

On codimension 1 del Pezzo foliations on varieties with mild singularities

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Mathematische Annalen

ISSN 0025-5831

Volume 360

Combined 3-4

Math. Ann. (2014) 360:769-798

DOI 10.1007/s00208-014-1053-3

Mathematische Annalen

Band 360 Heft 3–4 Dezember 2014

Begründet 1868 durch Alfred Clebsch · Carl Neumann
Fortgeführt durch Felix Klein · David Hilbert · Otto Blumenthal
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On codimension 1 del Pezzo foliations on varieties with mild singularities

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Received: 18 July 2013 / Revised: 22 March 2014 / Published online: 20 May 2014
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Abstract In this paper we extend to the singular setting the theory of Fano foliations developed in our previous paper (Araujo and Druel, Adv Math 238:70–118, 2013). A \mathbb{Q} -Fano foliation on a complex projective variety X is a foliation $\mathcal{F} \subsetneq T_X$ whose anti-canonical class $-K_{\mathcal{F}}$ is an ample \mathbb{Q} -Cartier divisor. In the spirit of Kobayashi–Ochiai Theorem, we prove that under some conditions the index $i_{\mathcal{F}}$ of a \mathbb{Q} -Fano foliation is bounded by the rank r of \mathcal{F} , and classify the cases in which $i_{\mathcal{F}} = r$. Next we consider \mathbb{Q} -Fano foliations \mathcal{F} for which $i_{\mathcal{F}} = r - 1$. These are called del Pezzo foliations. We classify codimension 1 del Pezzo foliations on mildly singular varieties.

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1 Introduction

The study of Fano manifolds is a classical theme in algebraic geometry. Much is known about these varieties, as they appear naturally in several different contexts. With the development of the Minimal Model Program in dimension ≥ 3 , it became clear that singularities are unavoidable in the birational classification of higher dimensional complex projective varieties. This led to the development of a powerful theory of *singularities of pairs* (see Definition 2.1 for basic notions, such as klt singularities). As a consequence of the Minimal Model Program, every smooth complex projective variety can be build up from *mildly singular* projective varieties X , for which K_X is \mathbb{Q} -Cartier and satisfies one of the following: $K_X < 0$, $K_X \equiv 0$ or $K_X > 0$. This makes imperative to extend the theory of Fano manifolds to the singular setting. A \mathbb{Q} -Fano variety is a normal projective variety X such that $-K_X$ is an ample \mathbb{Q} -Cartier divisor. Its index is the largest positive rational number i_X such that $-K_X \sim_{\mathbb{Q}} i_X H$ for a Cartier divisor H on X . When X is an n -dimensional smooth Fano variety, a classical result of Kobayashi and Ochiai asserts that $i_X \leq n + 1$. Moreover, equality holds if and only if $X \simeq \mathbb{P}^n$, and $i_X = n$ if and only if X is a quadric hypersurface in \mathbb{P}^{n+1} . We generalize this to the singular setting (see also [38] for related results).

Theorem 1.1 *Let Δ be an effective \mathbb{Q} -divisor on a normal projective variety X of dimension $n \geq 1$. Suppose that $K_X + \Delta$ is \mathbb{Q} -Cartier, and $-(K_X + \Delta) \sim_{\mathbb{Q}} ic_1(\mathcal{L})$ for an ample line bundle \mathcal{L} on X and $i \in \mathbb{Q}$.*

1. *If $i > n$, then $n < i \leq n + 1$, $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $\deg(\Delta) = n + 1 - i$.*
2. *If $i = n$, then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $\deg(\Delta) = 1$, or $\Delta = 0$ and $(X, \mathcal{L}) \simeq (Q_n, \mathcal{O}_{Q_n}(1))$, where Q_n is a (possibly singular) quadric hypersurface in \mathbb{P}^{n+1} .*

The Fano condition and the notion of index naturally find counterparts in the theory of holomorphic foliations. Given a holomorphic foliation $\mathcal{F} \subsetneq T_X$ of generic rank r on a normal complex projective variety X , we denote by $\det(\mathcal{F})$ the reflexive sheaf $(\wedge^r \mathcal{F})^{**}$, and by $-K_{\mathcal{F}}$ any Weil divisor on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \simeq \det(\mathcal{F})$. We say that \mathcal{F} is a *\mathbb{Q} -Fano foliation* if $-K_{\mathcal{F}}$ is an ample \mathbb{Q} -Cartier divisor on X . In this case, the *index of \mathcal{F}* is the largest positive rational number $i_{\mathcal{F}}$ such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}} H$ for a Cartier divisor H on X .

In our previous paper [1], inspired by the theory of Fano manifolds and the Minimal Model Program, we initiated the study of Fano foliations on complex projective manifolds. By [2, Theorem 1.1], the index $i_{\mathcal{F}}$ of a Fano foliation $\mathcal{F} \subsetneq T_X$ of rank r on a complex projective manifold X satisfies $i_{\mathcal{F}} \leq r$. Moreover, equality holds only if $X \simeq \mathbb{P}^n$. This may be viewed as a version for foliations of the classical Kobayashi–Ochiai Theorem. Fano foliations on \mathbb{P}^n with index equal to the rank were classified in [11, Théorème 3.8]. They are induced by linear projections $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$, and are classically known as *degree 0 foliations on \mathbb{P}^n* . In [1], we defined *del Pezzo foliations*

on complex projective manifolds X as Fano foliations $\mathcal{F} \subsetneq T_X$ of rank $r \geq 2$ and index $i_{\mathcal{F}} = r - 1$. Del Pezzo foliations \mathcal{F} on \mathbb{P}^n (classically known as *degree 1 foliations* on \mathbb{P}^n) were classified in [37, Theorem 6.2]:

1. Either \mathcal{F} is induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^{n-r}, 2)$, defined by $n - r$ linear forms and one quadratic form, or
2. \mathcal{F} is the linear pullback of a foliation on \mathbb{P}^{n-r+1} induced by a global holomorphic vector field.

The generic del Pezzo foliation of type (2) above does not have algebraic leaves. On the other hand, we showed in [1, Theorem 1.1] that if $X \not\cong \mathbb{P}^n$, then del Pezzo foliations on X have algebraic and rationally connected general leaves. We then worked toward the classification of del Pezzo foliations.

In this paper we address \mathbb{Q} -Fano foliations on possibly *singular* varieties. Our main goal is to extend the theory developed in [1] to the singular setting. There is at least one strong reason for studying this more general case. The birational classification of rank 1 foliations on smooth surfaces obtained in [9] was very much modeled after the birational classification of smooth surfaces. If one hopes to apply ideas from the Minimal Model Program to approach the problem of birational classification of holomorphic foliations in higher dimensions, then one may be led to study foliations on mildly singular varieties as well.

We first prove a version for foliations on mildly singular varieties of the Kobayashi–Ochiai Theorem. See Definition 4.1 for the notion of normal generalized cone, and [2, 5, 15, 30, 40] and for related results.

Theorem 1.2 *Let X be an n -dimensional \mathbb{Q} -factorial klt projective variety with $n \geq 2$, and $\mathcal{F} \subsetneq T_X$ a \mathbb{Q} -Fano foliation of rank r . Suppose that either X has Picard number $\rho(X) = 1$, or \mathcal{F} has rank $r = n - 1$. Then*

1. $i_{\mathcal{F}} \leq r$.
2. If $i_{\mathcal{F}} = r$, then X is a normal generalized cone over a polarized variety (Z, \mathcal{M}) with vertex $P \simeq \mathbb{P}^{r-1}$, and \mathcal{F} is induced by the natural map $X \dashrightarrow Z$. Moreover, Z is \mathbb{Q} -factorial and klt, $\rho(Z) = 1$ and $i_Z = k(i_X - r) > 0$ for some positive integer k .

Next we consider del Pezzo foliations, i.e., \mathbb{Q} -Fano foliations of rank $r \geq 2$ and index $i_{\mathcal{F}} = r - 1$. In the smooth case, del Pezzo foliations of codimension 1 on Fano manifolds with Picard number 1 were classified in [37, Proposition 3.7]: they are either degree 1 foliations on \mathbb{P}^n , or come from pencils of hyperplane sections of quadric hypersurfaces. We extend this classification to mildly singular varieties, without restriction on the Picard number.

Theorem 1.3 *Let $\mathcal{F} \subsetneq T_X$ be a codimension 1 del Pezzo foliation on a factorial klt projective variety X of dimension $n \geq 3$.*

1. Suppose that $\rho(X) = 1$. Then X is a \mathbb{Q} -Fano variety and one of the following holds.
 - (a) \mathcal{F} is a degree 1 foliation on $X \simeq \mathbb{P}^n$.
 - (b) X is isomorphic to a (possibly singular) quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$, and \mathcal{F} is a pencil of hyperplane sections of Q_n .

(c) X is a normal generalized cone over a projective normal surface Z with $\rho(Z) = 1$, and \mathcal{F} is the pullback by $X \dashrightarrow Z$ of a foliation induced by a nonzero global section of T_Z . Moreover, Z is factorial with klt singularities, $i_X > n - 2$ and $i_Z = k(i_X - n + 2)$ for some positive integer k .

2. Suppose that $\rho(X) \geq 2$. Then there is an exact sequence of vector bundles

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{V} \rightarrow 0$$

on \mathbb{P}^1 such that $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, and \mathcal{F} is the pullback via the relative linear projection $X \dashrightarrow Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{K})$ of a foliation on Z induced by a nonzero global section of $T_Z \otimes q^* \det(\mathcal{V})^*$. Here $q : Z \rightarrow \mathbb{P}^1$ denotes the natural projection. Moreover, one of the following holds.

- (a) $(\mathcal{E}, \mathcal{K}) \simeq (\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ for some positive integer a .
- (b) $(\mathcal{E}, \mathcal{K}) \simeq (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ for some positive integer a .
- (c) $(\mathcal{E}, \mathcal{K}) \simeq (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ for distinct positive integers a and b .

