma 8.3(2) applies. Replacing Y_1 by a quasi-étale cover, if necessary, we may assume that $\psi_1^{-1}(Y_1^\circ) =: X_1^\circ \cong Y_1^\circ \times F$ as varieties over Y_1°. We obtain a commutative diagram

$$\begin{array}{ccccc}
X_1^\circ \cong Y_1^\circ \times F & \hookrightarrow & X_1 & \xrightarrow{f,\ \text{finite}} & X \\
\downarrow & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi} \\
Y_1^\circ & \hookrightarrow & Y_1 & \xrightarrow[g,\ \text{finite}]{} & Y
\end{array}$$

By replacing Y_1 with a further finite cover, if necessary, we may assume without loss of generality that the finite cover $Y_1 \to Y$ is Galois. In particular, there is a finite group G acting on Y_1 with quotient Y. The group G also acts on X_1 since X_1 identifies with the normalization of $Y_1 \times_Y X$. Moreover, $X_1^\circ \cong Y_1^\circ \times F$ is G-invariant. Since $h^0(F, T_F) = 0$, G acts on F and its action on $Y_1^\circ \times F$ is the diagonal action. Let G_1 denote the kernel of the induced morphism of groups $G \to \mathrm{Aut}(F)$. Note that $X_1^\circ/G_1 \cong (Y_1^\circ/G_1) \times F$. By replacing Y_1 by Y_1/G_1, X_1 by X_1/G_1, and G by G/G_1, if necessary, we may assume that $G \subseteq \mathrm{Aut}(F)$. Set $X^\circ := \psi^{-1}(Y^\circ)$. We may obviously assume that $\dim Y \geq 1$ and $\dim F \geq 1$. Then the quotient map $Y_1^\circ \times F \to (Y_1^\circ \times F)/G \cong X_1^\circ$ is automatically étale in codimension 1, and hence quasi-étale by the Nagata–Zariski purity theorem. This immediately implies that $f\colon X_1 \to X$ is a quasi-étale cover as well. □

Step 6. End of proof. We maintain the notation of Claim 8.12. Let $\beta_1\colon X_2 \to X_1$ be a $\mathbb{Q}$-factorialization (see Section 2.2), and set $\mathscr{G}_2 := \beta_1^{-1} f^{-1}\mathscr{G}$. By Lemma 5.16 and Remark 5.17, $\mathscr{G}_2$ is a weakly regular foliation and $K_{\mathscr{G}_2} \sim_{\mathbb{Z}} 0$. It follows that $\mathscr{G}_2$ is induced by a surjective equidimensional morphism $\psi_2\colon X_2 \to Y_2$ onto a normal projective variety by Theorem 6.1 again.

Since β_1 is a small birational morphism, we may assume without loss of generality that $\psi_1 \circ \beta_1$ has integral fibers over Y_1°. In particular, the restriction of $\psi_1 \circ \beta_1$ to $X_2^\circ := (\psi_1 \circ \beta_1)^{-1}(Y_1^\circ)$ is equidimensional. It follows that $X_2^\circ \to Y_1^\circ$ is the family of leaves of $\mathscr{G}_{2|X_2^\circ}$, and hence there exist an open set $Y_2^\circ \subseteq Y_2$ and an isomorphism $Y_2^\circ \cong Y_1^\circ$ such that $X_2^\circ = \psi_2^{-1}(Y_2^\circ)$. Notice that we have a decomposition $T_{X_2} \cong \mathscr{G}_2 \oplus \beta_1^{-1} f^{-1}\mathscr{E}$ of T_{X_2} into involutive subsheaves. Set $F_2 := \beta_1^{-1}(F)$, and observe that $h^0(F_2, T_{F_2}) = h^0(F, T_F) = 0$ since $\beta_{1|F_2}$ is a small birational morphism. By Lemma 8.3(2), replacing Y_1 and Y_2 by further quasi-étale covers, if necessary, we may assume that $X_2^\circ \cong Y_2^\circ \times F_2$ as varieties over Y_2°. By Remark 2.16, we also have $q(F_2) = 0$. On the other hand, Y_2° has complement of codimension at least 2 in Y_2 since X_2° has complement of codimension at least 2 in X_2 and ψ_2 is equidimensional. So Lemma 8.6 applies, and we show that $X_2 \cong F_2 \times Y_2$. From Lemma 8.7 again, we conclude that $X_1 \cong F_1 \times Y_1$. This finishes the proof of Theorem 8.1. □

Proof of Theorem 1.5

We maintain the notation and assumptions of Theorem 1.5. By Proposition 4.24, $K_{\mathscr{G}}$ is torsion. By [63, Lemma 2.53] and Fact 2.10, there exists a quasi-étale cover $f: X_1 \to X$ with X_1 klt such that $f^*K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$. Moreover, $f^{-1}\mathscr{G}$ is canonical by Lemma 4.3 and $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} 0$. Applying Corollary 5.23, we see that $f^{-1}\mathscr{G}$ is weakly regular. Theorem 1.5 now follows from Theorem 8.1. □

8.2. Application

The purpose of this subsection is to prove the following decomposition result (see [29, Proposition 6.1] for a somewhat related result).

PROPOSITION 8.13

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a foliation on X with canonical singularities and $K_{\mathscr{G}} \equiv 0$. There exist normal projective varieties Y and Z with klt singularities, as well as a quasi-étale cover $f: Y \times Z \to X$ such that the following holds. The foliation $f^{-1}\mathscr{G}$ is the pullback via the projection $Y \times Z \to Y$ of a foliation $\mathscr{H}$ on Y with no positive-dimensional algebraic subvariety tangent to $\mathscr{H}$ passing through a general point of Y. In addition, $K_Z \sim_{\mathbb{Z}} 0$, $\mathscr{H}$ has canonical singularities, and $K_{\mathscr{H}} \equiv 0$.

We will need the following minor generalization of [66, Corollary 3.9].

LEMMA 8.14

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation on X with *canonical singularities and $K_{\mathscr{G}} \equiv 0$. Then $\mathscr{G}$ is semistable with respect to any polarization H on X.*

Proof
Let $\beta\colon Z \to X$ be a resolution of singularities. Suppose that $\mathscr{G}$ is not semistable with respect to H. Then $\beta^{-1}\mathscr{G}$ is not semistable with respect to β^*H. Applying [22, Theorem 4.7] to the maximally destabilizing subsheaf of $\beta^{-1}\mathscr{G}$ with respect to $\beta^*(H^{\dim X-1})$, we see that $\mathscr{G}$ is uniruled. But this contradicts Proposition 4.22, proving the lemma. □

Proof of Proposition 8.13
Recall from [66, Section 2.3] that there exist a normal projective variety Y, a dominant rational map $\varphi\colon X \dashrightarrow Y$, and a foliation $\mathscr{H}$ on Y such that the following hold.
(a) There is no positive-dimensional algebraic subvariety tangent to $\mathscr{H}$ passing through a general point of Y.
(b) $\mathscr{G}$ is the pullback of $\mathscr{H}$ via φ.
Let $\mathscr{E}$ be the foliation on X induced by φ. We may assume without loss of generality that Y is the space of leaves of $\mathscr{E}$. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism (see Section 3.6).

CLAIM 8.15
The canonical class $K_{\mathscr{E}}$ of $\mathscr{E}$ is torsion. In particular, $K_{\mathscr{E}}$ is $\mathbb{Q}$-Cartier.

Proof
Replacing X by a $\mathbb{Q}$-factorialization (see Section 2.2), we may assume in addition that X is $\mathbb{Q}$-factorial by Lemma 4.2.

Since $\beta^{-1}\mathscr{G} = \psi^{-1}\mathscr{H}$, there is an exact sequence

$$0 \to \beta^{-1}\mathscr{E} = T_{Z/Y} \to \beta^{-1}\mathscr{G} \to \psi^{[*]}\mathscr{H}.$$

Therefore, there exists an effective Weil divisor B on Z

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} K_{\beta^{-1}\mathscr{E}} + \psi^* K_{\mathscr{H}} + B.$$

It follows that

$$K_{\mathscr{G}} - K_{\mathscr{E}} \sim_{\mathbb{Z}} \varphi^* K_{\mathscr{H}} + C,$$

where $C := \beta_* B$ is effective. Notice that the pullback $\varphi^* K_{\mathscr{H}}$ is well defined since φ is well defined and equidimensional on some open set with complement of codimension at least 2 (see Section 2.7). Applying [22, Corollary 4.8] to the foliation induced by $\mathscr{H}$ on a resolution of Y, we see that $K_{\mathscr{H}}$ is pseudoeffective (see Remark 2.7

for this notion). This immediately implies that $\varphi^* K_{\mathscr{H}} + C$ is pseudoeffective as well. Let H be an ample Cartier divisor on X. Then we have $\mu_H(K_{\mathscr{G}}) \geq \mu_H(K_{\mathscr{E}})$. On the other hand, by Lemma 8.14, we also have $\mu_H(K_{\mathscr{G}}) \leq \mu_H(K_{\mathscr{E}})$, and hence $\mu_H(K_{\mathscr{E}}) = \mu_H(K_{\mathscr{G}}) = 0$. This implies that $K_{\mathscr{E}} \equiv 0$. By Proposition 4.22, $\mathscr{E}$ has canonical singularities since $\mathscr{G}$ does. Proposition 4.24 then implies that $K_{\mathscr{E}}$ is torsion, proving the claim. □

By Theorem 1.5 applied to $\mathscr{E}$ and [4, Lemma 6.7], there exist normal projective varieties Y_1 and Z_1, as well as a foliation $\mathscr{H}_1$ on Y_1 and a quasi-étale cover $f\colon Y_1 \times Z_1 \to X$ such that $f^{-1}\mathscr{E} = T_{Y_1 \times Z_1/Y_1}$ and such that $f^{-1}\mathscr{G}$ is the pull-back of $\mathscr{H}_1$ via the projection $\pi\colon Y_1 \times Z_1 \to Y_1$. Moreover, there is no positive-dimensional algebraic subvariety tangent to $\mathscr{H}_1$ passing through a general point of Y_1. We also have $\pi^* K_{\mathscr{H}_1} = K_{f^{-1}\mathscr{G}} - K_{f^{-1}\mathscr{E}} \equiv 0$. In particular, $\pi^* K_{\mathscr{H}_1}$ is $\mathbb{Q}$-Cartier. This in turn implies that $K_{\mathscr{H}_1}$ is $\mathbb{Q}$-Cartier by Lemma 2.6. Moreover, we have $K_{\mathscr{H}_1} \equiv 0$. By Lemma 4.5, $\mathscr{H}_1$ is canonical. This finishes the proof of Proposition 8.13. □

9. Algebraic integrability, I

In this section, we prove algebraicity criteria for leaves of algebraic foliations on uniruled varieties (see Proposition 9.3 and Theorem 9.4).

Example 9.1
Let $n \geq 2$ be an integer. Let $A = \mathbb{C}^{n-1}/\Lambda$ be a complex abelian variety, and let $\rho\colon \pi_1(A) \to \mathrm{PGL}(2,\mathbb{C})$ be a representation of the fundamental group $\pi_1(A) \cong \Lambda$ of A. Then the group $\pi_1(A)$ acts diagonally on $\mathbb{C}^{n-1} \times \mathbb{P}^1$ by $\gamma \cdot (z,p) = (\gamma(z), \rho(\gamma)(p))$. Set $X := (\mathbb{C}^{n-1} \times \mathbb{P}^1)/\pi_1(A)$, and denote by $\psi\colon X \to A \cong \mathbb{C}^{n-1}/\pi_1(A)$ the projection morphism, which is a $\mathbb{P}^1$-bundle. The foliation on $\mathbb{C}^{n-1} \times \mathbb{P}^1$ induced by the projection $\mathbb{C}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the action of $\pi_1(A)$ and yields a flat Ehresmann connection $\mathscr{G}$ on ψ. Note that ρ is the representation induced by $\mathscr{G}$. In particular, $\mathscr{G}$-invariant sections of ψ correspond to $\pi_1(A)$-invariant points on $\mathbb{P}^1$.

Let $H \subseteq \mathrm{PGL}(2,\mathbb{C})$ be the Zariski closure of the image of ρ. Observe that H is abelian, and that $\mathscr{G}$ is algebraically integrable if and only if H is finite.

Suppose that $\dim H > 0$. Let $p\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PGL}(2,\mathbb{C})$ denote the projection morphism, and set

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{C}^* \right\} \subset \mathrm{SL}(2,\mathbb{C}) \qquad \text{and}$$

$$U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \subset \mathrm{SL}(2,\mathbb{C}).$$

Using [55, Theorem 4.12] (or [26, Lemma 6.2]), one checks that H is conjugate to $p(T)$ or $p(U)$. In either case, there is a representation

$$\widetilde{\rho}\colon \pi_1(A) \to \mathrm{SL}(2,\mathbb{C}) \quad \text{such that } \rho = \widetilde{\rho} \circ p.$$

Thus, there is a flat rank 2 vector bundle $(\mathscr{E}, \nabla)$ on A such that $X \cong \mathbb{P}_A(\mathscr{E})$ over A. Moreover, if $\mathscr{G}_\nabla$ denotes the linear Ehresmann connection on the total space E of $\mathscr{E}^*$ induced by ∇, then $\mathscr{G}$ is the projection of $\mathscr{G}_{\nabla|E^\times}$ under the natural map $\pi\colon E^\times \to X$, where $E^\times := E \setminus 0_X$.

Suppose that $H = p(T)$. Then we may assume that $\mathscr{E} = \mathscr{M} \oplus \mathscr{M}^{-1}$ as flat vector bundles for some $\mathscr{M} \in \mathrm{Pic}^0(A)$. Since $\dim H > 0$ by assumption, $\mathscr{M}$ is not torsion. The $\mathscr{G}$-invariant sections D_1 and D_2 of ψ are the two sections given by the quotients $\mathscr{M} \oplus \mathscr{M}^{-1} \twoheadrightarrow \mathscr{M}$ and $\mathscr{E} = \mathscr{M} \oplus \mathscr{M}^{-1} \twoheadrightarrow \mathscr{M}^{-1}$. The structure of an algebraic group on $\mathbb{C}^n \times \mathbb{G}_m$ induces a structure of a commutative algebraic group G on $X^\circ := X \setminus (D_1 \cup D_2)$ fitting into an exact sequence

$$1 \to \mathbb{G}_m \to G \to A \to 1.$$

Moreover, X° is a principal $\mathbb{G}_m$-bundle over A, and the action of G on X° extends to an action on X. Finally, $\mathscr{G}_{|X^\circ}$ yields a flat $\mathbb{G}_m$-equivariant connection on $X^\circ \to A$.

Suppose that $H = p(U)$. Then there is a nontrivial exact sequence $0 \to \mathscr{O}_A \to \mathscr{E} \to \mathscr{O}_A \to 0$. The only $\mathscr{G}$-invariant section D of ψ corresponds to the quotient $\mathscr{E} \twoheadrightarrow \mathscr{O}_A$. In this case, there is a structure of a commutative algebraic group G on $X^\circ := X \setminus D$ fitting into an exact sequence

$$1 \to \mathbb{G}_a \to G \to A \to 1,$$

and X is an equivariant compactification of G. Moreover, X° is a principal $\mathbb{G}_a$-bundle over A, and $\mathscr{G}_{|X^\circ}$ yields a flat $\mathbb{G}_a$-equivariant connection on $X^\circ \to A$.

The proof of Proposition 9.3 below relies on an algebraicity criterion for leaves of algebraic foliations proved in [16, Theorem 2.9], which we recall now.

Let X be an algebraic variety over some field k of positive characteristic p, and let $\mathscr{G} \subseteq T_X$ be a subsheaf. We will denote by $\mathrm{F}_{\mathrm{abs}}\colon X \to X$ the absolute Frobenius morphism of X.

The sheaf of derivations $\mathrm{Der}_k(\mathscr{O}_X) \cong T_X$ is endowed with the pth power operation, which maps any local k-derivation D of $\mathscr{O}_X$ to its pth iterate $D^{[p]}$. When $\mathscr{G}$ is involutive, the map $\mathrm{F}_{\mathrm{abs}}^*\mathscr{G} \to T_X/\mathscr{G}$ which sends D to the class in $T_X/\mathscr{G}$ of $D^{[p]}$ is $\mathscr{O}_X$-linear. The sheaf $\mathscr{G}$ is said to be *closed under pth powers* if the map $\mathrm{F}_{\mathrm{abs}}^*\mathscr{G} \to T_X/\mathscr{G}$ vanishes.

Let $R \subset \mathbb{C}$ be a finitely generated $\mathbb{Z}$-algebra, with field of fractions K, and set $S := \mathrm{Spec}\, R$. For any closed point $s \in S$ with maximal ideal $\mathfrak{m}$, we let $k(s)$ be the finite field $R/\mathfrak{m}$. We denote by $\bar{s}\colon \operatorname{Spec} k(\bar{s}) \to S$ a geometric point of S lying over s with $k(\bar{s})$ an algebraic closure of $k(s)$. Given a scheme X over S, we let $X_{\mathbb{C}} := X \otimes \mathbb{C}$, and $X_{\bar{s}} := X \otimes k(\bar{s})$. Given a sheaf $\mathscr{G}$ on X, we let $\mathscr{G}_{\mathbb{C}} := \mathscr{G} \otimes \mathbb{C}$, and $\mathscr{G}_{\bar{s}} := \mathscr{G} \otimes k(\bar{s})$.

Let X be a normal complex projective variety, and let $\mathscr{G}$ be a foliation on X. Let $R \subset \mathbb{C}$ be a finitely generated $\mathbb{Z}$-algebra, with field of fractions K, and let $\mathbf{X}$ be a projective model of X over $\mathbf{S} := \operatorname{Spec} R$. Let also $\boldsymbol{\mathscr{G}}$ be a saturated subsheaf of the relative tangent sheaf $T_{\mathbf{X}/\mathbf{S}}$ such that $\boldsymbol{\mathscr{G}}_{\mathbb{C}}$ coincides with $\mathscr{G}$. We say that $\mathscr{G}$ is *closed under pth powers for almost all primes p* if there exists an open dense subset $\mathbf{U}$ of $\mathbf{S}$ such that for any closed point s in $\mathbf{U}$, the subsheaf $\boldsymbol{\mathscr{G}}_{\bar{s}}$ of $T_{\mathbf{X}_{\bar{s}}}$ is closed under pth powers, where p denotes the characteristic of $k(\bar{s})$. This condition is independent of the choices of $\mathbf{S}$ and $\mathbf{X}$.

THEOREM 9.2 ([16, Theorem 2.9])
Let G be an algebraic group over a number field K, whose neutral component is solvable. Let P be a principal G-bundle over a smooth connected variety B over K, and let $\mathscr{G}$ be a flat G-equivariant connection on $P \to B$. Suppose that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable.

PROPOSITION 9.3
Let A be a complex abelian variety, and let $\psi\colon X \to A$ be a $\mathbb{P}^1$-bundle. Suppose that there exists a flat holomorphic connection $\mathscr{G} \subset T_X$ on ψ, and that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable.

Proof
Set $n := \dim X$. Since ψ admits a flat holomorphic connection, there is a representation

$$\rho\colon \pi_1(A) \to \mathrm{PGL}(2, \mathbb{C})$$

of $\pi_1(A)$ such that $X \cong (\mathbb{C}^{n-1} \times \mathbb{P}^1)/\pi_1(A)$, where the group $\pi_1(A)$ acts diagonally on $\mathbb{C}^{n-1} \times \mathbb{P}^1$. Moreover, $\mathscr{G}$ is induced by the projection $\mathbb{C}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^1$.

Let $H \subseteq \mathrm{PGL}(2, \mathbb{C})$ be the Zariski closure of the image of ρ. Note that $\mathscr{G}$ is algebraically integrable if and only if H is finite.

We argue by contradiction and assume that $\dim H > 0$. We use the notation introduced in Example 9.1.