One of the key ingredients of our proofs of Theorems 1.2 and 1.3 is the following result, which illustrates the strong restrictions imposed by the existence of a \mathbb{Q} -Fano foliation on a variety X .

Theorem 1.4 *Let X be a klt projective variety, and $\mathcal{F} \subsetneq T_X$ a \mathbb{Q} -Fano foliation. Then $K_X - K_{\mathcal{F}}$ is not pseudo-effective.*

When $\rho(X) = 1$, Theorem 1.4 allows us to bound the index of X from below: $i_X > i_{\mathcal{F}}$. When $\rho(X) > 1$, it bounds from below the length of extremal rays of X . When \mathcal{F} is a codimension 1 del Pezzo foliation, these bounds are good enough for us to recover X . This is not the case for higher codimension or smaller index $i_{\mathcal{F}}$.

The paper is organized as follows.

In Sect. 2, we review basic definitions of singularities of pairs, gather some results about \mathbb{Q} -Fano varieties and singular del Pezzo varieties, and prove a vanishing result (Theorem 2.11) that has useful applications to the theory of \mathbb{Q} -Fano foliations. In particular, it allows us to prove Theorem 1.4.

In Sect. 3, we develop the basic theory of foliations and Pfaff fields on mildly singular varieties, and prove Theorem 1.4.

In Sect. 4, in the spirit of the Kobayashi–Ochiai Theorem, we investigate upper bounds for the index of \mathbb{Q} -Fano foliations. In particular, we prove Theorem 1.2.

Section 5 is devoted to del Pezzo foliations, and contains the proof of Theorem 1.3.

In Sect. 6, we discuss singularities of \mathbb{Q} -Fano foliations. In particular, we prove the following result, which says that codimension 1 \mathbb{Q} -Fano foliations on mildly singular varieties must be singular in codimension 2.

Theorem 1.5 *Let $\mathcal{F} \subsetneq T_X$ be a codimension 1 foliation on a klt projective variety X . Suppose that X and \mathcal{F} are both regular in codimension 2. Then $-K_{\mathcal{F}}$ is not ample.*

Notation and conventions. We always work over the field \mathbb{C} of complex numbers. Varieties are always assumed to be irreducible and reduced. We denote by $\text{Sing}(X)$ the singular locus of a variety X .

Let X be a normal variety. Given a Weil divisor D on X , we denote by $\mathcal{O}_X(D)$ the rank 1 reflexive sheaf defined by: $H^0(U, \mathcal{O}_X(D)|_U) = \{f \in \mathbb{C}(X) \setminus \{0\} \mid \text{div}(f) + D|_U \geq 0\} \cup \{0\}$ for any open subset $U \subset X$. Given a rank 1 reflexive sheaf \mathcal{F} on X , then, by abuse of notation, we denote by $c_1(\mathcal{F})$ any Weil divisor D on X such that $\mathcal{O}_X(D) \simeq \mathcal{F}$.

Given two \mathbb{Q} -Weil divisors D and D' on a normal variety X , we say that $D \leq D'$ if $D' - D$ is effective.

Let X be a normal variety, and D a \mathbb{Q} -Weil divisor on X . Let $\pi : Y \rightarrow X$ be a finite morphism such that, for any prime divisor $P \subset Y$, X is smooth at the generic point of $\pi(P)$. Then the pullback of D to Y is well defined, and we denote it either by π^*D or by $D|_Y$.

If \mathcal{E} is a locally free sheaf of \mathcal{O}_X -modules on a variety X , we denote by $\mathbb{P}_X(\mathcal{E})$ the Grothendieck projectivization $\text{Proj}_X(\text{Sym}(\mathcal{E}))$.

Given a sheaf \mathcal{F} of \mathcal{O}_X -modules on a variety X , we denote by \mathcal{F}^* the sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If r is the generic rank of \mathcal{F} , then we denote by $\det(\mathcal{F})$ the sheaf $(\wedge^r \mathcal{F})^{**}$. For $m \in \mathbb{N}$, we denote by $\mathcal{F}^{[m]}$ the sheaf $(\mathcal{F}^{\otimes m})^{**}$. If \mathcal{G} is another sheaf of \mathcal{O}_X -modules on X , then we denote by $\mathcal{F}[\otimes]\mathcal{G}$ the sheaf $(\mathcal{F} \otimes \mathcal{G})^{**}$. If $\pi : Y \rightarrow X$ is a morphism of varieties, then we write $\pi^{[*]}\mathcal{F}$ for $(\pi^*\mathcal{F})^{**}$.

If X is a normal variety and $X \rightarrow Y$ is any morphism, we denote by $T_{X/Y}$ the sheaf $(\Omega_{X/Y}^1)^*$. In particular, $T_X = (\Omega_X^1)^*$. If D is a reduced divisor on X and $q \in \mathbb{N}$, then we denote by $\Omega_X^{[q]}$ the sheaf $(\Omega_X^q)^{**}$, and by $\Omega_X^{[q]}(\log D)$ the sheaf of reflexive differential q -forms with logarithmic poles along D .

2 \mathbb{Q} -Fano varieties, del Pezzo varieties and a vanishing result

In this section we gather some results about \mathbb{Q} -Fano varieties and singular del Pezzo varieties. At the end of the section we prove a vanishing result that has useful applications to the theory of \mathbb{Q} -Fano foliations. We start by recalling some definitions of singularities of pairs, developed in the context of the Minimal Model Program.

2.1 Singularities of pairs

Definition 2.1 (See [34], section 2.3) Let X be a normal projective variety, and $\Delta = \sum a_i \Delta_i$ an effective \mathbb{Q} -divisor on X , i.e., Δ is a nonnegative \mathbb{Q} -linear combination of distinct prime Weil divisors Δ_i 's on X . Suppose that $K_X + \Delta$ is \mathbb{Q} -Cartier, i.e., some nonzero multiple of it is a Cartier divisor.

Let $f : \tilde{X} \rightarrow X$ be a log resolution of the pair (X, Δ) . This means that \tilde{X} is a smooth projective variety, f is a birational projective morphism whose exceptional locus is the union of prime divisors E_i 's, and the divisor $\sum E_i + \tilde{\Delta}$ has simple normal crossing support, where $\tilde{\Delta}$ denotes the strict transform of Δ in X . There are uniquely defined rational numbers $a(E_i, X, \Delta)$'s such that

$$K_{\tilde{X}} + \tilde{\Delta} = f^*(K_X + \Delta) + \sum_{E_i} a(E_i, X, \Delta)E_i.$$

The $a(E_i, X, \Delta)$'s do not depend on the log resolution f , but only on the valuations associated to the E_i 's.

We say that (X, Δ) is *canonical* if all $a_i \leq 1$, and, for some log resolution $f : \tilde{X} \rightarrow X$ of (X, Δ) , $a(E_i, X, \Delta) \geq 0$ for every f -exceptional prime divisor E_i . We say that (X, Δ) is *log terminal* (or *klt*) if all $a_i < 1$, and, for some log resolution $f : \tilde{X} \rightarrow X$ of (X, Δ) , $a(E_i, X, \Delta) > -1$ for every f -exceptional prime divisor E_i . If these conditions hold for some log resolution of (X, Δ) , then they hold for every log resolution of (X, Δ) .

We say that X is *canonical* (respectively *klt*) if so is $(X, 0)$. Note that if X is Gorenstein, i.e., K_X is Cartier, then X is canonical if and only if it is klt.

We will need the following observation.

Lemma 2.2 *Let r be a positive integer, and Δ an effective \mathbb{Q} -divisor on \mathbb{P}^r of degree ≤ 1 . Then (\mathbb{P}^r, Δ) is klt unless $\Delta = H$, where H is a hyperplane.*

Proof When $r = 1$, the result is clear. We assume from now on that $r \geq 2$.

Write $\Delta = \sum_{1 \leq i \leq k} d_i \Delta_i$, where $\Delta_i \subset \mathbb{P}^r$ is an integral hypersurface, $\Delta_i \neq \Delta_j$ if $i \neq j$ and the d_i 's are positive rational numbers. By assumption, $\deg(\Delta) = \sum_{1 \leq i \leq k} d_i \deg(\Delta_i) \leq 1$. Suppose that either $k \geq 2$, or $k = 1$ and $\deg(\Delta_1) \geq 2$. Let $p \in \mathbb{P}^r$ be a point. We shall show that (X, Δ) is klt at p .

Observe that

$$\text{mult}_p \Delta = \sum_{1 \leq i \leq k} d_i \text{mult}_p \Delta_i \leq \sum_{1 \leq i \leq k} d_i \deg(\Delta_i) = \deg(\Delta) \leq 1.$$

If $\text{mult}_p \Delta < 1$, then (\mathbb{P}^r, Δ) is klt at p by [36, Proposition 9.5.13].

If $\text{mult}_p \Delta = 1$, then $\text{mult}_p \Delta_i = \deg(\Delta_i)$ for every $1 \leq i \leq k$. Thus Δ_i is a cone in \mathbb{P}^r with vertex p for every $1 \leq i \leq k$.

Let $P \subset \mathbb{P}^r$ be a general plane passing through p . Then $\text{Supp}(\Delta)|_P$ is a union of $\sum_{1 \leq i \leq k} \deg(\Delta_i)$ distinct lines through p . Let $S \rightarrow P$ be the blow-up of P at p , with exceptional locus E . Then S is a log resolution of the pair $(P, \Delta|_P)$ and, $a(E, P, \Delta|_P) = 1 - \deg(\Delta) \geq 0$. Thus, $(P, \Delta|_P)$ is klt at p . This implies that (X, Δ) is klt at p , by [36, Corollary 9.5.11]. \square

2.2 \mathbb{Q} -Fano varieties

Definition 2.3 Let X be a normal projective variety such that K_X is \mathbb{Q} -Cartier. We say that X is *\mathbb{Q} -Fano* if $-K_X$ is ample. In this case, the *index of X* is the largest positive rational number i_X such that $-K_X \sim_{\mathbb{Q}} i_X H$ for an ample Cartier divisor H on X .