There is a dense open set $X^\circ \subset X$ that is a principal bundle over A with structure group $K = \mathbb{G}_m$ or $K = \mathbb{G}_a$, and $\mathscr{G}_{|X^\circ}$ yields a flat invariant connection on $\psi^\circ := \psi_{|X^\circ}\colon X^\circ \to A$, which is closed under pth powers for almost all primes p.

In addition, there is a flat rank 2 vector bundle $(\mathscr{E}, \nabla)$ on A such that $X \cong \mathbb{P}_A(\mathscr{E})$ over A. Moreover, if $\mathscr{G}_\nabla$ denotes the linear Ehresmann connection on the total space

E of $\mathscr{E}^*$ induced by ∇, then $\mathscr{G}$ is the projection of $\mathscr{G}_{\nabla|E^\times}$ under the natural map $\pi\colon E^\times \to X$, where $E^\times := E \setminus 0_X$ (see Example 9.1 above). Let $L \subset X$ be a leaf of $\mathscr{G}$. Since $\mathscr{G}_{\nabla|E^\times}$ is invariant under the natural $\mathbb{C}^*$-action on $E^\times$, we have $\pi^{-1}(L) \cong L \times \mathbb{C}^*$ and the restriction of $\mathscr{G}_\nabla$ to $\pi^{-1}(L)$ is given by the projection $L \times \mathbb{C}^* \to \mathbb{C}^*$. It follows that $\mathscr{G}$ is algebraically integrable if and only if so is $\mathscr{G}_\nabla$.

To show Proposition 9.3, let R be a subring of $\mathbb{C}$, finitely generated over $\mathbb{Q}$, and let $\mathbf{A}$ (resp., $\mathbf{K}$, $\mathbf{X}$, and $\mathbf{X}^\circ$) be a smooth projective model of A (resp., K, X, and X°) over $\mathbf{T} := \operatorname{Spec} R$. We may assume that there exist a rank 2 vector bundle $\boldsymbol{\mathscr{E}}$ on $\mathbf{X}$ and a relative flat connection $\boldsymbol{\nabla}\colon \boldsymbol{\mathscr{E}} \to \Omega^1_{\mathbf{X}/\mathbf{A}} \otimes \boldsymbol{\mathscr{E}}$ on $\boldsymbol{\mathscr{E}}$ such that $(\boldsymbol{\mathscr{E}}_{\mathbb{C}}, \boldsymbol{\nabla}_{\mathbb{C}}) \cong (\mathscr{E}, \nabla)$ and such that $\mathbf{X} = \mathbb{P}_{\mathbf{A}}(\boldsymbol{\mathscr{E}})$. Let $\boldsymbol{\mathscr{G}}$ be the subbundle of $T_{\mathbf{X}/\mathbf{T}}$ induced by $\boldsymbol{\nabla}$, so that $\boldsymbol{\mathscr{G}}_{\mathbb{C}}$ coincides with $\mathscr{G}$. Shrinking $\mathbf{T}$, if necessary, we may assume without loss of generality that for any closed point $t \in \mathbf{T}$, $\boldsymbol{\mathscr{G}}_{\bar t}$ is closed under pth powers for almost all primes p. We may finally also assume that $\mathbf{X}^\circ_{\bar t}$ is a principal bundle over $\mathbf{A}_{\bar t}$ with structure group $\mathbf{K}_{\bar t}$, and that the restriction of $\boldsymbol{\mathscr{G}}_{\bar t}$ to $\mathbf{X}^\circ_{\bar t}$ is $\mathbf{K}_{\bar t}$-equivariant.

Now, when $t \in \mathbf{T}$ is a closed point, its residue field is a number field, and hence $\boldsymbol{\mathscr{G}}_{\bar t}$ is algebraically integrable by Theorem 9.2 above. Let $\boldsymbol{\mathscr{G}}_{\boldsymbol{\nabla}}$ denote the linear Ehresmann connection on the total space $\mathbf{E}$ of $\boldsymbol{\mathscr{E}}^*$ induced by $\boldsymbol{\nabla}$. By construction, $\boldsymbol{\mathscr{G}}$ is the projection of $\boldsymbol{\mathscr{G}}_{\boldsymbol{\nabla}|\mathbf{E}^\times}$ under the map $\mathbf{E}^\times \to \mathbf{X}$, where $\mathbf{E}^\times := \mathbf{E} \setminus 0_{\mathbf{X}}$. Moreover, if $t \in \mathbf{T}$ is a closed point, then $(\boldsymbol{\mathscr{G}}_{\boldsymbol{\nabla}})_{\bar t}$ is algebraically integrable since $\boldsymbol{\mathscr{G}}_{\bar t}$ is. In other words, the flat connection $(\boldsymbol{\mathscr{E}}_{\bar t}, \boldsymbol{\nabla}_{\bar t})$ has finite monodromy representation. By [2, Théorème 7.2.2], we conclude that $(\mathscr{E}, \nabla)$ has finite monodromy representation as well, so that $\dim H = 0$. This yields a contradiction, completing the proof of the proposition. □

The following is the main result of this section.

THEOREM 9.4
Let X be a normal complex projective variety with terminal singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical and that it is closed under pth powers for almost all primes p. Suppose furthermore that K_X is not pseudoeffective, and that $K_{\mathscr{G}} \equiv 0$. Then $\mathscr{G}$ is algebraically integrable.

Before proving Theorem 9.4 below, we note the following corollary.

COROLLARY 9.5
Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$ and that $\mathscr{G}$ is closed under pth powers for almost all primes p. Suppose in addition that K_X is not pseudoeffective. Then $\mathscr{G}$ is algebraically integrable.

Proof

Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorial terminalization of X. By Proposition 4.10, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$. Moreover, $\beta^{-1}\mathscr{G}$ is obviously closed under pth powers for almost all primes p. Finally, K_Z is not pseudoeffective since $\beta_* K_Z \sim_{\mathbb{Z}} K_X$ and K_X is not pseudoeffective by assumption. The statement now follows from Theorem 9.4 applied to $\beta^{-1}\mathscr{G}$. □

Proof of Theorem 9.4

For the reader's convenience, the proof is subdivided into a number of relatively independent steps.

Step 1. Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorialization of X. By Lemma 4.2, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. Moreover, $\beta^{-1}\mathscr{G}$ is closed under pth powers for almost all primes p. Finally, K_Z is not pseudoeffective since $\beta_* K_Z \sim_{\mathbb{Z}} K_X$ and K_X is not pseudoeffective by assumption. Thus, replacing X by Z, if necessary, we may assume without loss of generality that the following holds.

Assumption 9.6

The variety X has $\mathbb{Q}$-factorial terminal singularities.

Step 2. Since K_X is not pseudoeffective by assumption, we may run a minimal model program for X and end with a Mori fiber space (see [14, Corollary 1.3.3]). Therefore, there exists a sequence of maps

$$\begin{array}{ccccccccccccccc} X := X_0 & \overset{\varphi_0}{\dashrightarrow} & X_1 & \overset{\varphi_1}{\dashrightarrow} & \cdots & \overset{\varphi_{i-1}}{\dashrightarrow} & X_i & \overset{\varphi_i}{\dashrightarrow} & X_{i+1} & \overset{\varphi_{i+1}}{\dashrightarrow} & \cdots & \overset{\varphi_{m-1}}{\dashrightarrow} & X_m \\ &&&&&&&&&&&& \downarrow{\scriptstyle \psi_m} \\ &&&&&&&&&&&& Y \end{array}$$

where the φ_i's are either divisorial contractions or flips, and ψ_m is a Mori fiber space. The spaces X_i are normal, $\mathbb{Q}$-factorial, and X_i has terminal singularities for all $0 \le i \le m$. Let $\mathscr{G}_i$ be the foliation on X_i induced by $\mathscr{G}$. Now, we show that $K_{\mathscr{G}_i} \equiv 0$ and that $\mathscr{G}_i$ is canonical by induction on $0 \le i \le m$.

For $i = 0$, the claim is true by assumption. Let $1 \le i \le m$.

Suppose first that φ_{i-1} is a divisorial contraction. By [59, Lemma 3.2.5(2)], there exists a $\mathbb{Q}$-Cartier divisor K_i on X_i such that $K_{\mathscr{G}_{i-1}} \sim_{\mathbb{Q}} \varphi_i^* K_i$. Note that we must have $K_i \sim_{\mathbb{Q}} K_{\mathscr{G}_i}$. This in turn implies that $\mathscr{G}_i$ has canonical singularities by Lemma 4.2(2). Moreover, $K_{\mathscr{G}_i} \equiv 0$ by the projection formula.

Suppose now that φ_{i-1} is the flip of a small extremal contraction $c_{i-1}\colon X_{i-1} \to Y_{i-1}$, and let $c_i\colon X_i \to Y_{i-1}$ be the natural morphism. By [59, Lemma 3.2.5(2)] again, there exists a $\mathbb{Q}$-Cartier divisor K_{i-1} on Y_{i-1} such that $K_{\mathscr{G}_{i-1}} \sim_{\mathbb{Q}} c_i^* K_{i-1}$. Note that we must have $K_{\mathscr{G}_i} \sim_{\mathbb{Q}} c_i^* K_{i-1}$. As before, we see that $\mathscr{G}_i$ is canonical using Lemma 4.2(2) and that $K_{\mathscr{G}_i} \equiv 0$.

Finally, one readily checks that $\mathscr{G}_m$ is closed under pth powers for almost all primes p. Hence, we may assume without loss of generality that the following holds.

Assumption 9.7
There exists a Mori fiber space $\psi\colon X \to Y$.

Step 3. First, we show that $\dim X - \dim Y = 1$. We argue by contradiction and assume that $\dim X - \dim Y \geq 2$. Let F be a general fiber of ψ. Note that F has terminal singularities, and that $K_F \sim_{\mathbb{Z}} K_{X|F}$ by the adjunction formula. Moreover, F is a Fano variety by construction. Note also that F is smooth in codimension 2 since it has terminal singularities. Let $\mathscr{H}$ be the foliation on F induced by $\mathscr{G}$. Since $c_1(\mathscr{N}_{\mathscr{G}}) \equiv -K_X$ is relatively ample, we see that $\mathscr{H}$ has codimension 1. By Proposition 3.6, we have $K_{\mathscr{H}} \sim_{\mathbb{Z}} K_{\mathscr{G}|F} - B$ for some effective Weil divisor B on F. Suppose that $B \neq 0$. Applying [22, Theorem 4.7] to the pullback of $\mathscr{H}$ on a resolution of F, we see that $\mathscr{H}$ is uniruled. This implies that $\mathscr{G}$ is uniruled as well since F is general. But this contradicts Proposition 4.22, and shows that $B = 0$. By Proposition 4.22 applied to $\mathscr{H}$, we see that $\mathscr{H}$ is canonical. Finally, one checks that $\mathscr{H}$ is closed under pth powers for almost all primes p.

Let $S \subseteq F$ be a 2-dimensional complete intersection of general elements of a very ample linear system $|H|$ on F. We may assume without loss of generality that S is smooth and contained in F_{reg}. Let $\mathscr{L}$ be the foliation by curves on S induced by $\mathscr{H}$. By Proposition 3.6, we have $\mathscr{N}_{\mathscr{L}} \cong \mathscr{N}_{\mathscr{H}|S}$. In particular, we have

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = K_F^2 \cdot H^{\dim F - 2} > 0. \tag{9.1}$$

On the other hand, by the Baum–Bott formula (see [20, Theorem 3.1]), we have

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_x \mathrm{BB}(\mathscr{L}, x), \tag{9.2}$$

where x runs through all singular points of $\mathscr{L}$, and $\mathrm{BB}(\mathscr{L}, x)$ denotes the Baum–Bott index of $\mathscr{L}$ at x (we refer to [20, Chapter 3] for this notion).

Let x be a singular point of $\mathscr{L}$. If $\mathscr{H}$ is regular at x, then there is a holomorphic function f defined in a neighborhood of x such that df vanishes at finitely many points and such that df defines $\mathscr{L}$ (see Proposition 3.6). It follows that $\mathrm{BB}(\mathscr{L}, x) = 0$. Suppose now that x is a singular point of $\mathscr{H}$. By general choice of S, we may assume that [66, Corollary 7.8] applies. The foliation $\mathscr{H}$ is defined at x by the local 1-form $\omega = pz_2\,dz_1 + qz_1\,dz_2$, where p and q are positive integers and $(z_1, \ldots, z_s)$ are analytic coordinates on F centered at x. By general choice of S, we may also assume without loss of generality that S intersects the singular locus of $\mathscr{H}$ transversely. It follows that $\mathscr{L}$ is defined at x by the local 1-form $\omega = pv\,du + qu\,dv$, where (u, v) are analytic coordinates on S centered at x, and hence $\mathrm{BB}(\mathscr{L}, x) = -\frac{(p-q)^2}{pq} \leq 0$. In either case, we have $\mathrm{BB}(\mathscr{L}, x) \leq 0$, and hence $c_1(\mathscr{N}_{\mathscr{L}})^2 \leq 0$ by equation (9.2). But this contradicts inequality (9.1) above, and shows that $\dim X - \dim Y = 1$.

Next, we show the following.

CLAIM 9.8

There exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least 2 such that $\psi^\circ := \psi_{|X^\circ}$ is a $\mathbb{P}^1$-bundle and such that $\mathscr{G}_{|X^\circ}$ yields a flat Ehresmann connection on ψ°, where $X^\circ := \psi^{-1}(Y^\circ)$.

Proof

Recall that X is smooth in codimension 2 since it has terminal singularities. Since $\dim X - \dim Y = 1$, there exists an open subset $Y^\circ \subseteq Y_{\mathrm{reg}}$ with complement of codimension at least 2 such that $X^\circ := \psi^{-1}(Y^\circ) \subseteq X_{\mathrm{reg}}$. From [3, Theorem 4.1], we conclude that $\psi^\circ := \psi_{|X^\circ}$ is a conic bundle.

Suppose that there is a codimension 2 irreducible component of the singular set of $\mathscr{G}$ which is mapped onto a divisor D by ψ.

Suppose first that ψ° is smooth over the generic point of D. Let $B \subset Y$ be a germ of analytic curve passing through a general point y of D and transverse to D at y, set $S := (\psi^\circ)^{-1}(B) \subset X$, and let $\mathscr{L}$ be the foliation by curves on S induced by $\mathscr{G}$. Denote by $\pi : S \to B$ the restriction of ψ° to S, and set $C := \pi^{-1}(y) \cong \mathbb{P}^1$. We have $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C = 2$ and either C is tangent to $\mathscr{G}$, or C is transverse to $\mathscr{G}$. In the latter case, $\mathscr{G}$ must be regular along C, yielding a contradiction. This shows that C is tangent to $\mathscr{G}$. We may assume without loss of generality that $\mathscr{G}$ intersects S transversely at a general point of C (see Proposition 3.6). This implies in particular that C is $\mathscr{L}$-invariant. Let $x \in C$ be a singular point of $\mathscr{L}$. If $\mathscr{G}$ is regular at x, then there is a holomorphic function f defined in a neighborhood of x in S such that df vanishes at finitely many points and such that df defines $\mathscr{L}$ (see Proposition 3.6). Suppose that C is given at x by the equation $t = 0$, and that $f(x) = 0$. Then $f = tg$ for some local holomorphic function g on S at x, and $-\mathrm{CS}(\mathscr{L}, C, x)$ is equal to the vanishing order of $g_{|C}$ at x, where $\mathrm{CS}(\mathscr{L}, C, x)$ denotes the Camacho–Sad index (we refer to [20, Chapter 3] for this notion). In particular, $\mathrm{CS}(\mathscr{L}, C, x) \leq 0$. Suppose now that $\mathscr{G}$ is singular at x. By [66, Corollary 7.8], $\mathscr{G}$ is defined at x by the local 1-form $\omega = pz_2\,dz_1 + qz_1\,dz_2$, where p and q are positive integers and $(z_1, \ldots, z_n)$ are analytic coordinates on X centered at x. This implies in particular that the singular locus of

$\mathscr{G}$ is smooth in a neighborhood of x. By general choice of B, we may also assume without loss of generality that S intersects the singular locus of $\mathscr{G}$ transversely. It follows that $\mathscr{L}$ is defined at x by the local 1-form $\omega = pv\,du + qu\,dv$, where (u, v) are local coordinates on S centered at x. Since C is $\mathscr{L}$-invariant, we may also assume that C is given at x by equation $u = 0$. But then $\mathrm{CS}(\mathscr{L}, C, x) = -\frac{p}{q} < 0$. On the other hand, by the Camacho–Sad formula, we have

$$C^2 = \sum_x \mathrm{CS}(\mathscr{L}, C, x),$$

where x runs through all singular points of $\mathscr{L}$. This yields a contradiction since $C^2 = 0$.

Suppose now that D is an irreducible component of the critical set of ψ°, and let $y \in D$ be a general point. Let also $C = C_1 \cup C_2$ be the corresponding fiber of ψ°. Note that $C_1 \neq C_2$, and that C_i is tangent to $\mathscr{G}$ since $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C_i = 1$. We argue as in the previous case. Consider a general germ of analytic curve passing through y, set $S := (\psi^\circ)^{-1}(B) \subset X$, and let $\mathscr{L}$ be the foliation by curves on S induced by $\mathscr{G}$. Denote also by $\pi\colon S \to B$ the restriction of ψ° to S. Note that S is smooth by general choice of B. As above, we may assume without loss of generality that $\mathscr{G}$ intersects S transversely at general points of C_1 and C_2. This again implies that C_i is $\mathscr{L}$-invariant. Let $x \in C$ be a singular point of $\mathscr{L}$. If x is a regular point of $\mathscr{G}$, then we have $\mathrm{CS}(\mathscr{L}, C, x) \le 0$ as before. If $\mathscr{G}$ is singular at x, then $\mathrm{CS}(\mathscr{L}, C, x) = -\frac{p}{q} < 0$ for some positive integers p and q if $x \neq C_1 \cap C_2$, and $\mathrm{CS}(\mathscr{L}, C, x) = -\frac{(p-q)^2}{pq}$ otherwise. Together with the Camacho–Sad formula and using $C^2 = 0$, this shows that $\mathscr{G}$ is regular at any point in $C \setminus C_1 \cap C_2$ and that $\mathscr{G}$ is defined at the intersection point x of C_1 and C_2 by the local 1-form $z_2\,dz_1 + z_1\,dz_2$, where $(z_1, \dots, z_n)$ are analytic coordinates on X centered at x. Lemma 5.27 then yields a contradiction since $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C_i = 1$.

This proves that the singular set of $\mathscr{G}$ is mapped in codimension at least 2 in Y.

Let $C \cong \mathbb{P}^1$ be an irreducible component of a fiber of ψ°, and suppose that $\mathscr{G}$ is regular along C. Recall that $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C \in \{1, 2\}$ and that C is tangent to $\mathscr{G}$ if $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C = 1$. By [30, Lemma 3.2], we see that C is not tangent to $\mathscr{G}$, and hence $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C = 2$. This immediately implies that $\mathscr{G}$ is transverse to ψ° at any point of C, finishing the proof of Claim 9.8. □

Step 4. By [37, Corollary 4.5] and [59, Lemma 5.1.5], Y has $\mathbb{Q}$-factorial klt singularities. Since $\operatorname{codim} Y \setminus Y^\circ \ge 2$ and $K_{\mathscr{G}} \equiv 0$, we must have $K_Y \equiv 0$. Applying [69, Corollary V 4.9], we conclude that K_Y is torsion. Let $Y_1 \to Y$ be the index 1 canonical cover, which is quasi-étale (see [63, Definition 2.52]). By construction, $K_{Y_1} \sim_{\mathbb{Z}} 0$. In particular, Y_1 has canonical singularities. By Theorem 2.14 applied to Y_1, we see that there exists an abelian variety A as well as a projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(Z) = 0$, and a quasi-étale cover $f: A \times Z \to Y$.

Recall that f branches only on the singular set of Y, so that $f^{-1}(Y^\circ)$ is smooth. On the other hand, since $f^{-1}(Y^\circ)$ has complement of codimension at least 2 in $A \times Z_{\mathrm{reg}}$, we have $\pi_1(A \times Z_{\mathrm{reg}}) \cong \pi_1(f^{-1}(Y^\circ))$. Now, consider the representation

$$\rho\colon \pi_1(A \times Z_{\mathrm{reg}}) \cong \pi_1\big(f^{-1}(Y^\circ)\big) \to \pi_1(Y^\circ) \to \mathrm{PGL}(2, \mathbb{C})$$

induced by $\mathscr{G}_{|X^\circ}$. By [40, Theorem I], the induced representation

$$\pi_1(Z_{\mathrm{reg}}) \to \pi_1(A) \times \pi_1(Z_{\mathrm{reg}}) \cong \pi_1(A \times Z_{\mathrm{reg}}) \to \mathrm{PGL}(2, \mathbb{C})$$

has finite image. Thus, replacing Z with a quasi-étale cover, if necessary, we may assume without loss of generality that ρ factors through the projection $\pi_1(A \times Z_{\mathrm{reg}}) \to \pi_1(A)$. Let P be the corresponding $\mathbb{P}^1$-bundle over A. The natural projection $P \to A$ comes with a flat connection $\mathscr{G}_P \subset T_P$. By the GAGA (*géométrie algébrique et géométrie analytique*) theorem, P is a projective variety. By assumption, its pullback to $A \times Z_{\mathrm{reg}}$ agrees with $f^{-1}(Y^\circ) \times_{Y^\circ} X^\circ$ over $f^{-1}(Y^\circ)$. Moreover, the pullbacks on $A \times Z_{\mathrm{reg}}$ of the foliations $\mathscr{G}$ and $\mathscr{G}_P$ agree as well, wherever this makes sense. In particular, $\mathscr{G}$ is algebraically integrable if and only if so is $\mathscr{G}_P$. Now, one checks that $\mathscr{G}_P$ is closed under pth powers for almost all primes p. Theorem 9.4 then follows from Proposition 9.3. ☐

10. Algebraic integrability, II

In this section, we address codimension 1 foliations with numerically trivial canonical class on mildly singular varieties X with pseudoeffective canonical divisor. An analogue of the Bogomolov vanishing theorem then says that X has numerical dimension $\nu(X) \le 1$ (see Lemma 12.5). We first describe codimension 1 foliations with numerically trivial canonical class on varieties with $\nu(X) = 0$ (see Lemma 10.1 and Proposition 10.2). We then give two algebraicity criteria for leaves of algebraic foliations on varieties with $\nu(X) = 1$ (see Theorems 10.4 and 10.5).

LEMMA 10.1

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with $K_{\mathscr{G}} \equiv 0$. Suppose in addition that $K_X \equiv 0$. There exist an abelian variety A, a normal projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(Z) = 0$, and a quasi-étale cover $f: A \times Z \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of a codimension 1 linear foliation on A via the projection $A \times Z \to A$.

Proof

By [69, Corollary V 4.9], K_X is torsion. Let $f: X_1 \to X$ be the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]). Recall from Fact 2.10 that X_1 is klt. Notice that $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} f^*K_{\mathscr{G}}$. In particular, $K_{f^{-1}\mathscr{G}}$ is $\mathbb{Q}$-Cartier, and $K_{f^{-1}\mathscr{G}} \equiv 0$. To prove the statement, we can therefore assume without loss of generality that $K_X \sim_{\mathbb{Z}} 0$. Then X has canonical singularities, so that Theorem 2.14 applies. Replacing X by a further quasi-étale cover, if necessary, we may therefore assume that there exist an abelian variety A, a normal projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$, and $\widetilde{q}(Z) = 0$ such that $X = A \times Z$.

Let $\beta\colon Z_1 \to Z$ be a resolution of singularities, and let m be a positive integer such that $\mathscr{L} := \mathscr{N}_{\mathscr{G}}^{[m]}$ is a line bundle. Note that $c_1(\mathscr{L}) \equiv 0$ by assumption. Since $\widetilde{q}(Z) = 0$, we have $h^0(Z, \Omega_Z^{[1]}) = 0$ by Hodge symmetry for klt spaces (see [43, Proposition 6.9]), and hence $h^0(Z_1, \Omega^1_{Z_1}) = 0$. It follows that $\beta^*(\mathscr{L}_{|Z})$ is torsion, and hence, $\mathscr{L}_{|Z_{\mathrm{reg}}}$ is torsion as well. Replacing Z by a further quasi-étale cover, if necessary, we may therefore assume that $\mathscr{N}_{\mathscr{G}|Z} \cong \mathscr{O}_Z$. In particular, we see that $\mathscr{N}_{\mathscr{G}}$ is a line bundle. Moreover, since $h^0(Z, \Omega_Z^{[1]} \otimes \mathscr{N}_{\mathscr{G}|Z}) = h^0(Z, \Omega_Z^{[1]}) = 0$, the fibers of the projection $A \times Z \to A$ are tangent to $\mathscr{G}$. Therefore, $\mathscr{G}$ is the pullback of a codimension 1 foliation $\mathscr{H}$ on A. Since $K_{\mathscr{G}} \equiv 0$ and $K_Z \sim_{\mathbb{Z}} 0$, we must have $K_{\mathscr{H}} \equiv 0$, and hence $\mathscr{H}$ is a linear foliation on A. This finishes the proof of the lemma. □

We will use Lemma 10.1 together with Proposition 10.2 below.

PROPOSITION 10.2

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities and $K_{\mathscr{G}} \equiv 0$. If $\nu(X) = 0$, then K_X is torsion.

Remark 10.3

We will prove that $K_X \equiv 0$. Then [69, Corollary V.4.9] implies that K_X is torsion.

Proof of Proposition 10.2

Applying Proposition 8.13, we may assume without loss of generality that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X.

We argue by contradiction and assume that K_X is not torsion. By [69, Corollary V 4.9], we have $\kappa(X) = 0$. Replacing X by a $\mathbb{Q}$-factorialization (see Section 2.2), we may assume that X is $\mathbb{Q}$-factorial by Lemma 4.2. By [28, Théorème 1.2], we may run a minimal model program for X and end with a minimal model. Therefore, there exists a sequence of maps

$$X := X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{i-1}}{\dashrightarrow} X_i \overset{\varphi_i}{\dashrightarrow} X_{i+1} \overset{\varphi_{i+1}}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} X_m$$

where the φ_i's are either divisorial contractions or flips, and K_{X_m} is torsion. The spaces X_i are normal, $\mathbb{Q}$-factorial, and X_i has klt singularities for all $0 \le i \le m$. Let $\mathscr{G}_i$ be the foliation on X_i induced by $\mathscr{G}$. Arguing as in Step 2 of the proof of Theorem 9.4, we see that $K_{\mathscr{G}_i} \equiv 0$ and that $\mathscr{G}_i$ has canonical singularities. Note that since K_X is not torsion, there is some i such that φ_i is a divisorial contraction. Let i_0 be the largest integer i such that ψ_i is a divisorial contraction. By replacing X by X_{i_0}, if necessary, we may assume that there exists a divisorial extremal contraction $\varphi\colon X \to Y$ and that K_Y is torsion.

Let $\mathscr{E}$ denote the foliation on Y induced by $\mathscr{G}$. By Lemma 10.1 and using the fact that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X, there exist an abelian variety A and a quasi-étale cover $f\colon A \to Y$ such that $f^{-1}\mathscr{E}$ is a codimension 1 linear foliation on A. Let X_1 be the normalization of the fiber product $X \times_Y A$, and denote by $\varphi_1\colon X_1 \to A$ and $f_1\colon X_1 \to X$ the natural morphisms. Let E be the exceptional locus of φ.

Suppose first that E is invariant under $\mathscr{G}$. Applying Lemma 3.4, we see that $K_{f_1^{-1}\mathscr{G}} \equiv 0$. Then the map $f_1^{-1}\mathscr{G} \to \varphi_1^*(f^{-1}\mathscr{E})$ induced by the tangent map $T\varphi_1\colon T_{X_1} \to \varphi_1^* T_A$ is an isomorphism. This yields a contradiction since every irreducible component of $f_1^{-1}(E)$ is φ_1-exceptional.

Suppose now that E is not invariant under $\mathscr{G}$, and let $\omega \in H^0(A, \Omega_A^1)$ defining $f^{-1}\mathscr{E}$. Observe that irreducible components of $f_1^{-1}(E)$ are not invariant under $\mathscr{G}$. This implies that $d\varphi_1(\omega)$ does not vanish in codimension 1 by [24, Proposition III.1.1, Première partie], and hence $\mathscr{N}_{f_1^{-1}\mathscr{G}} \cong \mathscr{O}_{X_1}$. This in turn implies that $c_1(\mathscr{N}_{\mathscr{G}}) \sim_{\mathbb{Q}} 0$ using Lemma 3.4. It follows that $K_X \sim_{\mathbb{Q}} K_{\mathscr{G}} - c_1(\mathscr{N}_{\mathscr{G}}) \equiv 0$, yielding a contradiction again, since $-K_X$ is φ-ample by construction.

This shows that K_X is torsion, completing the proof of the proposition. □

Next, we address weakly regular codimension 1 foliations with trivial canonical class on varieties X with $\nu(X) = 1$.

THEOREM 10.4

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a weakly regular codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}} \equiv 0$. Suppose in addition that X is smooth in codimension 2 and that $\nu(X) = 1$. Then $\mathscr{G}$ is algebraically integrable.

Proof

For the reader's convenience, the proof is subdivided into a number of relatively independent steps.

Step 1. Applying Proposition 8.13 together with Corollary 5.14 and Lemma 5.16, we may assume without loss of generality that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X.

Step 2. Let $\beta\colon Z \to X$ be a resolution of singularities with exceptional set E, and suppose that E is a divisor with simple normal crossings. Suppose in addition that the restriction of β to $\beta^{-1}(X_{\mathrm{reg}})$ is an isomorphism. Let E_1 be the reduced divisor on Z whose support is the union of all irreducible components of E that are invariant under $\beta^{-1}\mathscr{G}$. Note that $-c_1(\mathscr{N}_{\mathscr{G}}) \equiv K_X$ by assumption. By Proposition 4.9 and Remark 4.8, there exists a rational number $0 \le \varepsilon < 1$ such that

$$\nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big) = \nu\big(-c_1(\mathscr{N}_{\mathscr{G}})\big) = 1.$$

By [81, Theorem 6] applied to $\beta^{-1}\mathscr{G}$, we may assume that there exists an arithmetic irreducible lattice Γ of $\mathrm{PSL}(2,\mathbb{R})^N$ for some integer $N \ge 2$, as well as a morphism $\varphi\colon Z \to \mathfrak{H} := \mathbb{D}^N/\Gamma$ of quasiprojective varieties, such that $\mathscr{G} = \varphi^{-1}\mathscr{H}$, where $\mathscr{H}$ is a weakly regular codimension 1 foliation on $\mathfrak{H}$ induced by one of the tautological foliations on the polydisk $\mathbb{D}^N$. Note that φ is generically finite by Step 1.

By a result of Selberg, there exists a torsion-free subgroup Γ_1 of Γ of finite index. Set $\mathfrak{H}_1 := \mathbb{D}^N/\Gamma_1$, and denote by $\pi\colon \mathfrak{H}_1 \to \mathfrak{H}$ the natural finite morphism. Recall that $\mathfrak{H}$ has isolated quotient singularities. It follows that π is a quasi-étale cover since $N \ge 2$.

Let $F \subset Z$ be a prime divisor, and assume that F is not β-exceptional. Set $G := \beta(F)$. We show that $\dim\varphi(F) \ge 1$. We argue by contradiction, and assume that $\dim\varphi(F) = 0$. One checks that there is only one germ of $\mathscr{H}$-invariant hypersurface through any (singular) point of $\mathfrak{H}$. This implies that F must be invariant under $\beta^{-1}\mathscr{G}$ since otherwise, $\varphi(Z)$ would be contained in a leaf of $\mathscr{H}$, which is impossible since the foliation $\varphi^{-1}\mathscr{H}$ has codimension 1 by construction. Set $X^\circ := X_{\mathrm{reg}}$. By assumption, $\mathscr{G}_{|X^\circ}$ is regular, and hence $G^\circ := G \cap X^\circ$ is a smooth hypersurface with normal bundle $\mathscr{N}_{G^\circ/X^\circ} \cong \mathscr{N}_{\mathscr{G}|G^\circ}$. Let $S \subseteq X$ be a 2-dimensional complete intersection of general elements of a very ample linear system $|H|$ on X. We may assume without loss of generality that S is smooth and contained in X°, so that $\varphi\circ\beta^{-1}$ is regular in a neighborhood of S. Set $C := S \cap G$. Then $G \cdot C < 0$ since C is contracted by the generically finite morphism $\varphi\circ\beta^{-1}{}_{|S}$. On the other hand, we must have $G \cdot C = 0$ by [30, Lemma 3.2], yielding a contradiction. This proves that $\dim\varphi(F) \ge 1$.

Step 3. By Step 2, the natural map $Z \times_{\mathfrak{H}} \mathfrak{H}_1 \to Z$ is a quasi-étale cover away from the exceptional locus of β. In particular, it induces a quasi-étale cover $f_1\colon X_1 \to X$. Let $Z_1 \to Z \times_{\mathfrak{H}} \mathfrak{H}_1$ be a resolution of singularities. We obtain a diagram

$$\begin{array}{ccccc}
X_1 & \xleftarrow{\beta_1,\ \text{birational}} & Z_1 & \xrightarrow{\varphi_1,\ \text{generically finite}} & \mathfrak{H}_1 \\
{\scriptstyle f_1,\ \text{quasi-étale}}\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle \pi,\ \text{quasi-étale}} \\
X & \xleftarrow[\beta,\ \text{birational}]{} & Z & \xrightarrow[\varphi,\ \text{generically finite}]{} & \mathfrak{H}
\end{array}$$

Let $\mathscr{E}_i$ be the codimension 1 regular foliations on $\mathfrak{H}_1$ induced by the tautological foliations on $\mathbb{D}^N$ so that $\Omega^1_{\mathfrak{H}_1} \cong \bigoplus_{1\le i\le N} \mathscr{N}^*_{\mathscr{E}_i}$. Set $n := \dim X$. We may assume without loss of generality that the natural map $\bigoplus_{1\le i\le n} \varphi_1^*\mathscr{N}^*_{\mathscr{E}_i} \to \Omega^1_{Z_1}$ is generically injective. Now observe that the line bundle $\mathscr{N}^*_{\mathscr{E}_i}$ is Hermitian semipositive, so that $\varphi_1^*\mathscr{N}^*_{\mathscr{E}_i}$ is nef. On the other hand, we have $c_1(\varphi_1^*\mathscr{N}^*_{\mathscr{E}_1}) \times \cdots \times c_1(\varphi_1^*\mathscr{N}^*_{\mathscr{E}_n}) > 0$. This immediately implies that $\kappa(Z_1) = \nu(Z_1) = \dim Z_1$. It follows that $\kappa(X_1) = \nu(X_1) = \dim X_1$ since β_1 is a birational morphism. Applying [69, Proposition V.2.7], we see that $\nu(X) = \nu(X_1) = \dim X_1 = \dim X$ since $K_{X_1} \sim_{\mathbb{Q}} f_1^* K_X$. It follows that $\dim X = 1$ since $\nu(X) = 1$ by assumption. This finishes the proof of the theorem. □

The following is the main result of this section.

THEOREM 10.5
Let X be a normal complex projective variety with terminal singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}} \equiv 0$ and that $\nu(X) = 1$. Suppose in addition that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable.

Before proving Theorem 10.5 below, we note the following corollary.

COROLLARY 10.6
Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$, and that $\mathscr{G}$ is closed under pth powers for almost all primes p. Suppose in addition that $\nu(X) = 1$. Then $\mathscr{G}$ is algebraically integrable.

Proof
Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorial terminalization of X. By Proposition 4.10, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$. Moreover, $\beta^{-1}\mathscr{G}$ is closed under pth powers for almost all primes p. Finally, $\nu(Z) = 1$ since $K_Z \sim_{\mathbb{Q}} \beta^* K_X$ and $\nu(X) = 1$ by assumption (see [69, Proposition V.2.7]). The statement now follows from Theorem 10.5 applied to $\beta^{-1}\mathscr{G}$. □

Proof of Theorem 10.5
For the reader's convenience, the proof is subdivided into a number of relatively independent steps. Set $n = \dim X$.

Step 1. Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorialization of X. By Lemma 4.2, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. Moreover, $\beta^{-1}\mathscr{G}$ is closed under pth powers for almost all primes p. Finally, $\nu(Z) = 1$ since $K_Z \sim_{\mathbb{Q}} \beta^* K_X$ and $\nu(X) = 1$ by assumption (see [69, Proposition V.2.7]). Thus, replacing X by Z, if necessary, we may assume that X is $\mathbb{Q}$-factorial.

Step 2. By [28, Théorème 3.3], we may run a minimal model program for X and end with a minimal model in codimension 1. Therefore, there exists a sequence of maps

$$X := X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{i-1}}{\dashrightarrow} X_i \overset{\varphi_i}{\dashrightarrow} X_{i+1} \overset{\varphi_{i+1}}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} X_m$$

where the φ_i's are either divisorial contractions or flips, and K_{X_m} is movable. The spaces X_i are normal, $\mathbb{Q}$-factorial, X_i has terminal singularities for all $0 \le i \le m$, and $\nu(X_i) = 1$. Let $\mathscr{G}_i$ be the foliation on X_i induced by $\mathscr{G}$. Arguing as in Step 2 of the proof of Theorem 9.4, we see that $K_{\mathscr{G}_i} \equiv 0$ and that $\mathscr{G}_i$ has canonical singularities. Replacing X by X_m, we may therefore assume without loss of generality that X is $\mathbb{Q}$-factorial and K_X is movable.

Step 3. Let $S \subseteq X$ be a 2-dimensional complete intersection of general elements of a very ample linear system $|H|$ on X. We may assume without loss of generality that S is smooth and contained in X_{reg}. Let $\mathscr{L}$ be the foliation by curves on S induced by $\mathscr{G}$. By Proposition 3.6, we have $\mathscr{N}_{\mathscr{L}} \cong \mathscr{N}_{\mathscr{G}|S}$. In particular, we have

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = K_X^2 \cdot H^{n-2} \ge 0 \tag{10.1}$$

since K_X is movable by Step 2.

On the other hand, by the Baum–Bott formula (see [20, Theorem 3.1]), we have

$$c_1(\mathscr{N}_{\mathscr{L}})^2 = \sum_x \mathrm{BB}(\mathscr{L}, x), \tag{10.2}$$

where x runs through all singular points of $\mathscr{L}$, and $\mathrm{BB}(\mathscr{L}, x)$ denotes the Baum–Bott index of $\mathscr{L}$ at x.

Let x be a singular point of $\mathscr{L}$. Next, we compute the Baum–Bott index $\mathrm{BB}(\mathscr{L}, x)$ as in Step 3 of the proof of Theorem 9.4.

If $\mathscr{G}$ is regular at x, then $\mathrm{BB}(\mathscr{L}, x) = 0$. Suppose now that x is a singular point of $\mathscr{G}$. Then $\mathscr{G}$ is defined at x by the local 1-form $\omega = p z_2\, dz_1 + q z_1\, dz_2$, where p and q are positive integers and $(z_1, \ldots, z_n)$ are analytic coordinates on X centered at x, and $\mathrm{BB}(\mathscr{L}, x) = -\frac{(p-q)^2}{pq} \le 0$. In either case, we have $\mathrm{BB}(\mathscr{L}, x) \le 0$. Equations

(10.1) and (10.2) above then show that $\mathrm{BB}(\mathscr{L}, x) = 0$ for any singular point x of $\mathscr{L}$. It follows that there exists an open set $X^\circ \subseteq X$ with complement of codimension at least 3 such that $\mathscr{G}_{|X^\circ}$ is defined by closed holomorphic 1-forms with zero set of codimension at least 2 locally for the analytic topology.