The following geometric property of klt \mathbb{Q} -Fano varieties will be fundamental in our study of \mathbb{Q} -Fano foliations.

Theorem 2.4 ([28, Corollaries 1.3 and 1.5]) *Let (X, Δ) be a klt pair. If $-(K_X + \Delta)$ is nef and big, then X is rationally connected.*

The following result is certainly well known to experts. We include a proof for lack of adequate reference.

Lemma 2.5 *Let X be klt \mathbb{Q} -Fano variety, and \mathcal{L} a line bundle on X . If $c_1(\mathcal{L}) \equiv 0$, then $\mathcal{L} \simeq \mathcal{O}_X$. In particular, if $\rho(X) = 1$, then $\text{Pic}(X) \simeq \mathbb{Z}$.*

Proof Let $\pi : Y \rightarrow X$ be a resolution of singularities. Recall that X has rational singularities. The Leray spectral sequence implies that $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y)$ for all $i \geq 0$, and $h^i(X, \mathcal{L}) = h^i(Y, \pi^*\mathcal{L})$ for all $i \geq 0$. By the Kawamata–Viehweg vanishing Theorem, $h^i(X, \mathcal{O}_X) = 0$ and $h^i(X, \mathcal{L}) = 0$ for all $i \geq 1$ (see [31, Theorem 1.2.5]). The Grothendieck–Riemann–Roch Theorem and the fact that $c_1(\mathcal{L}) \equiv 0$ imply that $\chi(Y, \pi^*\mathcal{L}) = \chi(Y, \mathcal{O}_Y)$. Thus $\chi(Y, \pi^*\mathcal{L}) = 1$, and $h^0(X, \mathcal{L}) = h^0(Y, \pi^*\mathcal{L}) = 1$. The same argument shows that $h^0(X, \mathcal{L}^{\otimes -1}) = 1$ as well. The claim follows. \square

We end this subsection by proving Theorem 1.1.

Proof of Theorem 1.1 The result is clear if $n = 1$. So we assume that $n \geq 2$. Let $\pi : Y \rightarrow X$ be a resolution of singularities, with exceptional locus E . For every $t \in \mathbb{Z}$ we have

$$\begin{aligned} h^n(Y, \pi^*\mathcal{L}^{\otimes -t}) &\simeq h^0(Y, \mathcal{O}_Y(K_Y) \otimes \pi^*\mathcal{L}^{\otimes t}) \text{ by Serre duality} \\ &\leq h^0(Y \setminus E, \mathcal{O}_Y(K_Y) \otimes \pi^*\mathcal{L}^{\otimes t}) \\ &= h^0(X \setminus \pi(E), \mathcal{O}_X(K_X) \otimes \mathcal{L}^{\otimes t}) \text{ for } Y \setminus E \simeq X \setminus \pi(E) \\ &= h^0(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}^{\otimes t}) \text{ for } \text{codim}(\pi(E)) \geq 2. \end{aligned}$$

Notice that $K_X + tc_1(\mathcal{L}) \sim_{\mathbb{Q}} -\Delta + (t - i)c_1(\mathcal{L})$.

Suppose that $i > n$. Then, for every $t \in \{1, \dots, n\}$, we have $t - i < 0$, and thus $h^n(Y, \pi^*\mathcal{L}^{\otimes -t}) = h^0(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}^{\otimes t}) = 0$. Thus, by [22, Theorem 2.2], there is a birational morphism $\varphi : X \rightarrow \mathbb{P}^n$ such that $\mathcal{L} \simeq \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Since \mathcal{L} is ample, φ must be an isomorphism, proving (1).

Suppose that $i = n$. Then, for every $t \in \{1, \dots, n - 1\}$, we have $t - i < 0$, and thus $h^n(Y, \pi^*\mathcal{L}^{\otimes -t}) = h^0(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}^{\otimes t}) = 0$. Moreover, $h^n(Y, \pi^*\mathcal{L}^{\otimes -n}) = h^0(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}^{\otimes n})$ and $K_X + nc_1(\mathcal{L}) \sim_{\mathbb{Q}} -\Delta$. If $h^n(Y, \pi^*\mathcal{L}^{\otimes -n}) = 0$, then $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ again by [22, Theorem 2.2]. Otherwise, $\Delta = 0$, $-K_X \sim_{\mathbb{Z}} nc_1(\mathcal{L})$ and $h^n(Y, \pi^*\mathcal{L}^{\otimes -n}) = 1$. By [22, Theorem 2.3], either $g(X, \mathcal{L}) = 1$, where $g(X, \mathcal{L})$ denotes the sectional genus of (X, \mathcal{L}) , or there is a birational morphism $\psi : X \rightarrow Q_n$ onto a (possibly singular) quadric hypersurface in \mathbb{P}^{n+1} such that $\mathcal{L} \simeq \psi^*\mathcal{O}_{Q_n}(1)$. In the latter case, ψ must be an isomorphism since \mathcal{L} is ample. Finally note that $g(X, \mathcal{L}) = 1 + \frac{1}{2}(K_X + (n - 1)c_1(\mathcal{L})) \cdot c_1(\mathcal{L})^{n-1} = 1 - \frac{1}{2}c_1(\mathcal{L})^n \neq 1$. This proves (2). \square

2.3 Singular del Pezzo varieties

Definition 2.6 ([24, Introduction]) A del Pezzo variety is a pair (X, \mathcal{L}) , where X is a (not necessarily normal) projective variety and \mathcal{L} is an ample line bundle on X , such that:

1. X is Gorenstein,
2. $\omega_X \simeq \mathcal{L}^{\otimes -(\dim(X)-1)}$,
3. $h^i(X, \mathcal{L}^{\otimes t}) = 0$ for all $t \in \mathbb{Z}$ and $0 < i < \dim(X)$.

If X is a Gorenstein klt \mathbb{Q} -Fano variety, then, by the Kawamata–Viehweg vanishing Theorem (see [31, Theorem 1.2.5]), condition (3) above is satisfied. Therefore, X is a (normal) del Pezzo variety if and only if condition (2) holds.

Our classification of codimension 1 del Pezzo foliations in Theorem 1.3(1) will rely on the following results.

Proposition 2.7 *Let (X, \mathcal{L}) be a normal del Pezzo variety of dimension $n \geq 1$ and degree $d := \mathcal{L}^n$. Suppose that X is klt. Then $d \leq 9$.*

Proof We may assume $d \geq 3$, in which case \mathcal{L} is very ample by [24, Corollary 1.5].

Suppose that $i_X \neq n - 1$. Then $(X, \mathcal{L}) \simeq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ by Theorem 1.1, and so $d = 8$.

If $X \subset \mathbb{P}(H^0(X, \mathcal{L})^*)$ is not a cone, then $d \leq 9$ by [17–19] and [20, 2.9] (see also [23]).

From now on we assume that $X \subset \mathbb{P}(H^0(X, \mathcal{L})^*)$ is a (singular) cone over a normal projective variety W , with vertex a linear subspace P of $\mathbb{P}(H^0(X, \mathcal{L})^*)$. We may assume that W is not a cone. Notice that W is identified with a global complete intersection of general members of $|\mathcal{L}|$. Let \mathcal{M} be the line bundle on W induced by \mathcal{L} . Then W is Gorenstein and klt, $\mathcal{M}^{\dim(W)} = d$, and $i_W = k(i_X - r) = k(\dim(W) - 1)$, where $r = \dim(X) - \dim(W)$ and k is a positive integer. If $\dim(W) = 1$, then W is a smooth curve of genus one, contradicting the fact that X is klt. So $\dim(W) \geq 2$, and W is a normal del Pezzo variety of dimension ≥ 2 . Since W is not a cone, we must have $d = \mathcal{M}^{\dim(W)} \leq 9$ as above, completing the proof of the proposition. \square

Lemma 2.8 *Let (X, \mathcal{L}) be a normal del Pezzo variety of dimension $n \geq 2$ and degree $\mathcal{L}^n \leq 2$. Then $h^0(X, \Omega_X^{[1]} \otimes \mathcal{L}) = 0$.*

To prove Lemma 2.8, we will use the following description of reflexive forms on double covers.