Step 4. Suppose first that $q(X) = 0$. Then the Picard group of X is discrete, and thus $K_{\mathscr{G}}$ is torsion. Replacing X by the associated cyclic cover, which is quasi-étale (see [63, Definition 2.52]), and using Lemma 4.3 and Remark 2.9, we may assume without loss of generality that $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$. By Proposition 5.26, we see that $\mathscr{G}$ is weakly regular. Theorem 10.4 above then says that $\mathscr{G}$ is algebraically integrable.

Step 5. To prove the statement, we argue by induction on $\dim X$.

If $\dim X = 1$, then $\mathscr{G}$ is obviously algebraically integrable.

Suppose from now on that $\dim X \geq 2$. By Step 4, we may assume without loss of generality that $q(X) \neq 0$. Let

$$a_X \colon X \to \mathrm{A}$$

be the Albanese morphism, that is, the universal morphism to an abelian variety (see [76]). Since X has rational singularities (see [59, Theorem 1.3.6]), we have $\dim \mathrm{A} = q(X) > 0$ by [58, Lemma 8.1]. Let F be a very general fiber of the Stein factorization of $X \to a_X(X)$, and let $\mathscr{H}$ be the foliation on F induced by $\mathscr{G}$.

Suppose first that $\dim F > 0$. Then F has terminal singularities, and $K_F \sim_{\mathbb{Z}} K_{X|F}$ by the adjunction formula. In particular, K_F is movable. If $\mathscr{H} = T_F$, then $\mathscr{H}$ is algebraically integrable. Suppose that $\mathscr{H}$ has codimension 1. By Proposition 3.6, we have $K_{\mathscr{H}} \sim_{\mathbb{Z}} K_{\mathscr{G}|F} - B$ for some effective Weil divisor B on F. Suppose that $B \neq 0$. Applying [22, Theorem 4.7] to the pullback of $\mathscr{H}$ on a resolution of F, we see that $\mathscr{H}$ is uniruled. This implies that $\mathscr{G}$ is uniruled as well since F is general. But this contradicts Proposition 4.22, and shows that $B = 0$. From Lemma 12.5, we conclude that $\nu(F) \in \{0, 1\}$. By Proposition 4.22 applied to $\mathscr{H}$, we see that $\mathscr{H}$ is canonical. Notice that $\mathscr{H}$ is closed under pth powers for almost all primes p. If $\nu(F) = 0$, then $\mathscr{H}$ is algebraically integrable by Lemma 10.1 and Proposition 10.7 below. If $\nu(F) = 1$, then $\mathscr{H}$ is algebraically integrable by induction.

This shows that if $\dim F \geq 2$, then there is a positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X. By Proposition 8.13, replacing X by a quasi-étale cover, if necessary, we may assume that there exist normal projective varieties Y and Z with $\dim Y < \dim X$ such that $X = Y \times Z$, as well as a foliation $\mathscr{E}$ on Y such that $\mathscr{G}$ is the pullback of $\mathscr{E}$ via the projection $Y \times Z \to Y$. Moreover, $K_Z \sim_{\mathbb{Z}} 0$, $\mathscr{E}$ has canonical singularities, and $K_{\mathscr{E}} \equiv 0$. The induction hypothesis applied to $\mathscr{E}$ then says that $\mathscr{E}$ is algebraically integrable, so that $\mathscr{G}$ is algebraically integrable as well.

Suppose from now on that $\dim F \leq 1$ and that $\mathscr{H}$ is the foliation by points. Then the tangent map to a_X yields an inclusion map $\mathscr{G} \subseteq a_X^* T_{\mathrm{A}}$. On the other hand, by Lemma 8.14, $\mathscr{G}$ is semistable with respect to any polarization on X. This immediately implies that $\mathscr{G} \cong \mathscr{O}_X^{n-1}$ since $\mathscr{G}$ is reflexive and $K_{\mathscr{G}} \equiv 0$. Moreover, $\mathscr{G}$ is a sheaf of abelian Lie algebras. Let $\mathrm{Aut}^\circ(X)$ denote the neutral component of the automorphism group $\mathrm{Aut}(X)$ of X, and let $H \subseteq \mathrm{Aut}^\circ(X)$ be the connected commutative Lie subgroup with Lie algebra $H^0(X, \mathscr{G}) \subseteq H^0(X, T_X)$. Finally, let $G \subseteq \mathrm{Aut}^\circ(X)$ be its Zariski closure. Note that $\mathscr{G}$ is induced by H by construction. Moreover, G is a commutative algebraic group. If $H = G$, then $\mathscr{G}$ is algebraically integrable. Suppose from now on that $\dim G \geq \dim X$. Arguing as in the proof of [66, Lemma 9.3], one shows that $\dim G = \dim X$ and that X is an equivariant compactification of G. Since K_X is pseudoeffective and X has terminal singularities, X is not uniruled. It follows that $G = X$ is an abelian variety by Chevalley's structure theorem on algebraic groups. But this contradicts the assumption that $\nu(X) = 1$, completing the proof of the theorem. □

PROPOSITION 10.7

Let A be a complex abelian variety, and let $\mathscr{G}$ be a linear foliation on A. Suppose that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable.

Proof

By [34, Proposition 3.6], we may assume without loss of generality that X and $\mathscr{G}$ are defined over a number field. The statement then follows from [16, Theorem 2.3]. □

11. Foliations defined by closed rational 1-forms

In this section, we address codimension 1 foliations with numerically trivial canonical class defined by closed rational 1-forms with values in flat line bundles and whose zero sets have codimension at least 2. Recall that a *flat vector bundle* on a normal complex variety X is a vector bundle of rank r induced by a representation $\pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$. The following is the main result of the present section.

THEOREM 11.1

Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$. Suppose in addition that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$, whose zero set has codimension at least 2. Then there exist a normal projective equivariant compactification Z of a commutative algebraic group G of dimension at least 2, as well as a codimension 1 foliation $\mathscr{H} \cong \mathscr{O}_Z^{\dim Z - 1}$ on Z

induced by a codimension 1 *Lie subgroup of* G*, a normal projective variety* Y *with* $K_Y \sim_{\mathbb{Z}} 0$*, and a quasi-étale cover* $f: Y \times Z \to X$ *such that* $f^{-1}\mathscr{G}$ *is the pullback of* $\mathscr{H}$ *via the projection* $Y \times Z \to Z$*. Moreover, there is no positive-dimensional algebraic subvariety tangent to* $\mathscr{H}$ *passing through a general point of* Z*.*

Example 11.2
The setup and notation are as in Example 9.1. Suppose in addition that $\dim H > 0$.

Suppose that $H = p(T)$. Let z be a coordinate on $\mathbb{C} \subset \mathbb{P}^1$ such that the inverse images of D_1 and D_2 on the universal covering space $\mathbb{C}^{n-1} \times \mathbb{P}^1$ of X are given by equations $z = 0$ and $z = \infty$, respectively. Then $\frac{dz}{z}$ induces a closed logarithmic 1-form with poles along D_1 and D_2 and empty zero set defining $\mathscr{G}$. The 1-form $\frac{dz}{z^2}$ induces a closed rational 1-form on X with values in the flat line bundle $\mathscr{L} = \psi^*\mathscr{M}^{\otimes -2}$ whose divisor of zeros and poles is $-2D_1$.

Suppose that $H = p(U)$. Let z be a coordinate on $\mathbb{C} \subset \mathbb{P}^1$ such that the inverse image of D on the universal covering space $\mathbb{C}^{n-1} \times \mathbb{P}^1$ of X is given by equation $z = 0$. Then the 1-form $\frac{dz}{z^2}$ induces a closed rational 1-form on X defining $\mathscr{G}$ with divisor of zeros and poles $-2D$.

The proof of Theorem 11.1 makes use of the following result, which might be of independent interest.

THEOREM 11.3
Let X *be a normal complex projective variety with klt singularities. Let* ω *be a closed rational* 1*-form, and let* $\mathscr{G}$ *be the foliation on* X *defined by* ω*. Suppose that* $\mathscr{G}$ *has canonical singularities and that* $K_{\mathscr{G}} \equiv 0$*. Then one of the following holds.*

(1) *There exist a complete smooth curve* C*, a complex projective variety* Y *with canonical singularities and* $K_Y \sim_{\mathbb{Z}} 0$*, as well as a quasi-étale cover* $f: Y \times C \to X$ *such that* $f^{-1}\mathscr{G}$ *is induced by the projection* $Y \times C \to C$*.*

(2) *There exist a normal projective equivariant compactification* Z *of a commutative algebraic group* G *of dimension at least* 2*, as well as a codimension* 1 *foliation* $\mathscr{H} \cong \mathscr{O}_Z^{\dim Z - 1}$ *on* Z *induced by a codimension* 1 *Lie subgroup of* G*, a normal projective variety* Y *with* $K_Y \sim_{\mathbb{Z}} 0$*, and a quasi-étale cover* $f: Y \times Z \to X$ *such that* $f^{-1}\mathscr{G}$ *is the pullback of* $\mathscr{H}$ *via the projection* $Y \times Z \to Z$*. Moreover, there is no positive-dimensional algebraic subvariety tangent to* $\mathscr{H}$ *passing through a general point of* Z*.*

To prove Theorem 11.3 above, we will need the following auxiliary result.

LEMMA 11.4
Let X be a normal complex projective variety. Let ω be a closed rational 1-form, and let $\mathscr{G}$ be the foliation on X defined by ω. Then there is a finite-dimensional complex abelian Lie algebra V and a morphism of sheaves of Lie algebras $\mathscr{G} \to V \otimes \mathscr{O}_X$ whose kernel $\mathscr{E}$ is an algebraically integrable foliation on X.

Proof
Let $\beta\colon Z \to X$ be a resolution of singularities, and let D be the divisor on Z of codimension 1 poles of $d\beta(\omega)$. We may assume without loss of generality that D has simple normal crossings. Set $Z^\circ := Z \setminus \operatorname{Supp} D$, and let $z_0 \in Z^\circ$. Consider the multivalued holomorphic function $h(z) := \int_\gamma \omega$ on Z°, where γ is a path connecting z_0 and z contained in Z°. Then h yields local first integrals for $\beta^{-1}\mathscr{G}$. Let

$$\rho\colon \pi_1(Z^\circ) \to \mathbb{G}_a$$

be the representation of $\pi_1(Z^\circ)$ induced by analytic continuation of the multivalued holomorphic function h along loops with basepoint z_0. Now, let

$$\mathrm{qa}_{Z^\circ}\colon Z^\circ \to \mathrm{G}(Z^\circ) := H^0\big(Z, \Omega^1_Z(\log D)\big)^*/H_1(Z^\circ, \mathbb{Z})$$

be the quasi-Albanese morphism, that is, the universal morphism to a semiabelian variety (see [54]). Let also F° be a general fiber of qa_{Z°. The kernel of the natural morphism $\pi_1(Z^\circ) \to \pi_1(\mathrm{G}(Z^\circ))$ is generated by $[\pi_1(Z^\circ), \pi_1(Z^\circ)]$ together with finitely many torsion elements. It follows that the representation $\pi_1(F^\circ) \to \mathbb{G}_a$ given by the restriction of ρ to $\pi_1(F^\circ)$ is trivial. This in turn implies that the foliation induced by $\mathscr{G}$ on F° has algebraic leaves by [24, Théorème III.2.1, Première partie].

Let $\mathscr{E}_Z \subseteq \beta^{-1}\mathscr{G}$ be the foliation on Z such that $\mathscr{E}_{Z|Z^\circ}$ is the kernel of the morphism

$$\beta^{-1}\mathscr{G}_{|Z^\circ} \to (\mathrm{qa}_{Z^\circ})^* T_{\mathrm{G}(Z^\circ)} \cong H^0\big(Z, \Omega^1_Z(\log D)\big)^* \otimes \mathscr{O}_{Z^\circ},$$

and let $\mathscr{E}$ be the induced foliation on X. We have shown that $\mathscr{E}$ is algebraically integrable. On the other hand, any irreducible component of D is invariant under $\beta^{-1}\mathscr{G}$ by [24, Proposition III.1.1, Première partie]. This implies that the map

$$\beta^{-1}\mathscr{G} \to H^0\big(Z, \Omega^1_Z(\log D)\big)^* \otimes \mathscr{O}_Z$$

induced by the contraction morphism $T_Z \times \Omega^1_Z \to \mathscr{O}_Z$ is well defined. Therefore, we must have exact sequences

$$0 \to \mathscr{E}_Z \to \beta^{-1}\mathscr{G} \to H^0\big(Z, \Omega^1_Z(\log D)\big)^* \otimes \mathscr{O}_Z$$

and

$$0 \to \mathscr{E} \to \mathscr{G} \to H^0\big(Z, \Omega^1_Z(\log D)\big)^* \otimes \mathscr{O}_X.$$

This finishes the proof of the lemma. □

Proof of Theorem 11.3

We maintain the notation and assumptions of Theorem 11.3. By Proposition 8.13, there exist normal projective varieties Y and Z, a foliation $\mathscr{H}$ on Z such that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{H}$ passing through a general point of Z, and a quasi-étale cover $f: Y \times Z \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of $\mathscr{H}$ via the projection $Y \times Z \to Z$. Moreover, $\mathscr{H}$ has canonical singularities and $K_{\mathscr{H}} \equiv 0$.

If $\dim Z = 1$, then we are in case (1) of Theorem 11.3.

Suppose from now on that $\dim Z \geq 2$. Let $F \cong Z$ be a general fiber of the projection $Y \times Z \to Y$. Then $(f^{-1}\mathscr{G})_{|F} \cap T_F \cong \mathscr{H}$, and hence the restriction of $df(\omega)$ to F is a closed rational 1-form defining $\mathscr{H}$. Applying Lemma 11.4 above to $\mathscr{H}$, we see that there is a finite-dimensional complex abelian Lie algebra V and an injective morphism of sheaves of Lie algebras $\mathscr{H} \to V \otimes \mathscr{O}_Z$. This immediately implies that $\mathscr{H} \cong \mathscr{O}_Z^{\dim Z - 1}$ since $\mathscr{H}$ is reflexive and $K_{\mathscr{H}} \equiv 0$. Moreover, $\mathscr{H}$ is a sheaf of abelian Lie algebras. Let $\mathrm{Aut}^\circ(Z)$ denote the neutral component of the automorphism group $\mathrm{Aut}(Z)$ of Z, and let $H \subset \mathrm{Aut}^\circ(Z)$ be the connected commutative Lie subgroup with Lie algebra $H^0(Z, \mathscr{H}) \subset H^0(Z, T_Z)$. Finally, let $G \subseteq \mathrm{Aut}^\circ(Z)$ be its Zariski closure. Note that $\mathscr{H}$ is induced by H by construction, and that $\dim G \geq \dim Z$ since $\mathscr{H}$ is not algebraically integrable. Moreover, G is a commutative algebraic group. Arguing as in the proof of [66, Lemma 9.3], one shows that $\dim G = \dim Z$ and that Z is an equivariant compactification of G. Thus, we are in case (2) of Theorem 11.3. □

In the setting of Theorem 11.1, the twisted rational 1-form ω is not determined by $\mathscr{G}$ (see Example 11.2). The following result addresses this issue.

LEMMA 11.5

Let X be a normal complex projective variety, and let α and β be closed rational 1-forms with values in flat line bundles $\mathscr{L}$ and $\mathscr{M}$. Suppose that the divisors of zeros and poles of α and β coincide. Suppose in addition that $\alpha \wedge \beta = 0$. Then either $\mathscr{L} \cong \mathscr{M}$ as flat line bundles and $\alpha = c\beta$ for some $c \in \mathbb{C}^*$, or there is a nonzero $\gamma \in H^0(X, \Omega_X^{[1]})$ such that $\alpha \wedge \gamma = 0$.

Proof

Let D be the divisor of zeros and poles of α, and set $X^\circ := X_{\mathrm{reg}}$. There exist a covering $(U_i)_{i \in I}$ of X° by analytically open sets and closed meromorphic 1-forms α_i (resp., β_i) on U_i with divisor of zeros and poles $D_{|U_i}$ satisfying $\alpha_i = a_{ij}\alpha_j$ (resp.,

$\beta_i = b_{ij}\beta_j$) over $U_i \cap U_j$ with $a_{ij} \in \mathbb{C}^*$ (resp., $b_{ij} \in \mathbb{C}^*$) and such that $(\alpha_i)_{i\in I}$ (resp., $(\beta_i)_{i\in I}$) represents $\alpha_{|X^\circ}$ (resp., $\beta_{|X^\circ}$).

Since $\alpha_i \wedge \beta_i = 0$ by assumption, there exists a meromorphic function f_i on U_i such that $\alpha_i = f_i\beta_i$. Note that we must have $df_i \wedge \beta_i = 0$ since $d\alpha_i = d\beta_i = 0$. Now, observe that f_i is a nowhere vanishing holomorphic function since the divisors of zeros and poles of α_i and β_i coincide by assumption. On the other hand, we have $f_i = \frac{a_{ij}}{b_{ij}} f_j$ over $U_i \cap U_j$. If the functions f_i are constant, then $\mathscr{L} \cong \mathscr{M}$ as flat line bundles, and we can suppose without loss of generality that $a_{ij} = b_{ij}$ for all indices i and j. Then $\alpha = c\beta$ with $c = f_i = f_j$. Suppose that some f_i is not a constant function. Since $\frac{df_i}{f_i} = \frac{df_j}{f_j}$, there exists a nonzero holomorphic 1-form γ° on X° that restricts to $\frac{df_i}{f_i}$ over U_i. By construction, we have $\alpha_{|X^\circ} \wedge \gamma^\circ = 0$. The claim now follows from the GAGA theorem. □

To prove Theorem 11.1 in the case where X is uniruled, we will reduce to foliations defined by a closed rational 1-form using Proposition 11.9 below. We first consider the special case where X has terminal singularities.

PROPOSITION 11.6

Let X be a normal complex projective variety with terminal singularities, and let $\mathscr{G} \subset T_X$ be a codimension 1 foliation with canonical singularities. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ whose zero set has codimension at least 2. Suppose furthermore that K_X is not pseudoeffective and that $K_{\mathscr{G}} \equiv 0$. Then there exists a quasi-étale cover $f : X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form with zero set of codimension at least 2.

Proof

For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. By Proposition 8.13, there exist normal projective varieties Y and Z, a foliation $\mathscr{H}$ on Y such that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{H}$ passing through a general point of Y, and a quasi-étale cover $f : Y \times Z \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of $\mathscr{H}$ via the projection $Y \times Z \to Y$. Moreover, $\mathscr{H}$ has canonical singularities and $K_{\mathscr{H}} \equiv 0$. Let $F \cong Y$ be a general fiber of the projection $Y \times Z \to Z$. Then $(f^{-1}\mathscr{G})_{|F} \cap T_F \cong \mathscr{H}$, and hence the restriction of $df(\omega)$ to F is a closed rational 1-form with values in $\mathscr{L}_{|F}$ defining $\mathscr{H}$ whose zero set has codimension at least 2. Moreover, its pullback to $Y \times Z$ is a closed rational 1-form with values in the pullback of $\mathscr{L}_{|F}$ whose zero set has codimension at least 2 defining $\mathscr{G}$. Finally, K_Y is not pseudoeffective, and one checks that Y has terminal singularities using Fact 2.10. Therefore, replacing $\mathscr{G}$ by $\mathscr{H}$, if necessary, we may assume that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X.

Step 2. Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorialization of X. By Lemma 4.2, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}$. Moreover, $\beta^{-1}\mathscr{G}$ is given by a closed rational 1-form with values in the flat line bundle $\beta^*\mathscr{L}$ whose zero set has codimension at least 2. Finally, K_Z is not pseudoeffective since $\beta_* K_Z \sim_{\mathbb{Z}} K_X$ and K_X is not pseudoeffective by assumption.

Suppose now that there exists a quasi-étale cover $g\colon Z_1 \to Z$ such that $(g \circ \beta)^{-1}\mathscr{G}$ is given by a closed rational 1-form ω with zero set of codimension at least 2, and let $f : X_1 \to X$ be the normalization of X in the function field of Z_1. Denote by $\gamma\colon Z_1 \to X_1$ the induced birational map. Then f_1 is quasi-étale and $d\gamma(\omega)$ is a closed rational 1-form with zero set of codimension at least 2 defining $f^{-1}\mathscr{G}$. Thus, replacing X by Z, if necessary, we may assume without loss of generality that X has $\mathbb{Q}$-factorial terminal singularities.

Step 3. Since K_X is not pseudoeffective by assumption, we may run a minimal model program for X and end with a Mori fiber space (see [14, Corollary 1.3.3]). Therefore, there exists a sequence of maps

$$X := X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{i-1}}{\dashrightarrow} X_i \overset{\varphi_i}{\dashrightarrow} X_{i+1} \overset{\varphi_{i+1}}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} X_m \xrightarrow{\psi_m} Y$$

where the φ_i's are either divisorial contractions or flips, and ψ_m is a Mori fiber space. The spaces X_i are normal, $\mathbb{Q}$-factorial, and X_i has terminal singularities for all $0 \le i \le m$. Let $\mathscr{G}_i$ be the foliation on X_i induced by $\mathscr{G}$. Arguing as in Step 2 of the proof of Theorem 9.4, we see that $K_{\mathscr{G}_i} \equiv 0$ and that $\mathscr{G}_i$ has canonical singularities. Moreover, ω induces a closed rational 1-form ω_m on X_m with values in a flat line bundle $\mathscr{L}_m$, whose zero set has codimension at least 2. By construction, $\mathscr{G}_m$ is given by ω_m, and if $\mathscr{L}_m$ is a torsion flat line bundle, then so is $\mathscr{L}$.

We will show in Steps 4–7 that either $\mathscr{L}_m$ is torsion, or X_m is smooth, the polar locus of ω_m is a smooth hypersurface D_m and $\mathscr{G}_m$ can be defined by a (closed) logarithmic 1-form with poles along $D_m \sqcup D'_m$ for some smooth hypersurface D'_m. Taking this for granted, we now show that the conclusion of Proposition 11.6 holds for X.

If $\mathscr{L}_m$ is torsion, then so is $\mathscr{L}$. Therefore, there exists a quasi-étale cover $f : X_1 \to X$ such that $f^*\mathscr{L} \cong \mathscr{O}_{X_1}$ as flat line bundles. Then, $f^{-1}\mathscr{G}$ is given by the closed rational 1-form $df(\omega)$ whose zero set has codimension at least 2.

Suppose now that X_m is smooth, that the polar locus of ω_m is a smooth hypersurface D_m, and that $\mathscr{G}_m$ can be defined by a closed rational 1-form α_m with a pole along $D_m \sqcup D'_m$. Let Z be a resolution of the indeterminacy locus of the rational map $X \dashrightarrow X_m$, and let $p\colon Z \to X$ and $q\colon Z \to X_m$ be the natural maps. By construction, we have $p^*\mathscr{L} \cong q^*\mathscr{L}_m$. By assumption, there exists an effective divisor D on X such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X(D) \otimes \mathscr{L}$. Moreover, we have $\mathscr{N}_{\mathscr{G}_m} \cong \mathscr{O}_{X_m}(D_m) \otimes \mathscr{L}_m$. Since $K_{\mathscr{G}} \equiv 0$ and $K_{\mathscr{G}_m} \equiv 0$, we have $p^*(K_X + D) \equiv q^*(K_{X_m} + D_m)$. By [63, Lemma 3.38], there exists an effective q-exceptional $\mathbb{Q}$-divisor E on Z such that $p^*K_X = q^*K_{X_m} + E$. Moreover, $\operatorname{Supp} E$ contains the strict transforms of the divisors contracted by the rational map $X \dashrightarrow X_m$. Thus, we obtain

$$q^*D_m \equiv p^*D + E.$$

On the other hand, D_m is the pushforward of D on X_m. This implies that $q^*D_m - p^*D$ is q-exceptional. By the negativity lemma, we must have

$$q^*D_m = p^*D + E.$$

It follows that $q(\operatorname{Supp} E) \subset \operatorname{Supp} D_m$. In particular, the 1-form $dq(\alpha_m)$ has poles along $\operatorname{Supp} E$. Since $\operatorname{Supp} E$ contains the strict transforms of the divisors contracted by the rational map $X \dashrightarrow X_m$, we conclude that the closed rational 1-form on X induced by α_m has zero set of codimension at least 2. So, in either case, the conclusion of Proposition 11.6 holds for X.

For simplicity of notation, we will assume in the following that $X = X_m$, writing $\psi := \psi_m$.

Step 4. By assumption, there exist prime divisors $(D_i)_{1\le i\le r}$ on X and positive integers $(m_i)_{1\le i\le r}$ such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X(\sum_{1\le i\le r} m_i D_i) \otimes \mathscr{L}$. Let $I \subseteq \{1,\dots,r\}$ be the set of indices $i \in \{1,\dots,r\}$ such that $\psi(D_i) = Y$. Note that $I \neq \emptyset$ since $\sum_{1\le i\le r} m_i D_i \equiv -K_X$ is relatively ample.

Suppose that there exists $i \in I$ such that the residue of ω at a general point of D_i is nonzero. Then arguing as in [66, Section 8.2.1], one shows that $\mathscr{L}$ is torsion.

Suppose from now on that the residue of ω at a general point of D_i is zero for any $i \in I$.

Step 5. Let F be a general fiber of ψ. Note that F has terminal singularities and that $K_F \sim_{\mathbb{Z}} K_{X|F}$ by the adjunction formula. Moreover, F is a Fano variety by construction. Let $\mathscr{H}$ be the foliation on F induced by $\mathscr{G}$. Note that $\mathscr{H}$ has codimension 1 by Step 1. By Proposition 3.6, we have $K_{\mathscr{H}} \sim_{\mathbb{Z}} K_{\mathscr{G}|F} - B$ for some effective Weil divisor B on F. Suppose that $B \neq 0$. Applying [22, Theorem 4.7] to the pullback of $\mathscr{H}$ on a resolution of F, we see that $\mathscr{H}$ is uniruled. This implies that $\mathscr{G}$ is uniruled as well since F is general. But this contradicts Proposition 4.22, and shows that $B = 0$. By Proposition 4.22 applied to $\mathscr{H}$, we see that $\mathscr{H}$ is canonical. Since F is simply connected by [48, Corollary 1.14], $\mathscr{H}$ is given by a closed rational 1-form (possibly with codimension 1 zeros) with zero residues at general points of its codimension 1 poles. Lemma 11.7 then implies that $F \cong \mathbb{P}^1$.

Now, we have $\sum_{1 \le i \le r} m_i D_i \cdot F = 2$. Suppose that $D_1 \cdot F = D_2 \cdot F = 1$ or that $D_1 \cdot F = 2$. Let $y \in Y_{\text{reg}}$ be a general point, and let $U \subseteq Y_{\text{reg}}$ be an analytically open neighborhood of y such that $\psi^{-1}(U) \cong U \times \mathbb{P}^1$. We may assume that there exists a coordinate z on $\mathbb{C} \subset \mathbb{P}^1$ such that the poles of $\omega_{|\psi^{-1}(U)}$ are given by equations $z = 0$ and $z = \infty$. Then

$$\omega_{|U \times \mathbb{C}} = a\frac{dz}{z} + \frac{1}{z}(\alpha + z\beta + z^2\gamma),$$

where a is a holomorphic function on U, and α, β, and γ are holomorphic 1-forms on U. Observe that $a(y) \neq 0$ since $\mathscr{G}$ is generically transverse to F. This implies that ω has nonzero residue along D_i for any $i \in I$, yielding a contradiction. Therefore, we must have $\sharp I = 1$ and renumbering the D_i, if necessary, we may assume that $m_1 = 2$, and $D_1 \cdot F = 1$.

Recall that X is smooth in codimension 2 since it has terminal singularities. Since $\dim X - \dim Y = 1$, there exists an open subset $Y^\circ \subseteq Y_{\text{reg}}$ with complement of codimension at least 2 such that $X^\circ := \psi^{-1}(Y^\circ) \subseteq X_{\text{reg}}$. From [3, Theorem 4.1], we conclude that $\psi^\circ := \psi_{|X^\circ}$ is a conic bundle. It follows that ψ° is smooth since ψ is an elementary Mori contraction and $D_1 \cdot F = 1$.

Finally, we show that $r = 1$. We argue by contradiction and assume that $r \ge 2$. Set $C_i := \psi(D_i)$ for $i \in \{2, \ldots, r\}$. Note that C_i is a divisor on Y by construction. Moreover, D_i is invariant under $\mathscr{G}$ by [24, Proposition III.1.1, Première partie]. By [37, Corollary 4.5] and [59, Lemma 5.1.5], Y has $\mathbb{Q}$-factorial klt singularities. One readily checks that there exist positive rational numbers λ_i for $i \in \{2, \ldots, r\}$ and an effective $\mathbb{Q}$-divisor C on Y such that $K_Y + \sum_{2 \le i \le r} \lambda_i C_i + C \equiv 0$ using the fact that $\operatorname{codim} Y \setminus Y^\circ \ge 2$ and the assumption $K_{\mathscr{G}} \equiv 0$. It follows that K_Y is not pseudoeffective. In particular, Y is uniruled by [17] applied to a resolution of Y.

Let $Y \dashrightarrow R$ be the maximal rationally chain connected fibration. Recall that it is an almost proper map and that its general fibers are rationally chain connected. We show that the canonical divisor of a general fiber is not pseudoeffective. We argue by contradiction. Consider a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{\beta,\ \text{birational}} & Y \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\gamma,\ \text{birational}} & R \end{array}$$

where Y_1 and R_1 are projective manifolds. Let Γ and E be effective β-exceptional divisors on Y_1 such that (Y_1, Γ) is klt and $K_{Y_1} + \Gamma = \beta^* K_Y + E$. We may assume without loss of generality that $\Gamma + E$ has simple normal crossing support. Notice that general fibers of $Y_1 \to R_1$ are rationally chain connected by [48, Theorem 1.2]. Applying [39], we see that R_1 is not uniruled. This in turn implies that K_{R_1} is pseudoeffective by [17] again. Using [21, Theorem 4.13], we conclude that $K_{Y_1} + \Gamma$ is pseudoeffective, yielding a contradiction since K_Y is not pseudoeffective in our case.

Let now G be a very general fiber of the composed map $X \to Y \dashrightarrow R$. Note that G is projective with terminal singularities. Moreover, it is rationally chain connected and $\dim G \geq 2$ by construction. Arguing as in the first paragraph of this step and using Lemma 11.8, we see that we must have $G \cong \mathbb{P}^1$, yielding a contradiction. This proves that $r = 1$.

Step 6. We now show that, shrinking Y° further, if necessary, $\mathscr{G}_{|X^\circ}$ yields a flat Ehresmann connection on ψ°. Let $C \cong \mathbb{P}^1$ be any fiber of ψ°. We have $c_1(\mathscr{N}_{\mathscr{G}}) \cdot C = 2$ and either C is tangent to $\mathscr{G}$, or C is everywhere transverse to $\mathscr{G}$. Thus, we have to show that C is not tangent to $\mathscr{G}$.

Set $y := \psi(C) \in Y^\circ$. By assumption, there is an analytically open neighborhood U of y in Y° such that $\mathscr{G}$ is defined over $\psi^{-1}(U)$ by a closed rational 1-form ω_U with poles along $D_1 \cap \psi^{-1}(U)$. Shrinking U, if necessary, we may assume that $\psi^{-1}(U) \cong U \times \mathbb{P}^1$ and that there exists a coordinate z on $\mathbb{C} \subset \mathbb{P}^1$ such that the pole of ω_U is given by equation $z = \infty$. Then

$$\omega_{U\,|U\times\mathbb{C}} = a\,dz + \alpha + z\beta + z^2\gamma,$$

where a is a holomorphic function on U, and α, β, and γ are holomorphic 1-forms on U. Since $d\omega_U = 0$ by assumption, we must have $d\alpha = 0$, $\beta = da$, and $\gamma = 0$. Shrinking U further, we may assume that $\alpha = db$ for some holomorphic function b on U, so that $\omega_{U\,|U\times\mathbb{A}^1} = d(az + b)$. Set $u := \frac{1}{z}$. Then $\mathscr{G}$ is given by $d(\frac{a}{u} + b)$ in a neighborhood of $z = \infty$. If $a(y) = 0$, then $\mathscr{G}$ is not canonical in a neighborhood of $C \cap \{u = 0\}$ (see [67, Observation I.2.6] and [33, Proposition 2.10]), yielding a contradiction. This shows that $a(y) \neq 0$, and hence C is not tangent to $\mathscr{G}$. This proves that $\mathscr{G}_{|X^\circ}$ defines a flat Ehresmann connection on ψ°.

Step 7. Recall that Y has $\mathbb{Q}$-factorial klt singularities (see Step 5). Since $\operatorname{codim} Y \setminus Y^\circ \geq 2$ and $K_{\mathscr{G}} \equiv 0$, we must have $K_Y \equiv 0$. Applying [69, Corollary V.4.9], we conclude that K_Y is torsion. Let $Y_1 \to Y$ be the index 1 canonical cover, which is quasi-étale (see [63, Definition 2.52]). By construction, $K_{Y_1} \sim_{\mathbb{Z}} 0$. In particular, Y_1 has canonical singularities. By Theorem 2.14 applied to Y_1, we see that there exist an abelian variety, a projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(Z) = 0$, and a quasi-étale cover $f\colon A \times Z \to Y$. Recall that f branches only on the singular set of Y, so that $f^{-1}(Y^\circ)$ is smooth. On the other hand, since $f^{-1}(Y^\circ)$ has complement of codimension at least 2 in $A \times Z_{\mathrm{reg}}$, we have $\pi_1(A \times Z_{\mathrm{reg}}) \cong \pi_1(f^{-1}(Y^\circ))$. Now, consider the representation

$$\rho \colon \pi_1(A \times Z_{\text{reg}}) \cong \pi_1\big(f^{-1}(Y^\circ)\big) \to \pi_1(Y^\circ) \to \mathrm{PGL}(2, \mathbb{C})$$

induced by $\mathscr{G}_{|X^\circ}$. By [40, Theorem I], the induced representation

$$\pi_1(Z_{\text{reg}}) \to \pi_1(A) \times \pi_1(Z_{\text{reg}}) \cong \pi_1(A \times Z_{\text{reg}}) \to \mathrm{PGL}(2, \mathbb{C})$$

has finite image. Thus, replacing Z with a quasi-étale cover, if necessary, we may assume without loss of generality that ρ factors through the projection $\pi_1(A \times Z_{\text{reg}}) \to \pi_1(A)$. Let P be the corresponding $\mathbb{P}^1$-bundle over A. The natural projection $\pi \colon P \to A$ comes with a flat Ehresmann connection $\mathscr{G}_P \subset T_P$. By the GAGA theorem, P is a projective variety. By assumption, its pullback to $A \times Z_{\text{reg}}$ agrees with $f^{-1}(Y^\circ) \times_{Y^\circ} X^\circ$ over $f^{-1}(Y^\circ)$. Moreover, the pullbacks on $A \times Z_{\text{reg}}$ of the foliations $\mathscr{G}$ and $\mathscr{G}_P$ agree as well, wherever this makes sense. Since there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X by assumption, we must have $\dim Z = 0$.

Set $A^\circ := f^{-1}(Y^\circ)$, $P^\circ := \pi^{-1}(A^\circ)$, and $\mathscr{G}_{P^\circ} := \mathscr{G}_{P|P^\circ}$. Let $g^\circ \colon P^\circ \to X^\circ$ denote the natural morphism, which is an étale cover. Set $D_P^\circ := (g^\circ)^{-1}(D_1 \cap X^\circ)$. Then $\mathscr{G}_{P^\circ}$ is given by the closed rational 1-form $\omega_{P^\circ} := dg^\circ(\omega_{|X^\circ})$ with values in the flat line bundle $\mathscr{L}_{P^\circ} := (g^\circ)^*(\mathscr{L}_{|X^\circ})$. Moreover, the zero set of ω_{P° has codimension at least 2. Since P is smooth and $P \setminus P^\circ$ has codimension at least 2, $\mathscr{L}_{P^\circ}$ is the restriction to P° of a flat line bundle $\mathscr{L}_P$ on P, and ω_P° extends to a closed rational 1-form ω_P with values in $\mathscr{L}_P$ whose zero set has codimension at least 2. Note that the divisor of zeros and poles of ω_P is $-2D_P$, where D_P denotes the Zariski closure of D_P°. By [24, Proposition III.1.1, Première partie], D_P is a leaf of $\mathscr{G}_P$. It follows that D_P is a section of π.

Let $H \subseteq \mathrm{PGL}(2, \mathbb{C})$ be the Zariski closure of the image of ρ. We use the notation introduced in Examples 9.1 and 11.2. Recall from Step 1 that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}_P$ passing through a general point of P. Therefore, H is conjugate to $p(T)$ or $p(U)$. Observe that $\mathscr{G}_P$ is not given by a (closed) holomorphic form since it is transverse to π by Step 6.

If $H = p(U)$, then $\mathscr{L}_P \cong \mathscr{O}_P$ as flat line bundles by Lemma 11.5 together with Example 11.2. This in turn implies that $\mathscr{L}$ is torsion since the image of $\pi_1(P^\circ)$ in $\pi_1(X^\circ)$ has finite index.

Suppose from now on that $H = p(T)$. Let

$$a_Y \colon Y \to \mathrm{A}(Y)$$

be the Albanese morphism. Since X is a projective variety with rational singularities, $\mathrm{Pic}^\circ(Y)$ is an abelian variety, and $\mathrm{A}(Y) \cong (\mathrm{Pic}^\circ(Y))^\vee$. Moreover, the Albanese morphism is induced by the universal line bundle (see [58, Lemma 8.1]). In particular, there is a flat line bundle $\mathscr{L}_{\mathrm{A}(Y)}$ on $\mathrm{A}(Y)$ and a positive integer m such that $\mathscr{L}^{\otimes m} \cong \psi^*({a_Y}^*\mathscr{L}_{\mathrm{A}(Y)}^{\otimes m})$. This in turn implies that there exists a degree m étale cover $Y_1 \to Y$ such that the pullbacks of $\mathscr{L}$ and $\mathscr{L}_{\mathrm{A}(Y)}$ on Y_1 agree. Set $P_{\mathrm{A}(Y)} := \mathbb{P}_{\mathrm{A}(Y)}(\mathscr{O}_{\mathrm{A}(Y)} \oplus \mathscr{L}_{\mathrm{A}(Y)})$ and $A_1 := A \times_Y Y_1$. Lemma 11.5 together with Example 11.2 then implies that $A_1 \times_A P \cong A_1 \times_{\mathrm{A}(Y)} P_{\mathrm{A}(Y)}$ and that the pullback of $\mathscr{G}_P$ on $A_1 \times_A P$ is also the pullback of a foliation on $P_{\mathrm{A}(Y)}$. We conclude that $a_Y : Y \to \mathrm{A}(Y)$ is generically finite since there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}_P$ passing through a general point of P. Recall from [69, Corollary V.4.9] that $\kappa(Y) = 0$. Applying [57, Theorem 13] to the Stein factorization of $a_Y : Y \to \mathrm{A}(Y)$, we see that a_Y is an isomorphism. In particular, we can choose $A = Y$ above. Then the restriction of the tangent map $T\psi : T_X \to \psi^*T_A$ to $\mathscr{G}$ gives an isomorphism $\mathscr{G} \cong \psi^*T_A$, so that $\mathscr{G}$ induces a flat connection on ψ. Now, a classical result of complex analysis says that complex flows of vector fields on analytic spaces exist (see [56]). It follows that ψ is a locally trivial analytic fibration for the analytic topology. This shows that $X \cong P$ and that $\mathscr{G}$ identifies with $\mathscr{G}_P$. In particular, $\mathscr{G}$ is defined by a closed logarithmic 1-form, completing the proof of the proposition. ☐

LEMMA 11.7

Let X be a normal complex projective variety with klt singularities. Let ω be a closed rational 1-form on X, and let $\mathscr{G}$ be the foliation defined by ω. Suppose that $\mathscr{G}$ has canonical singularities and $K_{\mathscr{G}} \equiv 0$. Suppose in addition that the residues of ω at general points of its codimension 1 poles are zero. If X is Fano, then $X \cong \mathbb{P}^1$.