2.9 *Let X be a smooth projective variety, and D an effective reduced divisor on X such that $\mathcal{O}_X(D) \simeq \mathcal{R}^{\otimes 2}$ for some line bundle \mathcal{R} on X . Let $f : Y \rightarrow X$ be the double cover of X ramified over D . Then Y is normal, and $f_*\Omega_Y^{[1]}$ is a reflexive sheaf by [27, Corollary 1.7]. By [16, Lemme 1.9], $f_*\Omega_Y^{[1]} \simeq \Omega_X^1 \oplus \Omega_X^{[1]}(\log D) \otimes \mathcal{R}^{-1}$.*

Proof of Lemma 2.8 Suppose that $\mathcal{L}^n = 2$. By [23, Corollary 6.13], the linear system $|\mathcal{L}|$ induces a double cover $\pi : X \rightarrow \mathbb{P}^n$ ramified over a quartic hypersurface $B \subset \mathbb{P}^n$. Then $\pi_*\Omega_X^{[1]} \simeq \Omega_{\mathbb{P}^n}^1 \oplus \Omega_{\mathbb{P}^n}^{[1]}(\log B) \otimes \mathcal{O}_{\mathbb{P}^n}(-2)$. By the projection formula,

$$\begin{aligned} h^0(X, \Omega_X^{[1]} \otimes \mathcal{L}) &= h^0(\mathbb{P}^n, (\pi_*\Omega_X^{[1]}) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \\ &= h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1(1)) + h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{[1]}(\log B) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \\ &= 0. \end{aligned}$$

Suppose that $\mathcal{L}^n = 1$. By [23, Corollary 6.13], (X, \mathcal{L}) admits the following description. Set $Z := \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$, write $p : Z \rightarrow \mathbb{P}^{n-1}$ for the natural morphism, and $\mathcal{O}_Z(1)$ for the tautological line bundle. Let $B \in |\mathcal{O}_Z(3)|$ be a reduced divisor, $S \simeq \mathbb{P}^{n-1}$ the section of p corresponding to the surjection $\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}$, and set $D := B + S$. Observe that $S \cap B = \emptyset$, and $\mathcal{O}_Z(D) \simeq (\mathcal{O}_Z(2) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1))^{\otimes 2}$. Let $f : Y \rightarrow Z$ be the double cover of Z ramified over D , and set $E = f^{-1}(S) \simeq \mathbb{P}^{n-1}$. Then, for some choice of B as above, there is a birational morphism $\pi : Y \rightarrow X$ realizing Y as the blowup of X at a smooth point $x \in X$, with exceptional divisor E . Moreover, the composite morphism $\rho : Y \rightarrow Z \rightarrow \mathbb{P}^{n-1}$ is induced by the linear system $|\pi^* \mathcal{L} \otimes \mathcal{O}_Y(-E)|$.

By the projection formula, using that E is exceptional, we get

$$\begin{aligned} h^0(X, \Omega_X^{[1]} \otimes \mathcal{L}) &= h^0(X, \Omega_X^{[1]} \otimes \mathcal{L} \otimes \pi_* \mathcal{O}_Y(E)) \\ &= h^0(Y, \pi^* \Omega_X^{[1]} \otimes \pi^* \mathcal{L} \otimes \mathcal{O}_Y(E)). \end{aligned}$$

The short exact sequence

$$0 \rightarrow \pi^* \Omega_X^{[1]} \rightarrow \Omega_Y^{[1]} \rightarrow \Omega_E^1 \rightarrow 0$$

and the vanishing of $H^0(Y, \Omega_E^1 \otimes \pi^* \mathcal{L} \otimes \mathcal{O}_Y(E)) \simeq H^0(E, \Omega_E^1 \otimes \mathcal{O}_E(E)) \simeq H^0(\mathbb{P}^{n-1}, \Omega_{\mathbb{P}^{n-1}}^1 \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$ yield

$$h^0(Y, \pi^* \Omega_X^{[1]} \otimes \pi^* \mathcal{L} \otimes \mathcal{O}_Y(E)) = h^0(Y, \Omega_Y^{[1]} \otimes \pi^* \mathcal{L} \otimes \mathcal{O}_Y(E)).$$

Notice that $\pi^* \mathcal{L} \otimes \mathcal{O}_Y(E) \simeq f^* \mathcal{O}_Z(1) \otimes \rho^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Thus, by the projection formula,

$$\begin{aligned} h^0(Y, \Omega_Y^{[1]} \otimes \pi^* \mathcal{L} \otimes \mathcal{O}_Y(E)) &= h^0(Y, \Omega_Y^{[1]} \otimes f^* \mathcal{O}_Z(1) \otimes \rho^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \\ &= h^0(Z, f_* \Omega_Y^{[1]} \otimes \mathcal{O}_Z(1) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)). \end{aligned}$$

Since $f_* \Omega_Y^{[1]} \simeq \Omega_Z^1 \oplus \Omega_Z^{[1]}(\log(S+B)) \otimes \mathcal{O}_Z(-2) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)$, we conclude that

$$\begin{aligned} h^0(X, \Omega_X^{[1]} \otimes \mathcal{L}) &= h^0(Z, \Omega_Z^1 \otimes \mathcal{O}_Z(1) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \\ &\quad + h^0(Z, \Omega_Z^{[1]}(\log(S+B)) \otimes \mathcal{O}_Z(-1)). \end{aligned}$$

By restricting to a section of p corresponding to the surjection $\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}} \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(2)$, we see that $h^0(Z, \Omega_Z^1 \otimes \mathcal{O}_Z(1) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = 0$. To see the vanishing of the second summand above, notice that $\Omega_Z^{[1]}(\log(S+B)) \otimes \mathcal{O}_Z(-1) \subset \Omega_Z^{[1]}(\log B) \otimes \mathcal{O}_Z(S) \otimes \mathcal{O}_Z(-1)$. On the other hand, $\mathcal{O}_Z(S) \simeq \mathcal{O}_Z(1) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$, and so the desired vanishing follows from

$$h^0(Z, \Omega_Z^{[1]}(\log B) \otimes p^* \mathcal{O}_{\mathbb{P}^{n-1}}(-2)) = 0.$$

This completes the proof. □

2.4 A vanishing result

Rationally chain connected varieties with mild singularities do not carry nonzero reflexive forms (see [26]). We show that, under suitable conditions, the same holds for twisted reflexive forms.

The following observation will be used throughout the paper: if \mathcal{G} is a reflexive sheaf of \mathcal{O}_X -modules, and $i : U \hookrightarrow X$ is a dense open subset such that $\text{codim}(X \setminus U) \geq 2$, then $\mathcal{G} \simeq i_*\mathcal{G}|_U$. In particular, any section of \mathcal{G} on U extends to a section of \mathcal{G} on X .

Lemma 2.10 *Let (X, Δ) be a klt pair. Suppose that either $-(K_X + \Delta)$ is nef and big, or $K_X + \Delta \sim_{\mathbb{Q}} 0$ and Δ is big. Let D be an integral divisor on X such that $D \sim_{\mathbb{Q}} 0$. Then $h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) = 0$ for every positive integer q .*

Proof Let r be the smallest positive integer such that $rD \sim_{\mathbb{Z}} 0$, and let $\tau : X' \rightarrow X$ be the corresponding cyclic cover (see [34, Definition 2.52, Lemma 2.53]). Then τ is étale in codimension 1, and $\tau^{[*]}\mathcal{O}_X(D) \simeq \mathcal{O}_{X'}$. Set $\Delta' := \tau^*\Delta$, and notice that $\tau^*K_X = K_{X'}$. By [34, Proposition 5.20], (X', Δ') is klt.

Suppose that $-(K_X + \Delta)$ is nef and big. Then so is $-(K_{X'} + \Delta') \sim_{\mathbb{Q}} -\tau^*(K_X + \Delta)$. By Theorem 2.4, X' is rationally connected, and hence $h^0(X', \Omega_{X'}^{[q]}) = 0$ for any positive integer q by [26, Theorem 5.1]. On the other hand, any nonzero section of $\Omega_X^q[\otimes]\mathcal{O}_X(-D)$ induces a nonzero section of $\Omega_{X'}^{[q]}$. So we conclude that $h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) = 0$.

Suppose now that $K_X + \Delta \sim_{\mathbb{Q}} 0$ and Δ is big. Then there exist an ample \mathbb{Q} -divisor A and an effective \mathbb{Q} -divisor N on X such that $\Delta \sim_{\mathbb{Q}} A + N$, and (X, N) is klt. Then $-(K_X + N) \sim_{\mathbb{Q}} A$ is ample, and the claim follows from the previous case. \square

We are ready to state and prove our vanishing result.

Theorem 2.11 *Let (X, Δ) be a klt pair with Δ big and $K_X + \Delta$ pseudo-effective. Suppose that there exists an integral divisor D on X such that $K_X + \Delta \sim_{\mathbb{Q}} D$. Then $h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) = 0$ for every positive integer q .*

Proof We denote the dimension of X by n .

Consider the integral Weil divisor $\Gamma = D - K_X \sim_{\mathbb{Q}} \Delta$ on X .

Since $K_X + \Delta$ is pseudo-effective, the pair (X, Δ) has a minimal model $\varphi : X \dashrightarrow Y$ ([7, Corollary 1.4.2]). Set $\Delta_Y := \varphi_*\Delta$, $D_Y := \varphi_*D$ and $\Gamma_Y := \varphi_*\Gamma$, and notice that $K_Y = \varphi_*K_X$. Then (Y, Δ_Y) is klt, Δ_Y is big, and $K_Y + \Delta_Y \sim_{\mathbb{Q}} D_Y$ is nef. By the base point free Theorem ([7, Corollary 3.9.2]), $K_Y + \Delta_Y$ is semiample, and thus $\kappa := \kappa(\mathcal{O}_Y(K_Y + \Delta_Y)) \geq 0$.

Suppose first that $\kappa = 0$. Since $K_Y + \Delta_Y$ is semiample, $K_Y + \Delta_Y \sim_{\mathbb{Q}} 0$. By Lemma 2.10, for every positive integer q , $h^0(Y, \Omega_Y^q[\otimes]\mathcal{O}_Y(-D_Y)) = 0$. Since φ^{-1} does not contract any divisor, we conclude that

$$h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) = 0 \tag{2.1}$$

for every positive integer q .

Suppose from now on that $\kappa \geq 1$, and let $\pi : Y \rightarrow W$ be the surjective morphism onto a normal variety defined by the linear system $|mD_Y|$ for m sufficiently large and divisible. Denote by W_0 the smooth locus of W , and note that $\dim(W) = \kappa$.

Suppose that $h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) \neq 0$ for some $q \geq 1$. Let $\xi : \mathcal{O}_X(D) \hookrightarrow \Omega_X^{[q]}$ be the nonzero map associated to a nonzero global section of $\Omega_X^q[\otimes]\mathcal{O}_X(-D)$. Since φ^{-1} does not contract any divisors, ξ induces a nonzero map $\xi_Y : \mathcal{O}_Y(D_Y) \hookrightarrow \Omega_Y^{[q]}$. By restricting ξ_Y to the smooth locus Y_0 of Y , considering the perfect pairing $\tau_0 : \Omega_{Y_0}^{n-q} \otimes \Omega_{Y_0}^q \rightarrow \Omega_{Y_0}^n$, and extending back to Y , we see that ξ_Y induces a nonzero map $\eta_Y : \Omega_Y^{[n-q]} \rightarrow \mathcal{O}_Y(-\Gamma_Y)$.