Proof

By Theorem 11.3 and using the assumption that X is Fano, we may assume that there exist a normal projective equivariant compactification Z of a commutative linear algebraic group G and a quasi-étale cover $f : Z \to X$ such that $f^{-1}\mathscr{G}$ is induced by a codimension 1 Lie subgroup H of G. Moreover, there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X.

Recall that $G \cong (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$ for some nonnegative integers r and s. If $s \geq 2$, then $\dim \mathrm{Lie}\, H \cap \mathrm{Lie}(\mathbb{G}_a)^s \geq 1$, and hence $\mathscr{G}$ is uniruled. But this contradicts Proposition 4.22, and shows that $s \leq 1$. By [10, Theorem 2], X is a toric variety. It follows that X_{reg} has finite fundamental group by [72, Proposition 1.9].

Since ω is closed and since the residues of ω at general points of its codimension 1 poles are zero, we conclude that there is a meromorphic function f on X_{reg} such that $\omega = df$ using [24, Théorème III.2.1, Première partie]. By the Levi extension theorem, f extends to a meromorphic function on X, and thus, f is a rational function. This in turn implies that $\mathscr{G}$ is algebraically integrable. It follows that $\dim X = 1$, since there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X. This completes the proof of the lemma. ☐

LEMMA 11.8

Let X be a normal complex projective variety with klt singularities. Suppose that there exists a surjective morphism $\varphi\colon X \to B$ onto a normal rationally chain connected projective variety B with klt singularities whose fibers over an open set with complement of codimension at least 2 in Y are isomorphic to $\mathbb{P}^1$. Suppose furthermore that $K_B \not\equiv 0$. Let ω be a closed rational 1-form on X, and let $\mathscr{G}$ be the foliation defined by ω. Suppose that $\mathscr{G}$ has canonical singularities and $K_{\mathscr{G}} \equiv 0$. Suppose in addition that the residues of ω at general points of its codimension 1 poles are zero and that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X. Then $X \cong \mathbb{P}^1$.

Proof

By Theorem 11.3, we may assume that there exists a normal projective equivariant compactification Z of a commutative algebraic group G, as well as a quasi-étale cover $f : Z \to X$, such that $f^{-1}\mathscr{G}$ is induced by a codimension 1 Lie subgroup H of G. Notice that there is no positive-dimensional algebraic subvariety tangent to $f^{-1}\mathscr{G}$ passing through a general point of Z. By Chevalley's structure theorem on algebraic groups, there is an exact sequence of algebraic groups

$$1 \to G_{\mathrm{aff}} \to G \to A \to 1,$$

where A is an abelian variety and G_{aff} is a connected affine algebraic group. Using [48, Theorem 1.2] together with the rigidity lemma, we see that the morphism $G \to A$ extends to a morphism $\psi : Z \to A$.

Let F be a general fiber of ψ. Note that F has terminal singularities. Let $\mathscr{H}$ be the codimension 1 foliation on F induced by $f^{-1}\mathscr{G}$. By Proposition 3.6, we have $K_{\mathscr{H}} \equiv -B$ for some effective Weil divisor B on F. Suppose that $B \neq 0$. Applying [22, Theorem 4.7] to the pullback of $\mathscr{H}$ on a resolution of F, we see that $\mathscr{H}$ is uniruled. This implies that $f^{-1}\mathscr{G}$ is uniruled as well since F is general. But this contradicts Proposition 4.22, and shows that $B = 0$. By Proposition 4.22 applied to $\mathscr{H}$, we see that $\mathscr{H}$ is canonical. Arguing as in the proof of Lemma 11.7 above and using the fact that there is no positive-dimensional algebraic subvariety tangent to $f^{-1}\mathscr{G}$ passing through a general point of Z, we conclude that $\dim F = 1$ and that $\mathscr{H}$ is the foliation by points. Now, we may assume without loss of generality that $f^{-1}\mathscr{G} \cong \mathscr{O}_Z^{\oplus \dim Z - 1}$ (see Theorem 11.3). This immediately implies that $f^{-1}\mathscr{G}$ yields a flat Ehresmann connection ψ. In particular, ψ is a $\mathbb{P}^1$-bundle.

Let $Z \xrightarrow{\psi_1} B_1 \xrightarrow{g} B$ be the Stein factorization of the composed map $Z \to X \to B$. Observe that a general fiber F_1 of φ is contained in the smooth locus of X so that f is étale in a neighborhood of it. It follows that $f^{-1}(F_1)$ is the union of rational curves and hence it is contracted by ψ. Using the rigidity lemma, we then see there exists an isomorphism $g_1\colon A \to B_1$ such that $g_1 \circ \psi = \psi_1$. On the other hand, since $\varphi \circ f$ has reduced fibers over an open set with complement of codimension at least 2 in Y by assumption, we conclude that g is quasi-étale. In particular, we must have $K_B \sim_{\mathbb{Q}} 0$. This yields a contradiction since $K_B \not\equiv 0$ by assumption, completing the proof of the lemma. □

PROPOSITION 11.9

Let X be a normal complex projective variety with canonical singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X. Suppose that $\mathscr{G}$ is canonical with $K_{\mathscr{G}}$ Cartier and $K_{\mathscr{G}} \equiv 0$ and that K_X is not pseudoeffective. Suppose in addition that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ and whose zero set has codimension at least 2. Then there exists a quasi-étale cover $f: X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form with zero set of codimension at least 2.

Proof

By Proposition 8.13, there exist normal projective varieties Y and Z, a foliation $\mathscr{H}$ on Y such that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{H}$ passing through a general point of Y, and a quasi-étale cover $f: Y \times Z \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of $\mathscr{H}$ via the projection $Y \times Z \to Y$. Moreover, $K_Z \sim_{\mathbb{Z}} 0$, $\mathscr{H}$ has canonical singularities and $K_{\mathscr{H}} \equiv 0$. Note that K_Y is not pseudoeffective.

If $\dim Y = 1$, then $Y \cong \mathbb{P}^1$, and $\mathscr{G}$ is given by a (closed) logarithmic 1-form. Suppose from now on that $\dim Y \geq 2$.

Let $F \cong Y$ be a general fiber of the projection $Y \times Z \to Z$. Then $(f^{-1}\mathscr{G})_{|F} \cap T_F \cong \mathscr{H}$, and hence $K_{\mathscr{H}}$ is Cartier and the restriction of $df(\omega)$ to F is a closed rational 1-form with values in $\mathscr{L}_{|F}$ defining $\mathscr{H}$ whose zero set has codimension at least 2. Its pullback to $Y \times Z$ is a closed rational 1-form with values in the pullback of $\mathscr{L}_{|F}$ defining $\mathscr{G}$ whose zero set has codimension at least 2. Finally, one checks that Y has canonical singularities using Fact 2.10. Therefore, replacing $\mathscr{G}$ by $\mathscr{H}$, if necessary, we may assume that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X, and that $\dim X \geq 2$.

Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorial terminalization of X. By Proposition 4.10, $\beta^{-1}\mathscr{G}$ is canonical with $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$. Suppose first that $\mathscr{G}$ is closed under pth powers for almost all primes p. Then $\mathscr{G}$ is algebraically integrable by Corollary 9.5, yielding a contradiction. Thus, by Proposition 12.3 and Lemma 8.14, $\beta^{-1}\mathscr{G}$ is given by a closed rational 1-form α with values in a flat line bundle $\mathscr{M}$ whose zero set has codimension at least 2. Finally, K_Z is not pseudoeffective since $\beta_* K_Z \sim_{\mathbb{Z}} K_X$ and K_X is not pseudoeffective by assumption. By Proposition 11.6, we may therefore assume without loss of generality that $\mathscr{M}$ is torsion. Applying [79], we see that the natural map of topological fundamental groups $\pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Now, since $\mathscr{M}$ is flat, it is induced by a representation $\pi_1(Z) \to \mathbb{C}^*$. The latter then yields a (torsion) flat bundle $\mathscr{L}$ on X such that $\mathscr{M} \cong \beta^*\mathscr{L}$. Moreover, α induces a closed rational 1-form ω on X with values in $\mathscr{L}$ whose zero set has codimension at least 2. By construction, ω defines $\mathscr{G}$. This completes the proof of the proposition. □

Before proving Theorem 11.1 below, we address foliations defined by closed rational 1-forms with values in flat line bundles on projective varieties with pseudo-effective canonical class.

PROPOSITION 11.10
Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with canonical singularities and $K_{\mathscr{G}} \equiv 0$. Suppose that K_X is pseudoeffective, and that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ whose zero set has codimension at least 2. Then there exist an abelian variety A, a normal projective variety Z with $K_Z \sim_{\mathbb{Z}} 0$ and $\widetilde{q}(Z) = 0$, and a quasi-étale cover $f : A \times Z \to X$ such that $f^{-1}\mathscr{G}$ is the pullback of a codimension 1 linear foliation on A via the projection $A \times Z \to A$.

Proof
There exists an effective divisor D on X such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X(D) \otimes \mathscr{L}$. On the other hand, $c_1(\mathscr{N}_{\mathscr{G}}) \equiv -K_X$ since $K_{\mathscr{G}} \equiv 0$, and hence $-K_X \equiv D$. It follows that $K_X \equiv 0$ since K_X is pseudoeffective by assumption. Proposition 11.10 then follows from Lemma 10.1. □

Proof of Theorem 11.1
We maintain the notation and assumptions of Theorem 11.1. By Proposition 8.13, we may assume without loss of generality that there is no positive-dimensional algebraic subvariety tangent to $\mathscr{G}$ passing through a general point of X (see Step 1 of the proof of Proposition 11.6).

If K_X is pseudoeffective, then the statement follows easily from Proposition 11.10.

Suppose that K_X is not pseudoeffective. By Proposition 11.9, there exists a quasi-étale cover $f : X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form. Note that X_1 is canonical by Fact 2.10. By Lemma 4.3, $f^{-1}\mathscr{G}$ is canonical, and we obviously have $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} f^*K_{\mathscr{G}} \equiv 0$. Theorem 11.1 then follows from Theorem 11.3. □

Finally, we prove abundance in the setting of Proposition 11.6.

PROPOSITION 11.11

Let X be a normal complex projective variety with terminal singularities, and let $\mathscr{G} \subset T_X$ be a codimension 1 foliation with canonical singularities. Suppose that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle $\mathscr{L}$ whose zero set has codimension at least 2. Suppose furthermore that K_X is not pseudoeffective and that $K_{\mathscr{G}} \equiv 0$. Then $K_{\mathscr{G}}$ is torsion.

Proof

By Proposition 11.6, there exists a quasi-étale cover $f\colon X_1 \to X$ such that $f^{-1}\mathscr{G}$ is given by a closed rational 1-form. Note that X_1 is terminal by Fact 2.10. By Lemma 4.3, $f^{-1}\mathscr{G}$ is canonical, and we obviously have $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} f^*K_{\mathscr{G}} \equiv 0$. Proposition 11.11 then follows from Theorem 11.3. □

12. Proofs of Theorems 1.1 and 1.3 and proof of Corollary 1.4

The present section is devoted to the proofs of Theorems 1.1 and 1.3 and to the proof of Corollary 1.4. Lemma 12.1 and Proposition 12.3 below extend [66, Theorem 7.5] to the singular setting.

LEMMA 12.1

Let X be a normal projective variety over some algebraically closed field k of positive characteristic p with $\dim X \geq 2$, and let $\mathscr{G} \subset T_X$ be a codimension 1 foliation on X. Let ω be a rational 1-form ω defining $\mathscr{G}$, and let B denote the reduced effective divisor whose support is the union of codimension 1 zeros and poles of ω. If $\mathscr{G}$ is not closed under pth powers, then the following hold.

(1) *There exist a reduced effective Weil divisor D on X that does not contain any irreducible component of B in its support, and $\alpha \in H^0(X, \Omega_X^{[1]}(\log(B+D)))$ with $d\alpha = 0$ such that $d\omega = \alpha \wedge \omega$. If C is any irreducible component of $\operatorname{Supp}(B+D)$, then the residue of α at a general point of C is a constant function with values in the prime field $\mathbb{F}_p \subset k$.*

(2) *Suppose that X is smooth, and let A be a nef divisor such that $T_X(A)$ is generated by global sections. Suppose furthermore that $\mathscr{G}$ is semistable with respect to a nef and big divisor H on X and that $\mu_H(\mathscr{G}) \geq 0$. Then we have*

$$D \cdot H^{\dim X - 1} \leq c_1(\mathscr{N}_{\mathscr{G}}) \cdot H^{\dim X - 1} + (\dim X - 2) A \cdot H^{\dim X - 1} =: M.$$

Let C be an irreducible component of $\operatorname{Supp}(B+D)$ such that $C \cdot H^{\dim X - 1} \neq 0$, and let $m \in \mathbb{Z}$ be the vanishing order of ω along C. Then we have $\operatorname{res}_C \alpha \in \{m, \ldots, m+M\} \subseteq \mathbb{F}_p$.

Proof

Let X° denote an open set of X with complement of codimension at least 2, contained in the regular loci of X and $\mathscr{G}$. Let $(U_i)_{i\in I}$ be a finite covering of X° by open affine subsets, and let ω_i be a regular 1-form on U_i with zero set of codimension at least 2 such that $\omega_i \wedge \omega = 0$. Write $\omega_i = g_i\omega$ for some rational function g_i on X, and set $g_{ij} := \frac{g_i}{g_j}$. We have $\omega_i = g_{ij}\omega_j$, and hence g_{ij} is a nowhere vanishing regular function on $U_i \cap U_j$.

Set $\mathscr{G}^\circ := \mathscr{G}_{|X^\circ}$, and consider the nonzero map $\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}^\circ \to T_{X^\circ}/\mathscr{G}^\circ = \mathscr{N}_{\mathscr{G}|X^\circ} =: \mathscr{N}^\circ_{\mathscr{G}}$ induced by the pth power operation. Note that $\mathscr{N}^\circ_{\mathscr{G}}$ is a line bundle. Shrinking X°, if necessary, we may assume that there is an effective divisor D° on X° such that the above map induces a surjective morphism

$$\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}^\circ \twoheadrightarrow \mathscr{N}^\circ_{\mathscr{G}}(-D^\circ).$$

We may also assume without loss of generality that there exists a regular vector field ∂_i on U_i such that $f_i := \omega_i(\partial_i^p)$ is a defining equation of D° on U_i. By [66, Proposition 7.3], $\frac{1}{f_i}\omega_i$ is a closed rational 1-form, and hence $d\omega_i = \frac{df_i}{f_i} \wedge \omega_i$. On the other hand, by [66, Corollary 7.4], we must have

$$\frac{df_i}{f_i} - \frac{df_j}{f_j} = \frac{dg_{ij}}{dg_{ij}},$$

and hence

$$\frac{df_i}{f_i} - \frac{dg_i}{g_i} = \frac{df_j}{f_j} - \frac{dg_j}{g_j}.$$

This immediately implies that there exists $\alpha \in H^0(X, \Omega_X^{[1]}(\log(B+D)_{\mathrm{red}})$ with $d\alpha = 0$ such that α restricts to $-\frac{dg_i}{g_i} + \frac{df_i}{f_i}$ on U_i, where D denotes the Weil divisor on X such that $D_{|X^\circ} = D^\circ$. A straightforward computation then shows that

$$d\omega = \alpha \wedge \omega.$$

This proves (1).

To prove (2), observe that we must have

$$\begin{aligned}\mu_{\min}(\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}) &\leq \mu_H\big(\mathscr{N}_{\mathscr{G}}(-D)\big) = \mu_H(\mathscr{N}_{\mathscr{G}}) - \mu_H\big(\mathscr{O}_X(D)\big)\\ &\leq \mu_H(\mathscr{N}_{\mathscr{G}}) - \mu_H\big(\mathscr{O}_X(D_{\mathrm{red}})\big).\end{aligned}$$

On the other hand, by the proof of [64, Corollary 2.5]), we have

$$\mu_{\max}(\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}) - \mu_{\min}(\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}) \leq (\operatorname{rank}\mathscr{G} - 1)A \cdot H^{\dim X - 1}.$$

Now, we must have $\mu_{\max}(\mathrm{F}^*_{\mathrm{abs}}\mathscr{G}) \geq 0$ since $\mu_H(\mathscr{G}) \geq 0$ by assumption. The claim then follows easily. □

Remark 12.2
The notation is as in the proof of Lemma 12.1. By [66, Corollary 7.4], there is a rational function h_{ij} on X such that $f_i = g_{ij} f_j h_{ij}^p$. Note that h_{ij} is regular on $U_i \cap U_j$ since both f_i and f_j are local equations of D° on $U_i \cap U_j$, and that the h_{ij} automatically satisfy the cocycle condition. Therefore, there exists a rank 1 reflexive sheaf $\mathscr{L}$ on X as well as an effective Weil divisor such that $\mathscr{N}_{\mathscr{G}} \cong \mathscr{O}_X(D) \boxtimes \mathscr{L}^{\otimes p}$.

PROPOSITION 12.3
Let X be a normal complex projective variety, and let $\mathscr{G}$ be a codimension 1 foliation on X. Let $\beta\colon Z \to X$ be a resolution of singularities. Suppose that $\mathscr{G}$ is semistable with respect to some ample divisor H on X with $\mu_H(\mathscr{G}) \geq 0$. Then either $\mathscr{G}$ is closed under pth powers for almost all primes p, or $\beta^{-1}\mathscr{G}$ is given by a closed rational 1-form with values in a flat line bundle whose codimension 1 zeros are β-exceptional.

Remark 12.4
In the setup of Proposition 12.3, suppose in addition that X has klt singularities. Suppose that $\beta^{-1}\mathscr{G}$ is given by a closed rational 1-form ω_Z with values in a flat line bundle $\mathscr{L}_Z$ whose zero set has codimension at least 2. Applying [79], we see that there exists a flat line bundle $\mathscr{L}_X$ on X such that $\mathscr{L}_Z \cong \beta^*\mathscr{L}_X$. Then ω_Z induces a closed rational 1-form ω_X on X with values in $\mathscr{L}_X$ defining $\mathscr{G}$. Moreover, its zero set has codimension at least 2.

Proof of Proposition 12.3
Proposition 12.3 follows from Lemma 12.1 using the spreading out technique, which we recall now. Assume that $\dim X \geq 2$.