Claim. Over $\pi^{-1}(W_0)$, the map η_Y factors through the canonical map $\Omega_Y^{[n-q]} \rightarrow \Omega_{Y/W}^{[n-q]}$.

To prove the claim, it is enough to show this factorization over $\pi^{-1}(W_0) \cap Y_0$. Fix $y \in Y_0$ such that $w = \pi(y) \in W_0$. At y , the kernel of $\Omega_{Y_0}^{n-q} \rightarrow \Omega_{Y_0/W}^{n-q}$ is generated by $(n - q)$ -forms of the form $\beta \wedge \pi^*\gamma$, where β is a local section of $\Omega_{Y_0}^{n-q-1}$ at y , and γ is a local section of Ω_W^1 at w . So we must show that $(\eta_Y)_y(\beta \wedge \pi^*\gamma) = 0$. Let ω be a local generator of $\Omega_{Y_0}^n$ at y , and e a local generator of $\mathcal{O}_Y(\Gamma_Y)$ at y . Then $\omega \otimes e$ is a local generator of $\mathcal{O}_Y(D_Y)$ at y , and

$$(\eta_Y)_y(\beta \wedge \pi^*\gamma) \otimes \omega \otimes e = (\tau_0)_y((\xi_Y)_y(\omega \otimes e) \wedge \beta \wedge \pi^*\gamma).$$

In order to prove the claim, we will show that

$$(\xi_Y)_y(\omega \otimes e) \wedge \pi^*\gamma = 0. \tag{2.2}$$

Given $s_0 \in H^0(Y, \mathcal{O}_Y(mD_Y))$ generating $\mathcal{O}_Y(mD_Y)$ at y , we can find (h_1, \dots, h_κ) a local system of coordinates at w such that $s_i = \pi^*h_i$, $s_0 \in H^0(Y, \mathcal{O}_Y(mD_Y))$. In order to prove (2.2), it is enough to show that, for every $1 \leq i \leq \kappa$, $(\xi_Y)_y(\omega \otimes e) \wedge \pi^*dh_i = 0$. This follows from the proof of the Bogomolov–Sommese vanishing Theorem (see [39, Theorem III]). For the convenience of the reader, we sketch the proof. Let $p : T \rightarrow Y$ be a resolution of singularities of Y . By [26, Theorem 4.3], there exists a nonzero map $p^*\Omega_Y^{[q]} \rightarrow \Omega_T^q$. Thus, there exists a nonzero map $p^{[*]}\mathcal{O}_Y(D_Y) \rightarrow \Omega_T^{[q]} \simeq \Omega_T^q$. Replacing Y with T , and $\mathcal{O}_Y(D_Y)$ with $p^{[*]}\mathcal{O}_Y(D_Y)$, we may assume that Y is smooth. By taking successively m -th root out of the s_i 's we may also assume that $m = 1$, and that $s_0 = \omega \otimes e$. For $0 \leq i \leq \kappa$, set $\alpha_i := \xi_Y(s_i) \in H^0(Y, \Omega_Y^1)$. Then $d\alpha_i = 0$. On the other hand, $d\alpha_i = -\pi^*dh_i \wedge \alpha_0$. This proves the claim.

Next we show that $q = \kappa$. Let $w \in W_0$ be a general point, and set $F := \pi^{-1}(w)$. Then $(F, \Delta_{Y|F})$ is klt, and the adjunction formula gives $D_{Y|F} \sim_{\mathbb{Q}} K_F + \Delta_{Y|F} \sim_{\mathbb{Q}} 0$. Since η_Y factors through the canonical map $\Omega_Y^{[n-q]} \rightarrow \Omega_{Y/W}^{[n-q]}$, it induces a nonzero map $\Omega_F^{[n-q]} \rightarrow \mathcal{O}_F(-\Gamma_{Y|F})$. Thus $h^0(F, \Omega_F^{\dim(F)-n+q}[\otimes]\mathcal{O}_F(-D_{Y|F})) \neq 0$, and Lemma 2.10 yields $\dim(F) = n - q$. Hence $q = \kappa$.

Let $C \subset W$ be a general complete intersection curve. Then $C \subset W_0$, C is smooth, and $Y_C = \pi^{-1}(C) \subset Y$ is a normal projective variety equipped with a flat morphism $\pi_C : Y_C \rightarrow C$ whose general fibers are reduced. Moreover, η_Y restricts to a nonzero

map $\eta_{Y_C} : \Omega_{Y_C/C}^{[n-q]} \rightarrow \mathcal{O}_{Y_C}(-\Gamma_{Y|Y_C})$. By [8, Theorem 2.1'], there exists a smooth complete curve B , together with a finite morphism $B \rightarrow C$, such that $\varpi : Z \rightarrow B$ has reduced fibers, where Z is the normalization of the fiber product $Y_C \times_C B$, and $\varpi : Z \rightarrow B$ is the natural morphism. Let $r : Y_C \times_C B \rightarrow Y_C$ denote the natural morphism, and set $(Y_C \times_C B)_0 := r^{-1}(Y_C \setminus \text{Sing}(X))$. The map η_{Y_C} induces a nonzero map $\Omega_{(Y_C \times_C B)_0/B}^{[n-q]} \rightarrow \mathcal{O}_{(Y_C \times_C B)_0}(-\Gamma_{Y|(Y_C \times_C B)_0})$. By [2, Proposition 4.5], this extends to a nonzero map

$$\eta_Z : \Omega_{Z/B}^{[n-q]} \rightarrow \mathcal{O}_Z(-\Gamma_{Y|Z}),$$

using the fact that $(Y_C \times_C B) \setminus (Y_C \times_C B)_0$ has codimension at least two in $Y_C \times_C B$. By [14, Lemme 4.4], $\Omega_{Z/B}^{[n-q]} \simeq \mathcal{O}_Z(K_{Z/B})$. If we denote by F a general fiber of ϖ , then $K_F + \Delta_{Y|F} \sim_{\mathbb{Q}} 0$ and $\Gamma_{Y|Z} \sim_{\mathbb{Q}} \Delta_{Y|Z}$. Thus there exists an effective divisor Ψ on Z such that $\varpi(\text{Supp}(\Psi)) \subsetneq B$ and

$$-(K_{Z/B} + \Psi) = \Gamma_{Y|Z} \sim_{\mathbb{Q}} \Delta_{Y|Z}.$$

Since $\Delta_{Y|Z}$ is big, we can write $\Delta_{Y|Z} \sim_{\mathbb{Q}} A + N$, with A an ample \mathbb{Q} -divisor and N an effective \mathbb{Q} -divisor. Since $(Z, \Delta_{Y|Z})$ is klt over the generic point of B , so is $(Z, (1 - \varepsilon)\Delta_{Y|Z} + \varepsilon N)$ for $\varepsilon > 0$ sufficiently small. On the other hand,

$$-(K_{Z/B} + \Psi + (1 - \varepsilon)\Delta_{Y|Z} + \varepsilon N) \sim_{\mathbb{Q}} \varepsilon A$$

is ample, contradicting [2, Theorem 3.1]. We conclude that

$$h^0(X, \Omega_X^q[\otimes]\mathcal{O}_X(-D)) = 0.$$

□

3 Foliations

3.1 Foliations and Pfaff fields

Definition 3.1 Let X be a normal variety. A *foliation* on X is a nonzero coherent subsheaf $\mathcal{F} \subsetneq T_X$ satisfying

1. \mathcal{F} is closed under the Lie bracket, and
2. \mathcal{F} is saturated in T_X (i.e., T_X/\mathcal{F} is torsion free).

The *rank* r of \mathcal{F} is the generic rank of \mathcal{F} .

The *canonical class* $K_{\mathcal{F}}$ of \mathcal{F} is any Weil divisor on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \simeq \det(\mathcal{F})$.

3.2 (Foliations defined by q -forms) *Let \mathcal{F} be a codimension q foliation on an n -dimensional normal variety X . The normal sheaf of \mathcal{F} is $N_{\mathcal{F}} := (T_X/\mathcal{F})^{**}$. The q -th wedge product of the inclusion $N_{\mathcal{F}}^* \hookrightarrow (\Omega_X^1)^{**}$ gives rise to a nonzero global section*

$\omega \in H^0(X, \Omega_X^q[\otimes] \det(N_{\mathcal{F}}))$ whose zero locus has codimension at least 2 in X . Such ω is locally decomposable and integrable. To say that ω is locally decomposable means that, in a neighborhood of a general point of X , ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \dots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for all $i \in \{1, \dots, q\}$.

Conversely, let \mathcal{L} be a reflexive sheaf of rank 1 on X , and $\omega \in H^0(X, \Omega_X^q[\otimes] \mathcal{L})$ a global section whose zero locus has codimension at least 2 in X . Suppose that ω is locally decomposable and integrable. Then we define a foliation of rank $r = n - q$ on X as the kernel of the morphism $T_X \rightarrow \Omega_X^{q-1}[\otimes] \mathcal{L}$ given by the contraction with ω . These constructions are inverse of each other.

Definition 3.3 We say that a foliation \mathcal{F} on a normal variety is \mathbb{Q} -Gorenstein if its canonical class $K_{\mathcal{F}}$ is \mathbb{Q} -Cartier. We say it is Gorenstein if $K_{\mathcal{F}}$ is Cartier.