Let ω be a rational 1-form defining $\beta^{-1}\mathscr{G}$. Let E be an effective β-exceptional divisor such that $A := m_0\beta^*H - E$ is ample for some positive integer m_0, and let B be a reduced effective divisor that contains codimension 1 zeros and poles of ω and all β-exceptional divisors in its support. Replacing H by m_0H, we may assume that $m_0 = 1$. Let m be a positive integer such that $T_Z(mA)$ is generated by global sections, and set

$$M := c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) \cdot (\beta^*H)^{\dim X - 1} + m(\dim X - 2)A \cdot (\beta^*H)^{\dim X - 1}.$$

Let $R \subset \mathbb{C}$ be a finitely generated $\mathbb{Z}$-algebra, and let $\mathbf{X}$ (resp., $\mathbf{Z}$) be a projective (resp., smooth projective) model of X (resp., Z) over $\mathbf{S} := \operatorname{Spec} R$. Let $\boldsymbol{\beta}\colon \mathbf{Z} \to \mathbf{X}$ be a projective birational morphism such that $\boldsymbol{\beta}_{\mathbb{C}}$ coincides with β. Let $\boldsymbol{\mathscr{G}}$ (resp., $\boldsymbol{\beta}^{-1}\boldsymbol{\mathscr{G}}$) be a saturated subsheaf of the relative tangent sheaf $T_{\mathbf{X}/\mathbf{S}}$ (resp., $T_{\mathbf{Z}/\mathbf{S}}$), flat over $\mathbf{S}$, such that $\boldsymbol{\mathscr{G}}_{\mathbb{C}}$ (resp., $\boldsymbol{\beta}^{-1}\boldsymbol{\mathscr{G}}_{\mathbb{C}}$) coincides with $\mathscr{G}$ (resp., $\beta^{-1}\mathscr{G}$). We may assume that for any closed point $s \in \mathbf{S}$, $\mathbf{X}_{\bar{s}}$ is normal, and that $\boldsymbol{\mathscr{G}}_{\bar{s}}$ (resp., $\boldsymbol{\beta}^{-1}\boldsymbol{\mathscr{G}}_{\bar{s}}$) is a foliation on $\mathbf{X}_{\bar{s}}$ (resp., $\mathbf{Z}_{\bar{s}}$). Let $\mathbf{H}$ (resp., $\mathbf{A}$) be an ample Cartier divisor on $\mathbf{X}$ (resp., $\mathbf{Z}$) such that $\mathbf{H}_{\mathbb{C}} \sim_{\mathbb{Z}} H$ (resp., $\mathbf{A}_{\mathbb{C}} \sim_{\mathbb{Z}} A$). Suppose that $\mathbf{A} = \boldsymbol{\beta}^* H - \mathbf{E}$ for some effective $\boldsymbol{\beta}$-exceptional relative divisor $\mathbf{E}$ over $\mathbf{S}$, and that $E_{\bar{s}}$ is $\boldsymbol{\beta}_{\bar{s}}$-exceptional for any geometric point $s \in \mathbf{S}$. Finally, let $\mathbf{B}$ be a reduced effective relative divisor on $\mathbf{Z}$ over $\mathbf{S}$, and let $\boldsymbol{\omega}$ be rational section of $\Omega^1_{\mathbf{Z}/\mathbf{T}}$ such that $\mathbf{B}_{\mathbb{C}} = B$ and $\boldsymbol{\omega}_{\mathbb{C}} = \omega$. Shrinking $\mathbf{S}$, if necessary, we may assume that $\boldsymbol{\omega}_{\bar{s}}$ defines $\boldsymbol{\beta}^{-1}\boldsymbol{\mathscr{G}}_{\bar{s}}$ for any geometric point $s \in \mathbf{S}$, and that $\mathbf{B}_{\bar{s}}$ contains the codimension 1 zeros and poles of $\boldsymbol{\omega}_{\bar{s}}$. We can also choose $\mathbf{B}$ so that $\mathbf{B}$ contains all $\boldsymbol{\beta}$-exceptional prime divisors in its support.

Since semistability with respect to an ample divisor is an open condition in flat families of sheaves (see the proof of [52, Proposition 2.3.1]), we may assume that the sheaves $\mathscr{G}_{\bar{s}}$ are semistable with respect to $\mathbf{H}_{\bar{s}}$, with slopes $\mu_{\mathbf{H}_{\bar{s}}}(\mathscr{G}_{\bar{s}}) = \mu_H(\mathscr{G}) \geq 0$. It follows that $\boldsymbol{\beta}_{\bar{s}}^{-1}\boldsymbol{\mathscr{G}}_{\bar{s}}$ is semistable with respect to $\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}}$ with slope $\mu_{\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}}}(\boldsymbol{\beta}^{-1}\boldsymbol{\mathscr{G}}_{\bar{s}}) = 0$.

By [45, Lemme 2.4] and [44, Proposition 4.1], there exists a quasiprojective $\mathbf{S}$-scheme $\mathrm{Div}^{\leq M}_{\mathbf{Z}/\mathbf{S}}$ parameterizing effective relative Cartier divisors on $\mathbf{Z}$ over $\mathbf{S}$ with $\mathbf{A}$-degree at most M. Using [47, Théorème 12.2.1], we see that there is an open set $\mathbf{T} \subset \mathrm{Div}^{\leq M}_{\mathbf{Z}/\mathbf{S}}$ parameterizing geometrically reduced divisors that do not contain any irreducible component of some $\mathbf{B}_{\bar{s}}$ in their supports. Replacing $\mathbf{T}$ by $\mathbf{T}_{\mathrm{red}}$, if necessary, we may assume that $\mathbf{T}$ is reduced. By generic flatness, we can suppose that $\mathbf{T}$ is flat over $\mathbf{S}$. Let $\mathbf{D} \subset \mathbf{T} \times_{\mathbf{S}} \mathbf{Z}$ be the universal effective relative Cartier divisor, and denote by $\boldsymbol{\pi}\colon \mathbf{T} \times_{\mathbf{S}} \mathbf{Z} \to \mathbf{Z}$ and $\boldsymbol{\upsilon}\colon \mathbf{T} \times_{\mathbf{S}} \mathbf{Z} \to \mathbf{T}$ the projections. Set $\mathbf{C} := \boldsymbol{\pi}^*\mathbf{B}$. Observe that $\mathbf{C}$ is a reduced relative effective Cartier divisor on $\mathbf{T} \times_{\mathbf{S}} \mathbf{Z}$ over $\mathbf{T}$. Write $\Omega^{[1]}_{\mathbf{T}\times_{\mathbf{S}}\mathbf{Z}/\mathbf{T}}(\log(\mathbf{C} + \mathbf{D}))$ for the reflexive sheaf on $\mathbf{T} \times_{\mathbf{S}} \mathbf{Z}$ whose restriction to the open set $\mathbf{Z}^\circ_{\mathbf{T}}$ where $\mathbf{C} + \mathbf{D}$ has relative simple normal crossings over $\mathbf{T}$ is $\Omega^1_{\mathbf{Z}^\circ_{\mathbf{T}}/\mathbf{T}}(\log(\mathbf{C}_{|\mathbf{Z}^\circ_{\mathbf{T}}} + \mathbf{D}_{|\mathbf{Z}^\circ_{\mathbf{T}}}))$, and set

$$\mathbf{U} := \boldsymbol{\upsilon}_*\Omega^{[1]}_{\mathbf{T}\times_{\mathbf{S}}\mathbf{Z}/\mathbf{T}}\big(\log(\mathbf{C} + \mathbf{D})\big).$$

By generic flatness and the base change theorem, we see that, replacing $\mathbf{T}$ with a finite disjoint union of locally closed subsets, we may assume without loss of generality that $\mathbf{U}$ is flat over $\mathbf{T}$, and that the formation of $\boldsymbol{\upsilon}_*\Omega^{[1]}_{\mathbf{T}\times_{\mathbf{S}}\mathbf{Z}/\mathbf{T}}(\log(\mathbf{C} + \mathbf{D}))$ commutes with arbitrary base change. We will also assume that, for any geometric point $\bar{t} \in \mathbf{T}$, the restriction of $\Omega^{[1]}_{\mathbf{T}\times_{\mathbf{S}}\mathbf{Z}/\mathbf{T}}(\log(\mathbf{C} + \mathbf{D}))$ to the fiber of the projection $\mathbf{T} \times_{\mathbf{S}} \mathbf{Z} \to \mathbf{T}$ over $\bar{t}$ is reflexive, so that

$$\Omega^{[1]}_{\mathbf{T}\times_{\mathbf{S}}\mathbf{Z}/\mathbf{T}}\big(\log(\mathbf{C} + \mathbf{D})\big)_{|\mathbf{Z}_{\bar{s}}} \cong \Omega^{[1]}_{\mathbf{Z}_{\bar{s}}}\big(\log(\mathbf{B}_{\bar{s}} + \mathbf{D}_{\bar{t}})\big),$$

where $\bar{s} \in \mathbf{S}$ is the image of $\bar{t}$ in $\mathbf{S}$. By [47, Corollaire 9.7.9], there exist a finite set $I \subset \mathbb{N}$ and a decomposition

$$\mathbf{T} = \bigsqcup_{i \in I} \mathbf{T}_i$$

of T into locally closed subsets such that any geometric fiber of $\mathbf{D}_i := \mathbf{D} \times_{\mathbf{T}} \mathbf{T}_i \to \mathbf{T}_i$ has i irreducible components, and such that any geometric fiber of $\mathbf{C}_i := \mathbf{C} \times_{\mathbf{T}} \mathbf{T}_i \to \mathbf{T}_i$ has $n(i)$ irreducible components. Let $\mathbf{D}_i^\circ$ (resp., $\mathbf{C}_i^\circ$) denote the open set where $\mathbf{D}_i \to \mathbf{T}_i$ (resp., $\mathbf{C}_i \to \mathbf{T}_i$) is smooth. We can choose $\mathbf{T}_i$ so that there exist sections $a_{i,1}, \dots, a_{i,i}$ (resp., $b_{i,1}, \dots, b_{i,n(i)}$) of $\mathbf{D}_i^\circ \to \mathbf{T}_i$ (resp., $\mathbf{C}_i^\circ \to \mathbf{T}_i$) with $a_{i,j}(\bar{t}) \in \mathbf{D}^\circ_{\bar{t},i,j}$ (resp., $b_{i,j}(\bar{t}) \in \mathbf{C}^\circ_{\bar{t},i,j}$), where the $\mathbf{D}^\circ_{\bar{t},i,1}, \dots, \mathbf{D}^\circ_{\bar{t},i,i}$ (resp., $\mathbf{C}^\circ_{\bar{t},i,1}, \dots, \mathbf{C}^\circ_{\bar{t},i,n(i)}$) are the irreducible components of the corresponding fiber of $\mathbf{D}_i^\circ \to \mathbf{T}_i$ (resp., $\mathbf{C}_i^\circ \to \mathbf{T}_i$).

Let $\bar{t} \in T$ be a geometric point, and denote by $\bar{s}$ the image of $\bar{t}$ in $\mathbf{S}$. Let $l_{\bar{t},i,j}$ (resp., $m_{\bar{t},i,j}$) be the vanishing order of $\boldsymbol{\omega}_{\bar{s}}$ along $\mathbf{D}^\circ_{\bar{t},i,j}$ (resp., $\mathbf{C}^\circ_{\bar{t},i,j}$).

Let $\mathbf{P} \subseteq \mathbf{U}$ be the closed subset defined by the conditions

(1) $d\alpha_{\bar{t}} = 0$,

(2) $d\boldsymbol{\omega}_{\bar{s}} = \alpha_{\bar{t}} \wedge \boldsymbol{\omega}_{\bar{s}}$,

(3) $\operatorname{res}_{\mathbf{D}^\circ_{\bar{t},i,j}} \alpha_{\bar{t}}(a_{i,j}(\bar{t})) \in \{l_{\bar{t},i,j}, \dots, l_{\bar{t},i,j} + M\} \subset k(\bar{t})$ for all indices $1 \le j \le i$, and

(4) $\operatorname{res}_{\mathbf{C}^\circ_{\bar{t},i,j}} \alpha_{\bar{t}}(b_{i,j}(\bar{t})) \in \{m_{\bar{t},i,j}, \dots, m_{\bar{t},i,j} + M\} \subset k(\bar{t})$ for all indices $1 \le j \le n(i)$ such that $\mathbf{C}^\circ_{\bar{t},i,j}$ is not $\boldsymbol{\beta}_{\bar{s}}$-exceptional,

where $\bar{t} \in \mathbf{T}_i$ and $\alpha_{\bar{t}} \in H^0(\mathbf{Z}_{\bar{s}}, \Omega^{[1]}_{\mathbf{Z}_{\bar{s}}}(\log(\mathbf{B}_{\bar{s}} + \mathbf{D}_{\bar{t}})))$.

Fix a closed point s in $\mathbf{S}$, and denote by $p > 0$ the characteristic of $k(\bar{s})$. Suppose that $\mathscr{G}_{\bar{s}}$ is not closed under pth powers. This immediately implies that $\boldsymbol{\beta}^{-1}\mathscr{G}_{\bar{s}}$ is not closed under pth powers as well. Applying Lemma 12.1 to $\boldsymbol{\beta}^{-1}\mathscr{G}_{\bar{s}}$ and $\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}}$, we conclude that there exist a reduced effective Cartier divisor $\mathbf{D}(\bar{s})$ on $\mathbf{Z}_{\bar{s}}$ that does not contain any irreducible component of $\mathbf{B}_{\bar{s}}$ in its support, and

$$\alpha_{\bar{t}} \in H^0\big(\mathbf{Z}_{\bar{s}}, \Omega^{[1]}_{\mathbf{Z}_{\bar{s}}}\big(\log\big(\mathbf{B}_{\bar{s}} + \mathbf{D}(\bar{s})\big)\big)\big)$$

with $d\alpha_{\bar{t}} = 0$ such that $d\boldsymbol{\omega}_{\bar{s}} = \alpha_{\bar{t}} \wedge \boldsymbol{\omega}_{\bar{s}}$. Moreover, the functions $\operatorname{res}_{\mathbf{D}^\circ_{\bar{t},i,j}} \alpha_{\bar{t}}$ and $\operatorname{res}_{\mathbf{C}^\circ_{\bar{t},i,j}} \alpha_{\bar{t}}$ are constant, with values in $\{l_{\bar{t},i,j}, \dots, l_{\bar{t},i,j} + M\} \subset k(\bar{s})$ and $\{m_{\bar{t},i,j}, \dots, m_{\bar{t},i,j} + M\} \subset k(\bar{s})$, respectively, if $\mathbf{C}^\circ_{\bar{t},i,j}$ is not $\boldsymbol{\beta}_{\bar{s}}$-exceptional. We also have $\mathbf{D}(\bar{s}) \cdot (\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}})^{\dim \mathbf{Z}_{\bar{s}} - 1} \le M$. Notice that there is no $\boldsymbol{\beta}_{\bar{s}}$-exceptional prime divisor contained in $\operatorname{Supp} \mathbf{D}(\bar{s})$ by construction. It follows that

$$\begin{aligned}\mathbf{D}(\bar{s}) \cdot \mathbf{A}_{\bar{s}}^{\dim \mathbf{Z}_{\bar{s}} - 1} &= \mathbf{D}(\bar{s}) \cdot (\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}})^{\dim \mathbf{Z}_{\bar{s}} - 1} \\ &\quad - \sum_{0 \le i \le \dim \mathbf{Z}_{\bar{s}} - 2} \mathbf{D}(\bar{s}) \cdot \mathbf{E}_{\bar{s}} \cdot \mathbf{A}_{\bar{s}}^i \cdot (\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}})^{\dim \mathbf{Z}_{\bar{s}} - 2 - i} \\ &\le \mathbf{D}(\bar{s}) \cdot (\boldsymbol{\beta}_{\bar{s}}^*\mathbf{H}_{\bar{s}})^{\dim \mathbf{Z}_{\bar{s}} - 1} \le M,\end{aligned}$$

and hence, $\alpha_{\bar{t}}$ yields a closed point in $\mathbf{P}$ over $\bar{t} := [\mathbf{D}(\bar{s})] \in \mathbf{T}$.

If the set of closed points $\bar{s}$ in $\mathbf{S}$ such that $\mathscr{G}_{\bar{s}}$ is not closed under pth powers is Zariski-dense, then the image of $\mathbf{P} \to \mathbf{S}$ contains the generic point of $\mathbf{S}$ since it is a constructible set by a theorem of Chevalley (see [46, Corollaire 1.8.5]). It follows that there exists a closed logarithmic 1-form α on Z such that $d\omega = \alpha \wedge \omega$. Let C be any prime divisor on Z. Since the residue $\mathrm{res}_C\, \alpha$ of α at a general point of C is a constant function, we must have $\mathrm{res}_C\, \alpha \in \{m, \dots, m+M\} \subset \mathbb{Z}$ if C is not β-exceptional, where m denotes the order of vanishing of ω along C.

Let $(U_i)_{i\in I}$ be a covering of Z by analytically open sets such that $\alpha_{|U_i} = d \ln f_i + dg_i$, where f_i (resp., g_i) is a meromorphic (resp., holomorphic) function on U_i. Then

$$\omega_i := \frac{1}{f_i \exp(g_i)} \omega_{|U_i}$$

is a closed rational 1-form with zero set of codimension at least 2 and $\omega_i = c_{ij}\omega_j$ on $U_i \cap U_i$ for some $c_{ij} \in \mathbb{C}$. This completes the proof of the proposition. □

We end the preparation for the proofs of our main results with the following observation.

LEMMA 12.5

Let X be a normal complex projective variety with klt singularities, and let $\mathscr{G}$ be a codimension 1 foliation on X with $K_{\mathscr{G}}$ $\mathbb{Q}$-Cartier and $K_{\mathscr{G}} \equiv 0$. Then $\nu(X) \leq 1$.

Proof

Let $\beta\colon Z \to X$ be a resolution of singularities with exceptional set E, and suppose that E is a divisor with simple normal crossings. Let E_1 be the reduced divisor on Z whose support is the union of all irreducible components of E that are invariant under $\beta^{-1}\mathscr{G}$. Note that $-c_1(\mathscr{N}_{\mathscr{G}}) \equiv K_X$ by assumption. By Proposition 4.9 and Remark 4.8, there exists a rational number $0 \leq \varepsilon < 1$ such that

$$\nu(X) = \nu\big(-c_1(\mathscr{N}_{\mathscr{G}})\big) = \nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big).$$

On the other hand, by [69, Proposition V.2.7(1)], we have

$$\nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + \varepsilon E_1\big) \leq \nu\big(-c_1(\mathscr{N}_{\beta^{-1}\mathscr{G}}) + E_1\big).$$

The lemma now follows from [81, Proposition 9.3]. □

Proof of Theorem 1.1

By [63, Lemma 2.53] and Fact 2.10, there exists a quasi-étale cover $f: X_1 \to X$ with X_1 canonical such that $f^*K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$. By Lemma 4.3, $f^{-1}\mathscr{G}$ is canonical and $K_{f^{-1}\mathscr{G}} \sim_{\mathbb{Z}} 0$. Replacing X by X_1, if necessary, we may assume that $K_{\mathscr{G}} \sim_{\mathbb{Z}} 0$.

By Proposition 12.3 and Lemma 8.14, either $\mathscr{G}$ is closed under pth powers for almost all primes p, or it is given by a closed rational 1-form ω with values in a flat line bundle whose zero set has codimension at least 2 (see also Remark 12.4). In the latter case, the statement follows from Theorem 11.1.

Suppose that $\mathscr{G}$ is closed under pth powers for almost all primes p. By Lemma 12.5, we have $\nu(X) \le 1$. If $\nu(X) = -\infty$, then Corollary 9.5 says that $\mathscr{G}$ is algebraically integrable. If $\nu(X) = 1$, then $\mathscr{G}$ has algebraic leaves as well by Corollary 10.6. In either case, the statement follows from Theorem 1.5. If $\nu(X) = 0$, then Theorem 1.1 follows from Lemma 10.1 and Proposition 10.2. □

Proof of Theorem 1.3

By Proposition 12.3 and Lemma 8.14, either $\mathscr{G}$ is closed under pth powers for almost all primes p, or it is given by a closed rational 1-form ω with values in a flat line bundle whose zero set has codimension at least 2 (see also Remark 12.4).

Suppose first that $\mathscr{G}$ is closed under pth powers for almost all primes p. By Lemma 12.5, we have $\nu(X) \le 1$. If $\nu(X) = -\infty$, Theorem 9.4 and Corollary 9.5 then imply that $\mathscr{G}$ is algebraically integrable. If $\nu(X) = 1$, then $\mathscr{G}$ has algebraic leaves by Theorem 10.5 and Corollary 10.6. In either case, $K_{\mathscr{G}}$ is torsion by Proposition 4.24. If $\nu(X) = 0$, then $K_{\mathscr{G}}$ is torsion by Lemma 10.1 and Proposition 10.2.

Suppose now that $\mathscr{G}$ is given by a closed rational 1-form ω with values in a flat line bundle whose zero set has codimension at least 2. If $\nu(X) \ge 0$, then the statement follows from Proposition 11.10. If $\nu(X) = -\infty$, then Theorem 1.3 follows from Proposition 11.11. □

Proof of Corollary 1.4

Arguing as in the proof of Theorem 1.1, we see that we may assume without loss of generality that $K_{\mathscr{G}}$ is Cartier. Lemma 5.9 then implies that $\mathscr{G}$ is canonical, so that Theorem 1.1 applies. In particular, to prove Corollary 1.4, it suffices to consider the case where X is an equivariant compactification of a commutative algebraic group G of dimension at least 2 and $\mathscr{G} \cong \mathscr{O}_X^{\dim X - 1}$ is induced by a codimension 1 Lie subgroup $H \subset G$.