Definition 3.4 Let X be a variety, and r a positive integer. A Pfaff field of rank r on X is a nonzero map $\eta : \Omega_X^r \rightarrow \mathcal{L}$, where \mathcal{L} is a reflexive sheaf of rank 1 on X such that $\mathcal{L}^{[m]}$ is invertible for some integer $m \geq 1$. The singular locus S of η is the closed subscheme of X whose ideal sheaf \mathcal{I}_S is the image of the induced map $\Omega_X^r[\otimes] \mathcal{L}^* \rightarrow \mathcal{O}_X$.

A closed subscheme Y of X is said to be invariant under η if

1. no irreducible component of Y is contained in the singular locus of η , and
2. the restriction $\otimes^m \eta|_Y : \otimes^m \Omega_{X|Y}^r \rightarrow \mathcal{L}^{[m]}|_Y$ factors through the natural map $\otimes^m \Omega_{X|Y}^r \rightarrow \otimes^m \Omega_Y^r$, where $m \geq 1$ is such that $\mathcal{L}^{[m]}$ is invertible. In other words, there is a commutative diagram

$$\begin{array}{ccc}
 \otimes^m \Omega_{X|Y}^r & \xrightarrow{\otimes^m \eta|_Y} & \mathcal{L}^{[m]}|_Y, \\
 \downarrow & \nearrow & \\
 \otimes^m \Omega_Y^r & &
 \end{array}$$

where the vertical map is the natural one.

Suppose that Y is reduced and set $Y_0 := Y \setminus \text{Sing}(Y)$. Observe that $\mathcal{L}|_{Y_0}$ is locally free (see [27, Proposition 1.9]). Suppose that condition (1) above is satisfied. Then, Y is invariant under η if and only if the restriction $\eta|_{Y_0} : \Omega_{X|Y_0}^r \rightarrow \mathcal{L}|_{Y_0}$ factors through the natural map $\Omega_{X|Y_0}^r \rightarrow \Omega_{Y_0}^r$.

Notice that a \mathbb{Q} -Gorenstein foliation \mathcal{F} of rank r on normal variety X naturally gives rise to a Pfaff field of rank r on X :

$$\eta : \Omega_X^r = \wedge^r(\Omega_X^1) \rightarrow \wedge^r(T_X^*) \rightarrow \wedge^r(\mathcal{F}^*) \rightarrow \det(\mathcal{F}^*) \simeq \mathcal{O}_X(K_{\mathcal{F}}).$$

Definition 3.5 Let \mathcal{F} be a \mathbb{Q} -Gorenstein foliation on a normal variety X . The singular locus of \mathcal{F} is defined to be the singular locus S of the associated Pfaff field. We say that \mathcal{F} is regular at a point $x \in X$ if $x \notin S$. We say that \mathcal{F} is regular if $S = \emptyset$.

Example 3.6 ([1], Remark 3.12) The notion of regularity of foliations discussed above does not say anything about the singularities of the ambient space. For instance, let Y be a smooth variety, T any normal variety, and set $X := Y \times T$, with natural projection $p : X \rightarrow Y$. Set $\mathcal{F} := p^*T_Y \subset T_X$. Then \mathcal{F} is a regular Gorenstein foliation, while X may be very singular.

Our definition of Pfaff field is more general than the one usually found in the literature, where \mathcal{L} is required to be invertible. This generalization is needed in order to treat \mathbb{Q} -Gorenstein foliations whose canonical classes are not Cartier. Under the more restrictive definition, it was proved in [2, Proposition 4.5] that Pfaff fields extend uniquely to the normalization. The next lemma generalizes this to our current setting.

Lemma 3.7 *Let X be a normal variety, and \mathcal{L} a torsion free sheaf of rank 1 on X such that $\mathcal{L}^{[m]}$ is locally free for some integer $m \geq 1$. Let r be a positive integer, and $\eta : \Omega_X^r \rightarrow \mathcal{L}$ a Pfaff field. Let $F \subset X$ be an invariant integral closed subscheme, and $\tilde{e} : \tilde{F} \rightarrow X$ its normalization. Then the map $\otimes^m \Omega_F^r \rightarrow \mathcal{L}^{[m]}|_F$ extends uniquely to a generically surjective map $\otimes^m \Omega_{\tilde{F}}^r \rightarrow \tilde{e}^* \mathcal{L}^{[m]}$.*

Proof It is enough to prove the claim on some open cover of F . Thus, by replacing X with an open subset if necessary, we may assume that $\mathcal{L}^{[m]} \simeq \mathcal{O}_X$. Let $\pi : Y \rightarrow X$ be the corresponding cyclic cover (see [34, Definition 2.52, Lemma 2.53]), with Galois group G . Then π is étale in codimension one, and $\pi^{[*]} \mathcal{L} \simeq \mathcal{O}_Y$ is locally free. Thus η induces a Pfaff field $\zeta : \Omega_Y^r \rightarrow \pi^{[*]} \mathcal{L}$ on Y , and $Z := \pi^{-1}(F) \subset Y$ is invariant under ζ . Let $\tilde{f} : \tilde{Z} \rightarrow Y$ be the normalization of Z , and denote by $\tilde{\pi} : \tilde{Z} \rightarrow \tilde{F}$ the induced morphism. Then ζ induces a map $\Omega_Z^r \rightarrow \pi^{[*]} \mathcal{L}|_Z$, which extends uniquely to a generically surjective map $\Omega_{\tilde{Z}}^r \rightarrow \tilde{f}^* \pi^{[*]} \mathcal{L}$ by [2, Proposition 4.5]. So we obtain a nonzero map

$$\xi : \otimes^m \tilde{\pi}^* \Omega_{\tilde{F}}^r \xrightarrow{\otimes^m d\tilde{\pi}} \otimes^m \Omega_{\tilde{Z}}^r \rightarrow \otimes^m \tilde{f}^* \pi^{[*]} \mathcal{L} \simeq \tilde{\pi}^* \tilde{e}^* \mathcal{L}^{[m]}.$$

Observe that the natural action of G on Z extends to an action on \tilde{Z} such that $\tilde{Z}/G \simeq \tilde{Y}$. Moreover, $\otimes^m \tilde{\pi}^* \Omega_{\tilde{F}}^r$ and $\tilde{\pi}^* \tilde{e}^* \mathcal{L}$ are naturally G -linearized sheaves on \tilde{Z} and ξ is G -equivariant. Our claim follows. \square

3.2 Algebraically integrable foliations

Definition 3.8 Let X be a normal variety. A foliation \mathcal{F} on X is said to be *algebraically integrable* if the leaf of \mathcal{F} through a general point of X is an algebraic variety. In this situation, by abuse of notation we often use the word *leaf* to mean the closure in X of a leaf of \mathcal{F} .

The following Lemma is well known to experts. See for instance [1, Lemma 3.2].

Lemma 3.9 *Let X be a normal projective variety, and \mathcal{F} an algebraically integrable foliation on X . There is a unique irreducible closed subvariety W of $\text{Chow}(X)$ whose general point parametrizes the closure of a general leaf of \mathcal{F} (viewed as a reduced*

and irreducible cycle in X). In other words, if $U \subset W \times X$ is the universal cycle, with universal morphisms $\pi : U \rightarrow W$ and $e : U \rightarrow X$, then e is birational, and, for a general point $w \in W$, $e(\pi^{-1}(w)) \subset X$ is the closure of a leaf of \mathcal{F} .

Notation 3.10 We say that the subvariety W provided by Lemma 3.9 is the closure in $\text{Chow}(X)$ of the subvariety parametrizing general leaves of \mathcal{F} . We call the induced rational map $X \dashrightarrow W$ a rational first integral for \mathcal{F} .

In [1, Definition 3.4] we introduced the notion of general log leaf for algebraically integrable Gorenstein foliations. Next we extend this to the \mathbb{Q} -Gorenstein case.

Definition 3.11 Let X be a normal projective variety, \mathcal{F} a \mathbb{Q} -Gorenstein algebraically integrable foliation of rank r on X , and $\eta : \Omega^r_X \rightarrow \mathcal{O}_X(K_{\mathcal{F}})$ the corresponding Pfaff field. Let $F \subset X$ be the closure of a general leaf of \mathcal{F} , and $\tilde{e} : \tilde{F} \rightarrow X$ the normalization of F . Let $m \geq 1$ be the Cartier index of $K_{\mathcal{F}}$, i.e., the smallest positive integer m such that $mK_{\mathcal{F}}$ is Cartier. By [1, Lemma 2.7], F is invariant under η , i.e., the restriction $\otimes^m \eta|_F : \otimes^m \Omega^r_{X|F} \rightarrow \mathcal{O}_X(mK_{\mathcal{F}})|_F$ factors through the natural map $\otimes^m \Omega^r_{X|F} \rightarrow \otimes^m \Omega^r_Y$. By Lemma 3.7, the induced map $\otimes^m \Omega^r_F \rightarrow \mathcal{O}_X(mK_{\mathcal{F}})|_F$ extends uniquely to a generically surjective map $\tilde{\eta} : \otimes^m \Omega^r_{\tilde{F}} \rightarrow \tilde{e}^* \mathcal{O}_X(mK_{\mathcal{F}})$. Hence there is a canonically defined effective Weil \mathbb{Q} -divisor $\tilde{\Delta}$ on \tilde{F} such that $mK_{\tilde{F}} + m\tilde{\Delta} \sim_{\mathbb{Z}} \tilde{e}^* mK_{\mathcal{F}}$. Namely, $m\tilde{\Delta}$ is the divisor of zeroes of $\tilde{\eta}$.

We call the pair $(\tilde{F}, \tilde{\Delta})$ a general log leaf of \mathcal{F} .

Remark 3.12 Let X be a normal projective variety, and \mathcal{F} an algebraically integrable \mathbb{Q} -Gorenstein foliation of rank r on X . Let W be the closure in $\text{Chow}(X)$ of the subvariety parametrizing general leaves of \mathcal{F} , and $U \subset W \times X$ the universal cycle. Denote by $e : U \rightarrow X$ the natural morphism. Let m be the Cartier index of $K_{\mathcal{F}}$. Then \mathcal{F} induces a map $\otimes^m \Omega^r_U \rightarrow e^* \mathcal{O}_X(mK_{\mathcal{F}})$, which factors through the natural morphism $\otimes^m \Omega^r_U \rightarrow \otimes^m \Omega^r_{U/W}$ (see [1, Remark 3.8 and Lemma 3.2]).

Let \tilde{W} and \tilde{U} be the normalizations of W and U , respectively, and denote by $\tilde{\pi} : \tilde{U} \rightarrow \tilde{W}$ and $\tilde{e} : \tilde{U} \rightarrow X$ the induced morphisms. By Lemma 3.7, the map $\otimes^m \Omega^r_U \rightarrow e^* \mathcal{O}_X(mK_{\mathcal{F}})$ extends uniquely to a nonzero map $\otimes^m \Omega^r_{\tilde{U}} \rightarrow \tilde{e}^* \mathcal{O}_X(mK_{\mathcal{F}})$. As before, this map factors through the natural map $\otimes^m \Omega^r_{\tilde{U}} \rightarrow \otimes^m \Omega^r_{\tilde{U}/\tilde{W}}$, yielding a generically surjective map

$$\otimes^m \Omega^r_{\tilde{U}/\tilde{W}} \rightarrow \tilde{e}^* \mathcal{O}_X(mK_{\mathcal{F}}).$$

Thus there is a canonically defined effective Weil \mathbb{Q} -divisor Δ on \tilde{U} such that $K_{\tilde{U}/\tilde{W}} + \Delta \sim_{\mathbb{Q}} \tilde{e}^* K_{\mathcal{F}}$.

Let w be a general point of \tilde{W} , set $\tilde{U}_w := \tilde{\pi}^{-1}(w)$ and $\Delta_w := \Delta|_{\tilde{U}_w}$. Then (\tilde{U}_w, Δ_w) coincides with the general log leaf $(\tilde{F}, \tilde{\Delta})$ defined above.

The same construction can be carried out for any base change $V \rightarrow W$.

Next we define notions of singularity for \mathbb{Q} -Gorenstein algebraically integrable foliations according to the singularity type of their general log leaf.

Definition 3.13 Let X be a normal projective variety, \mathcal{F} a \mathbb{Q} -Gorenstein algebraically integrable foliation on X , and $(\tilde{F}, \tilde{\Delta})$ its general log leaf. We say that \mathcal{F} has *log terminal singularities along a general leaf* if $(\tilde{F}, \tilde{\Delta})$ is log terminal.

Proposition 3.14 *Let \mathcal{F} be an algebraically integrable \mathbb{Q} -Gorenstein foliation on a normal projective variety X . Suppose that \mathcal{F} has log terminal singularities along a general leaf. Then $\det(\mathcal{F})$ is not nef and big.*

Proof The proof is the same as the proof of [1, Proposition 5.8]. One only needs to replace the use of [2, Proposition 4.5] with Lemma 3.7. □

3.3 \mathbb{Q} -Fano foliations

Definition 3.15 Let X be a normal projective variety, and \mathcal{F} a rank r \mathbb{Q} -Gorenstein foliation on X .

We say that \mathcal{F} is a *\mathbb{Q} -Fano foliation* if $-K_{\mathcal{F}}$ is ample. In this case, the index of \mathcal{F} is the largest positive rational number $i_{\mathcal{F}}$ such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}} H$ for a Cartier divisor H on X .

We say that a \mathbb{Q} -Fano foliation \mathcal{F} is a *del Pezzo foliation* if $r \geq 2$ and $i_{\mathcal{F}} = r - 1$.

Remark 3.16 Let \mathcal{F} be \mathbb{Q} -Fano foliation of rank r and index $i_{\mathcal{F}}$ on a normal projective variety X , and let H be an ample divisor on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}} H$. Suppose that $K_{\mathcal{F}}$ is Cartier. In general, we do not have $-K_{\mathcal{F}} \sim_{\mathbb{Z}} i_{\mathcal{F}} H$. However, this holds if X is a klt \mathbb{Q} -Fano variety by Lemma 2.5.

Proof of Theorem 1.4 Let X be a klt projective variety, and $\mathcal{F} \subsetneq T_X$ a \mathbb{Q} -Fano foliation. We assume to the contrary that $K_X - K_{\mathcal{F}}$ is pseudo-effective. Set $n := \dim(X)$, and denote by r the rank of \mathcal{F} , $1 \leq r \leq n - 1$. The r -th wedge product of the inclusion $\mathcal{F} \subsetneq T_X$ gives rise to a nonzero map $\xi : \mathcal{O}_X(K_X) \otimes \det(\mathcal{F}) \hookrightarrow \Omega_X^{[n-r]}$. Thus $h^0(X, \Omega_X^{n-r}[\otimes] \mathcal{O}_X(-K_X + K_{\mathcal{F}})) \neq 0$. Choose a \mathbb{Q} -divisor $\Delta \sim_{\mathbb{Q}} -K_{\mathcal{F}}$ such that the pair (X, Δ) is klt, and set $D := K_X - K_{\mathcal{F}}$. Then D is an integral divisor on X , $K_X + \Delta \sim_{\mathbb{Q}} D$, and $h^0(X, \Omega_X^{n-r}[\otimes] \mathcal{O}_X(-D)) \neq 0$. This contradicts Theorem 2.11 since $n - r \geq 1$. □

The following is an immediate consequence of Theorem 1.4.

Corollary 3.17 *Let X be a klt projective variety with Picard number $\rho(X) = 1$, and \mathcal{F} a \mathbb{Q} -Fano foliation of index $i_{\mathcal{F}}$ on X . Then X is a \mathbb{Q} -Fano variety, and its index satisfies $i_X > i_{\mathcal{F}}$.*

4 The Kobayashi–Ochiai Theorem for foliations

For Fano foliations on smooth projective varieties, the index is at most the rank, and equality holds only for degree 0 foliations on \mathbb{P}^n ([1, Theorem 2.10]). The goal of this section is to extend this result to the singular case. We start by noticing that by allowing singularities we get more examples of \mathbb{Q} -Fano foliations with index equal to the rank, which we describe now.

Definition 4.1 Let \mathcal{M} be an ample line bundle on a normal projective variety Z , and r a positive integer. Consider the \mathbb{P}^r -bundle $Y = \mathbb{P}_Z(\mathcal{M} \oplus \mathcal{O}_Z^{\oplus r})$, with natural projection $\pi : Y \rightarrow Z$. The tautological line bundle $\mathcal{O}_Y(1)$ is semiample on Y . For $m \gg 0$, the linear system $|\mathcal{O}_Y(m)|$ induces a birational morphism $e : Y \rightarrow X$ onto a normal projective variety. The morphism e contracts the divisor $E = \mathbb{P}_Z(\mathcal{O}_Z^{\oplus r}) \subset Y$ corresponding to the projection $\mathcal{M} \oplus \mathcal{O}_Z^{\oplus r} \rightarrow \mathcal{O}_Z^{\oplus r}$ onto $e(E) \simeq \mathbb{P}^{r-1}$, and induces an isomorphism $Y \setminus E \simeq X \setminus e(E)$. Following [10], we call X the *normal generalized cone over the base (Z, \mathcal{M}) with vertex $e(E) \simeq \mathbb{P}^{r-1}$* . If $r = 1$, then X is called a normal cone over the base (Z, \mathcal{M}) .

Remark 4.2 Let the notation be as in Definition 4.1.

If $\rho(Z) = 1$ and Z is \mathbb{Q} -factorial, then the same holds for X .

If X is factorial, then so is Z .

Suppose that Z is klt, and $K_Z \sim_{\mathbb{Q}} -(1 + a)c_1(\mathcal{M})$ for some $a \in \mathbb{Q}$. Then Y is also klt, and $K_Y \sim_{\mathbb{Q}} \pi^*K_Z - (r + 1)c_1(\mathcal{O}_Y(1)) + \pi^*c_1(\mathcal{M}) \sim_{\mathbb{Q}} -(r + 1)c_1(\mathcal{O}_Y(1)) - a\pi^*c_1(\mathcal{M})$. Let \mathcal{L} be a line bundle on X such that $e^*\mathcal{L} \simeq \mathcal{O}_Y(1)$. Then $e_*\pi^*c_1(\mathcal{M}) \sim_{\mathbb{Z}} c_1(\mathcal{L})$, and $E \sim_{\mathbb{Z}} c_1(\mathcal{O}_Y(1)) - \pi^*c_1(\mathcal{M}) \sim_{\mathbb{Z}} e^*c_1(\mathcal{L}) - \pi^*c_1(\mathcal{M})$. Thus $K_X = e_*K_Y \sim_{\mathbb{Q}} -(r + a + 1)c_1(\mathcal{L})$ is \mathbb{Q} -Cartier, and $K_Y \sim_{\mathbb{Q}} e^*K_X + aE$. By [33, Lemma 3.10)], we conclude that X is klt if and only if $a > -1$.

4.3 Let the notation be as in Definition 4.1 and Remark 4.2. Let \mathcal{F} be the foliation of rank r on X induced by the rational map $X \dashrightarrow Z$. Then $-K_{\mathcal{F}} = rc_1\mathcal{L}$.

We will show that, under suitable conditions, the \mathbb{Q} -Fano foliations described in 4.3 are the only ones for which the index equals the rank (Theorem 1.2).

Proposition 4.4 Let X be a klt projective variety of dimension n . If X admits a \mathbb{Q} -Fano foliation $\mathcal{F} \subsetneq T_X$ of index $i_{\mathcal{F}} \geq n - 1$, then $\rho(X) = 1$.