If X is not uniruled, then G must be an abelian variety by Chevalley's structure theorem on algebraic groups, and $\mathscr{G}$ is a linear foliation on X, so that we are in case (2) of Corollary 1.4.

Suppose from now on that X is uniruled. Let $\beta\colon Z \to X$ be an equivariant resolution of X with exceptional set E, and assume that E is a divisor with simple normal crossings and that β induces an isomorphism over X_{reg}. By Corollary 4.21, there is an inclusion $\beta^*\mathscr{G} \subset T_Z(-\log E)$. In particular, we must have $\beta^*\mathscr{G} \subset \beta^{-1}\mathscr{G}$. Since $\mathscr{G}$ is canonical, we conclude that $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Z}} \beta^* K_{\mathscr{G}}$ and that $\beta^*\mathscr{G} \cong \beta^{-1}\mathscr{G}$. This implies that any irreducible component of E is invariant under $\beta^{-1}\mathscr{G}$. This also implies that $\beta^{-1}\mathscr{G}$ is regular by Lemma 5.15. Since Z is uniruled by assumption, there exists a $\mathbb{P}^1$-bundle structure $\varphi\colon Z \to Y$ onto a complex projective manifold Y with $K_Y \sim_{\mathbb{Z}} 0$ such that $\beta^{-1}\mathscr{G}$ induces a flat connection on φ. This follows either from [80] or from the proof of [30, Proposition 5.1] (see also [30, Lemma 3.8]). Suppose that $E \neq \emptyset$, and let E_1 be an irreducible component of E. Then E_1 is smooth and $\mathscr{N}_{E_1/Z}$ is flat since E_1 is invariant under $\beta^{-1}\mathscr{G}$. But this contradicts the fact that E_1 is β-exceptional. It follows that X is as in case (1) of Corollary 1.4. This finishes the proof of the corollary. □

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References

[1] F. AMBRO, *The moduli b-divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403. MR 2134273. DOI 10.1112/S0010437X04001071. *(99, 126)*

[2] Y. ANDRÉ, "Sur la conjecture des p-courbures de Grothendieck-Katz et un problème de Dwork" in *Geometric Aspects of Dwork Theory, Vol. I, II*, de Gruyter, Berlin, 2004, 55–112. MR 2023288. *(99, 164)*

[3] M. ANDREATTA and J. A. WIŚNIEWSKI, "A view on contractions of higher-dimensional varieties" in *Algebraic Geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math. **62**, Amer. Math. Soc., Providence, 1997, 153–183. MR 1492522. *(167, 183)*

[4] C. ARAUJO and S. DRUEL, *On Fano foliations*, Adv. Math. **238** (2013), 70–118. MR 3033631. DOI 10.1016/j.aim.2013.02.003. *(96, 112, 133, 140, 144, 150, 161)*

[5] ———, *On codimension 1 del Pezzo foliations on varieties with mild singularities*, Math. Ann. **360** (2014), no. 3–4, 769–798. MR 3273645. DOI 10.1007/s00208-014-1053-3. *(96, 110, 126)*

[6] ———, "On Fano foliations 2" in *Foliation Theory in Algebraic Geometry*, Simons Symp., Springer, Cham, 2016, 1–20. MR 3644241. *(96)*

[7] ———, *Codimension 1 Mukai foliations on complex projective manifolds*, J. Reine Angew. Math. **727** (2017), 191–246. MR 3652251. DOI 10.1515/crelle-2014-0110. *(96)*

[8] ———, *Characterization of generic projective space bundles and algebraicity of foliations*, Comment. Math. Helv. **94** (2019), no. 4, 833–853. MR 4046006. DOI 10.4171/cmh/475. *(112, 121)*

[9] C. ARAUJO, S. DRUEL, and S. J. KOVÁCS, *Cohomological characterizations of projective spaces and hyperquadrics*, Invent. Math. **174** (2008), no. 2, 233–253. MR 2439607. DOI 10.1007/s00222-008-0130-1. *(133, 144)*

[10] I. ARZHANTSEV and P. KOTENKOVA, *Equivariant embeddings of commutative linear algebraic groups of corank one*, Doc. Math. **20** (2015), 1039–1053. MR 3424473. *(97, 186)*

[11] V. BALAJI and J. KOLLÁR, *Holonomy groups of stable vector bundles*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 183–211. MR 2426347. DOI 10.2977/prims/1210167326. *(147, 148)*

[12] P. F. BAUM and R. BOTT, "On the zeros of meromorphic vector-fields" in *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, Springer, New York, 1970, 29–47. MR 0261635. *(139)*

[13] A. BEAUVILLE, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782. MR 0730926. *(96)*

[14] C. BIRKAR, P. CASCINI, C. D. HACON, and J. MCKERNAN, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. MR 2601039. DOI 10.1090/S0894-0347-09-00649-3. *(101, 165, 181)*

[15] F. BOGOMOLOV and M. MCQUILLAN, "Rational curves on foliated varieties" in *Foliation Theory in Algebraic Geometry*, Simons Symp., Springer, Cham, 2016, 21–51. MR 3644242. *(126, 141)*

[16] J.-B. BOST, *Algebraic leaves of algebraic foliations over number fields*, Publ. Math. Inst. Hautes Études Sci. **93** (2001), 161–221. MR 1863738. DOI 10.1007/s10240-001-8191-3. *(99, 162, 163, 176)*

[17] S. BOUCKSOM, J.-P. DEMAILLY, M. PĂUN, and T. PETERNELL, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248. MR 3019449. DOI 10.1090/S1056-3911-2012-00574-8. *(183, 184)*

[18] M. BRION, *Differential forms on quotients by reductive group actions*, Proc. Amer. Math. Soc. **126** (1998), no. 9, 2535–2539. MR 1451789. DOI 10.1090/S0002-9939-98-04320-2. *(130)*

[19] ———, "Some basic results on actions of nonaffine algebraic groups" in *Symmetry and Spaces*, Progr. Math. **278**, Birkhäuser, Boston, 2010, 1–20. MR 2562620. DOI 10.1007/978-0-8176-4875-6_1. *(151)*

[20] M. BRUNELLA, *Birational Geometry of Foliations*, Publ. Mat. IMPA, Rio de Janeiro, 2004. MR 2114696. *(96, 112, 166, 167, 174)*

[21] F. CAMPANA, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630. MR 2097416. *(184)*

[22] F. CAMPANA and M. PĂUN, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49. MR 3949026. DOI 10.1007/s10240-019-00105-w. *(125, 160, 166, 175, 182, 187)*

[23] P. CASCINI and C. SPICER, *MMP for co-rank one foliation on threefolds*, preprint, arXiv:1808.02711v1 [math.AG]. *(96, 97, 98)*

[24] D. CERVEAU and J.-F. MATTEI, *Formes intégrables holomorphes singulières*, Astérisque **97**, Soc. Math. France, Paris, 1982. MR 0704017. *(171, 178, 183, 185, 186)*

[25] B. CLAUDON, *Positivité du cotangent logarithmique et conjecture de Shafarevich-Viehweg*, Astérisque **390** (2017), 27–63, Séminaire Bourbaki 2015/2016, no. 1105. MR 3666021. *(153)*

[26] K. CORLETTE and C. SIMPSON, *On the classification of rank-two representations of quasiprojective fundamental groups*, Compos. Math. **144** (2008), no. 5, 1271–1331. MR 2457528. DOI 10.1112/S0010437X08003618. *(162)*

[27] J.-P. DEMAILLY, T. PETERNELL, and M. SCHNEIDER, *Pseudo-effective line bundles on compact Kähler manifolds*, Internat. J. Math. **12** (2001), no. 6, 689–741. MR 1875649. DOI 10.1142/S0129167X01000861. *(137)*

[28] S. DRUEL, *Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle*, Math. Z. **267** (2011), no. 1–2, 413–423. MR 2772258. DOI 10.1007/s00209-009-0626-4. *(170, 174)*

[29] ———, *On foliations with nef anti-canonical bundle*, Trans. Amer. Math. Soc. **369** (2017), no. 11, 7765–7787. MR 3695844. DOI 10.1090/tran/6873. *(142, 143, 156, 159)*

[30] ———, *Some remarks on regular foliations with numerically trivial canonical class*, Épijournal Géom. Algébrique **1** (2017), art. ID 4. MR 3743107. *(109, 148, 152, 168, 172, 197)*

[31] ———, *A decomposition theorem for singular spaces with trivial canonical class of dimension at most five*, Invent. Math. **211** (2018), no. 1, 245–296. MR 3742759. DOI 10.1007/s00222-017-0748-y. *(96, 148, 154)*

[32] S. DRUEL and H. GUENANCIA, *A decomposition theorem for smoothable varieties with trivial canonical class*, J. Éc. polytech. Math. **5** (2018), 117–147. MR 3732694. DOI 10.5802/jep.65. *(150)*

[33] S. DRUEL and W. OU, *Codimension one foliations with trivial canonical class on singular spaces II*, preprint, arXiv:1912.07727v2 [math.AG]. *(98, 184)*

[34] T. EKEDAHL, N. I. SHEPHERD-BARRON, and R. TAYLOR, *A conjecture on the existence of compact leaves of algebraic foliations*, preprint, 1999, http://www.dpmms.cam.ac.uk/nisb/. *(99, 176)*

[35] R. ELKIK, *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), no. 1, 1–6. MR 0621766. DOI 10.1007/BF01393930. *(104, 136)*

[36] E. FLORIS, *Inductive approach to effective b-semiampleness*, Int. Math. Res. Not. IMRN **2014**, no. 6, 1465–1492. MR 3180598. DOI 10.1093/imrn/rns260. *(99, 126)*

[37] O. FUJINO, *Applications of Kawamata's positivity theorem*, Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), no. 6, 75–79. MR 1712648. *(168, 183)*

[38] O. GABBER, Q. LIU, and D. LORENZINI, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270. MR 3347315. DOI 10.1215/00127094-2877293. *(103)*

[39] T. GRABER, J. HARRIS, and J. STARR, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67. MR 1937199. DOI 10.1090/S0894-0347-02-00402-2. *(184)*

[40] D. GREB, H. GUENANCIA, and S. KEBEKUS, *Klt varieties with trivial canonical class: Holonomy, differential forms, and fundamental groups*, Geom. Topol. **23** (2019), no. 4, 2051–2124. MR 3988092. DOI 10.2140/gt.2019.23.2051. *(99, 169, 185)*

[41] D. GREB, S. KEBEKUS, and S. J. KOVÁCS, *Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties*, Compos. Math. **146** (2010), no. 1, 193–219. MR 2581247. DOI 10.1112/S0010437X09004321. *(124)*

[42] D. GREB, S. KEBEKUS, S. J. KOVÁCS, and T. PETERNELL, *Differential forms on log canonical spaces*, Publ. Math. Inst. Hautes Études Sci. **114** (2011), 87–169. MR 2854859. DOI 10.1007/s10240-011-0036-0. *(102, 115, 116, 118, 130, 140, 143)*

[43] D. GREB, S. KEBEKUS, and T. PETERNELL, "Singular spaces with trivial canonical class" in *Minimal Models and Extremal Rays (Kyoto, 2011)*, Adv. Stud. Pure Math. **70**, Math. Soc. Japan, Tokyo, 2011. MR 3617779. DOI 10.2969/aspm/07010067. *(104, 105, 106, 170)*

[44] A. GROTHENDIECK, "Technique de descente et théorèmes d'existence en géométrie algébrique, V: Les schémas de Picard: théorèmes d'existence" in *Séminaire Bourbaki, Vol. 7*, no. 232, Soc. Math. France, Paris, 1995, 143–161. MR 1611170. *(193)*

[45] ———, "Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert" in *Séminaire Bourbaki, Vol. 6 (1960/1961)*, no. 221, Soc. Math. France, Paris, 1995, 249–276. MR 1611822. *(193)*

[46] A. GROTHENDIECK and J. DIEUDONNÉ, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, I*, Publ. Math. Inst. Hautes Études Sci. **20** (1964). MR 0173675. *(195)*

[47] ———, *Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III*, Publ. Math. Inst. Hautes Études Sci. **28** (1966). MR 0217086. *(111, 122, 153, 193)*

[48] C. D. HACON and J. MCKERNAN, *On Shokurov's rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136. MR 2309156. DOI 10.1215/S0012-7094-07-13813-4. *(182, 184, 187)*

[49] R. HARTSHORNE, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer, New York, 1977. MR 0463157. *(127)*

[50] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176. MR 0597077. DOI 10.1007/BF01467074. *(115, 129, 131, 139, 154)*

[51] H. HOLMANN, "On the stability of holomorphic foliations with all leaves compact" in *Variétés analytiques compactes (Nice, 1977)*, Lecture Notes in Math. **683**, Springer, Berlin, 1978, 217–248. MR 0517527. *(126)*

[52] D. HUYBRECHTS and M. LEHN, *The Geometry of Moduli Spaces of Sheaves*, Asp. Math. **E31**, Vieweg, Braunschweig, 1997. MR 1450870. DOI 10.1007/978-3-663-11624-0. *(101, 148, 149, 193)*

[53] J.-M. HWANG and E. VIEHWEG, *Characteristic foliation on a hypersurface of general type in a projective symplectic manifold*, Compos. Math. **146** (2010), no. 2, 497–506. MR 2601637. DOI 10.1112/S0010437X09004412. *(142, 149)*

[54] S. IITAKA, *Logarithmic forms of algebraic varieties*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 525–544. MR 0429884. *(178)*

[55] I. KAPLANSKY, *An Introduction to Differential Algebra*, 2nd ed., Hermann, Paris, 1976. MR 0460303. *(162)*

[56] W. KAUP, *Infinitesimale Transformationsgruppen komplexer Räume*, Math. Ann. **160** (1965), 72–92. MR 0181761. DOI 10.1007/BF01364336. *(152, 157, 186)*

[57] Y. KAWAMATA, *Characterization of abelian varieties*, Compos. Math. **43** (1981), no. 2, 253–276. MR 0622451. *(186)*

[58] ———, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46. MR 0814013. DOI 10.1515/crll.1985.363.1. *(99, 105, 175, 185)*

[59] Y. KAWAMATA, K. MATSUDA, and K. MATSUKI, "Introduction to the minimal model problem" in *Algebraic Geometry (Sendai, 1985)*, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987, 283–360. MR 0946243. DOI 10.2969/aspm/01010283. *(101, 165, 168, 175, 183)*

[60] S. KEBEKUS, *Pull-back morphisms for reflexive differential forms*, Adv. Math. **245** (2013), 78–112. MR 3084424. DOI 10.1016/j.aim.2013.06.013. *(102, 145)*

[61] J. KOLLÁR, "Singularities of pairs" in *Algebraic Geometry (Santa Cruz, 1995)*, Proc. Sympos. Pure Math. **62**, Amer. Math. Soc. Providence, 1997, 221–287. MR 1492525. *(104, 113, 143, 156)*

[62] ———, *Singularities of the Minimal Model Program*, Cambridge Tracts in Math. **200**, Cambridge Univ. Press, Cambridge, 2013. MR 3057950. DOI 10.1017/CBO9781139547895. *(101)*

[63] J. KOLLÁR and S. MORI, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, Cambridge, 1998. MR 1658959. DOI 10.1017/CBO9780511662560. *(101, 103, 105, 110, 115, 121, 132, 141, 144, 146, 154, 159, 168, 170, 175, 182, 184, 195)*

[64] A. LANGER, *Semistable sheaves in positive characteristic*, Ann. of Math. (2) **159** (2004), no. 1, 251–276. MR 2051393. DOI 10.4007/annals.2004.159.251. *(191)*

[65] R. LAZARSFELD, *Positivity in Algebraic Geometry, I: Classical Setting: Line Bundles and Linear Series*, Ergeb. Math. Grenzgeb. (3) **48**, Springer, Berlin, 2004. MR 2095471. DOI 10.1007/978-3-642-18808-4. *(157)*

[66] F. LORAY, J. V. PEREIRA, and F. TOUZET, *Singular foliations with trivial canonical class*, Invent. Math. **213** (2018), no. 3, 1327–1380. MR 3842065. DOI 10.1007/s00222-018-0806-0. *(96, 98, 99, 124, 125, 133, 159, 160, 166, 167, 176, 179, 182, 190, 191, 192, 197)*

[67] M. MCQUILLAN, *Canonical models of foliations*, Pure Appl. Math. Q. **4** (2008), no. 3, 877–1012. MR 2435846. DOI 10.4310/PAMQ.2008.v4.n3.a9. *(96, 112, 118, 141, 184)*

[68] A. MÉZARD, "Fundamental group" in *Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998)*, Progr. Math. **187**, Birkhäuser, Basel, 2000, 141–155. MR 1768098. *(152)*

[69] N. NAKAYAMA, *Zariski-Decomposition and Abundance*, MSJ Mem. **14**, Math. Soc. Japan, Tokyo, 2004. MR 2104208. *(103, 105, 117, 125, 168, 169, 170, 173, 174, 184, 186, 195)*

[70] M. S. NARASIMHAN and C. S. SESHADRI, *Holomorphic vector bundles on a compact Riemann surface*, Math. Ann. **155** (1964), 69–80. MR 0166799. DOI 10.1007/BF01350891. *(149)*

[71] ———, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540–567. MR 0184252. DOI 10.2307/1970710. *(147, 149)*

[72] T. ODA, *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, Ergeb. Math. Grenzgeb. (3) **15**, Springer, Berlin, 1988. MR 0922894. *(186)*

[73] M. REID, "Chapters on algebraic surfaces" in *Complex Algebraic Geometry (Park City, 1993)*, IAS/Park City Math. Ser. **3**, Amer. Math. Soc., Providence, 1997, 5–159. MR 1442522. DOI 10.1090/pcms/003/02. *(125)*

[74] K. SAITO, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR 0586450. *(102)*

[75] A. SEIDENBERG, *Differential ideals in rings of finitely generated type*, Amer. J. Math. **89** (1967), no. 1, 22–42. MR 0212027. DOI 10.2307/2373093. *(109, 122, 124)*

[76] J.-P. SERRE, *Exposés de séminaires (1950–1999)*, Doc. Math. (Paris) **1**, Soc. Math. France, Paris, 2001. MR 1942136. *(175)*

[77] Y.-T. SIU and G. TRAUTMANN, *Gap-Sheaves and Extension of Coherent Analytic Subsheaves*, Lecture Notes in Math. **172**, Springer, Berlin, 1971. MR 0287033. *(139)*

[78] C. SPICER, *Higher dimensional foliated Mori theory*, Compos. Math. **156** (2020), no. 1, 1–38. MR 4036447. DOI 10.1112/s0010437x19007681. *(96)*

[79] S. TAKAYAMA, *Local simple connectedness of resolutions of log-terminal singularities*, Internat. J. Math. **14** (2003), no. 8, 825–836. MR 2013147. DOI 10.1142/S0129167X0300196X. *(188, 192)*

[80] F. TOUZET, *Feuilletages holomorphes de codimension un dont la classe canonique est triviale*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 4, 655–668. MR 2489636. DOI 10.24033/asens.2078. *(96, 97, 197)*

[81] ———, "On the structure of codimension 1 foliations with pseudoeffective conormal bundle" in *Foliation Theory in Algebraic Geometry*, Simons Symp., Springer, Cham, 2016, 157–216. MR 3644247. *(99, 172, 195)*

[82] K. UENO, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Math. **439**, Springer, Berlin, 1975. MR 0506253. *(105)*

[83] O. ZARISKI, *Studies in equisingularity, I: Equivalent singularities of plane algebroid curves*, Amer. J. Math. **87** (1965), no. 2, 507–536. MR 0177985. DOI 10.2307/2373019. *(140)*

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