Proof Let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}}c_1(\mathcal{L})$. By Theorem 1.4, $K_X + i_{\mathcal{F}}c_1(\mathcal{L})$ is not nef. If $\rho(X) \geq 2$, then, by [3, Theorem 2.1], there is a surjective morphism $p : X \rightarrow Y$ onto a smooth projective curve Y such that $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ for a general fiber F of p . Let $\ell \subset F \simeq \mathbb{P}^{n-1}$ be a general line. Then X is smooth in a neighborhood of ℓ and $\mathcal{F}|_{\ell} \subset T_{X|\ell} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2} \oplus \mathcal{O}_{\mathbb{P}^1}$. Thus, either $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-3} \subset \mathcal{F}|_{\ell}$, or $\mathcal{F}|_{\ell} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}$. So we must have $\mathcal{F} = T_{X/Y}$, which contradicts Proposition 3.14. \square

Proposition 4.5 Let X be a normal projective variety, and $\mathcal{F} \subsetneq T_X$ an algebraically integrable \mathbb{Q} -Fano foliation of rank r . Then

1. $i_{\mathcal{F}} \leq r$.
2. If $i_{\mathcal{F}} = r$, then the general log leaf of \mathcal{F} satisfies $(\tilde{F}, \tilde{\Delta}) \simeq (\mathbb{P}^r, H)$, where H is a hyperplane in \mathbb{P}^r .

Proof Let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}}c_1(\mathcal{L})$.

We denote by $\tilde{e} : \tilde{F} \rightarrow X$ the natural morphism. Recall from Definition 3.11 that

$$-(K_{\tilde{F}} + \tilde{\Delta}) \sim_{\mathbb{Q}} -\tilde{e}^*K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}}c_1(\tilde{e}^*\mathcal{L}).$$

Suppose that $i_{\mathcal{F}} > r$. Then $\tilde{F} \simeq \mathbb{P}^r$ and $\deg(\tilde{\Delta}) = r + 1 - i$ by Theorem 1.1. In particular $(\tilde{F}, \tilde{\Delta})$ is klt by Lemma 2.2. But this contradicts Proposition 3.14. So $i_{\mathcal{F}} \leq r$.

Suppose $i_{\mathcal{F}} = r$. By Theorem 1.1, one of the following holds.

- Either $(\tilde{F}, \tilde{e}^*\mathcal{L}) \simeq (\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ and $\deg(\tilde{\Delta}) = 1$, or
- $\tilde{\Delta} = 0$ and $(\tilde{F}, \tilde{e}^*\mathcal{L}) \simeq (Q_r, \mathcal{O}_{Q_r}(1))$ where Q_r is a possibly singular quadric hypersurface in \mathbb{P}^{r+1} .

In the first case, we conclude that $\tilde{\Delta} = H$ is a hyperplane in \mathbb{P}^r by Lemma 2.2 and Proposition 3.14. The latter case does not occur by Proposition 3.14. □

The next lemma is useful to prove algebraic integrability of \mathbb{Q} -Fano foliations of high index. First we recall the notion of slope of a torsion-free sheaf.

4.6 *Let X be an n -dimensional projective variety, and \mathcal{L} an ample line bundle on X . The slope with respect to \mathcal{L} of a torsion-free sheaf \mathcal{F} of rank r on X is $\mu_{\mathcal{L}}(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot \mathcal{L}^{n-1}}{r}$.*

Lemma 4.7 ([1], Proposition 7.5) *Let X be a normal projective variety, \mathcal{L} an ample line bundle on X , and $\mathcal{F} \subsetneq T_X$ a foliation on X . Suppose that $\mu_{\mathcal{L}}(\mathcal{F}) > 0$, and let $C \subset X$ be a general complete intersection curve. Then either*

- \mathcal{F} is algebraically integrable with rationally connected general leaves, or
- there exists an algebraically integrable subfoliation $\mathcal{G} \subsetneq \mathcal{F}$ with rationally connected general leaf such that $\det(\mathcal{G}) \cdot C \geq \det(\mathcal{F}) \cdot C$.

Proposition 4.8 *Let X be a normal \mathbb{Q} -factorial projective variety, and assume that $\text{Pic}(X)/\text{tors} \simeq \mathbb{Z}$. Let $\mathcal{F} \subsetneq T_X$ be a \mathbb{Q} -Fano foliation of rank r and index $i_{\mathcal{F}} > r - 1$. Then \mathcal{F} is algebraically integrable with rationally connected general leaves.*

Proof Let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}} c_1(\mathcal{L})$. Then $\text{Pic}(X)/\text{tors} = \mathbb{Z}[\mathcal{L}]$.

By Lemma 4.7, either \mathcal{F} is algebraically integrable with rationally connected general leaves, or there is an algebraically integrable foliation $\mathcal{G} \subsetneq \mathcal{F}$ of rank $s \leq r - 1$ and index $i_{\mathcal{G}} \geq i_{\mathcal{F}}$. In the latter case, $i_{\mathcal{G}} \geq i_{\mathcal{F}} > r - 1 \geq s$. But this contradicts Proposition 4.5. Our claim follows. □

The following corollary is an immediate consequence of Propositions 4.5 and 4.8.

Corollary 4.9 *Let X be a normal \mathbb{Q} -factorial projective variety such that $\text{Pic}(X)/\text{tors} \simeq \mathbb{Z}$. Let $\mathcal{F} \subsetneq T_X$ be a \mathbb{Q} -Fano foliation of rank r . Then*

1. $i_{\mathcal{F}} \leq r$.
2. If $i_{\mathcal{F}} = r$, then \mathcal{F} is algebraically integrable, and the general log leaf of \mathcal{F} satisfies $(\tilde{F}, \tilde{\Delta}) \simeq (\mathbb{P}^r, H)$, where H is a hyperplane in \mathbb{P}^r .

In the situation of Corollary 4.9(2), we want to prove that X is a normal generalized cone. The idea is to look at the family of leaves $\pi : U \rightarrow W$ described in Lemma 3.9. The first step is to show that π is a \mathbb{P}^r -bundle. For that we need the following characterization of projective bundles, which extends [21, Lemma 2.12] to the singular setting.

Proposition 4.10 *Let X be a normal variety, and $p : X \rightarrow Y$ an equidimensional projective morphism of relative dimension k onto a normal variety. Suppose that there exists a p -ample line bundle \mathcal{L} on X such that, for a general point $y \in Y$, $(X_y, \mathcal{L}|_{X_y}) \simeq (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Then $(X, \mathcal{L}) \simeq (\mathbb{P}_Y(\mathcal{E}), \mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1))$ as varieties over Y , where $\mathcal{E} := p_*\mathcal{L}$ is a vector bundle of rank $r + 1$.*

Proof By [29, Proposition 3.1], all fibers of p are irreducible and generically reduced. Moreover, the normalization of any fiber is isomorphic to \mathbb{P}^k , and $\mathcal{L}|_{\mathbb{P}^k} \simeq \mathcal{O}_{\mathbb{P}^k}(1)$. Apply [35, Theorem 12] to conclude that $p : X \rightarrow Y$ is a \mathbb{P}^k -bundle. \square

Proof of Theorem 1.2 Let X be an n -dimensional \mathbb{Q} -factorial klt projective variety with $n \geq 2$, and $\mathcal{F} \subsetneq T_X$ a \mathbb{Q} -Fano foliation of rank r .

Suppose first $\rho(X) = 1$. By Corollary 3.17, X is a klt \mathbb{Q} -Fano variety. By Lemma 2.5, $\text{Pic}(X) \simeq \mathbb{Z}$. Corollary 4.9 then implies that $i_{\mathcal{F}} \leq r$.

Suppose that $i_{\mathcal{F}} = r$, and let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} rc_1(\mathcal{L})$. By Corollary 4.9, \mathcal{F} is algebraically integrable, and the general log leaf of \mathcal{F} satisfies $(\tilde{F}, \tilde{\Delta}) \simeq (\mathbb{P}^r, H)$, where H is a hyperplane in \mathbb{P}^r . Let $\tilde{e} : \tilde{F} \rightarrow X$ denote the natural morphism. Then $\tilde{e}^*\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^r}(1)$.

Let W be the normalization of the closure in $\text{Chow}(X)$ of the subvariety parametrizing general leaves of \mathcal{F} , and U the normalization of the universal cycle over W , with universal family morphisms:

$$\begin{array}{ccc} U & \xrightarrow{e} & X \\ \pi \downarrow & & \\ W & & \end{array}$$

The line bundle $e^*\mathcal{L}$ is π -ample and restricts to $\mathcal{O}_{\mathbb{P}^r}(1)$ on a general fiber $U_w \simeq \mathbb{P}^r$ of π . By Proposition 4.10, π is a \mathbb{P}^r -bundle. More precisely, setting $\mathcal{E} := \pi_*e^*\mathcal{L}$, U is isomorphic to $\mathbb{P}_W(\mathcal{E})$ over W , and under this isomorphism $e^*\mathcal{L}$ corresponds to the tautological line bundle $\mathcal{O}_{\mathbb{P}_W(\mathcal{E})}(1)$.

We will show that the above diagram realizes X as a normal generalized cone over $(W, \text{det}(\mathcal{E}))$.

By Remark 3.12, there is a canonically defined \mathbb{Q} -Weil divisor B on U such that $K_{U/W} + B \sim_{\mathbb{Q}} e^*K_{\mathcal{F}}$. Notice that since $K_{U/W}$ and $e^*K_{\mathcal{F}}$ are \mathbb{Q} -Cartier divisors, so is B . Since $\tilde{\Delta}$ is integral, the support of B has a unique irreducible component B' that dominates W . Moreover, B' appears with coefficient 1 in B . Let E be the exceptional locus of e . Note that E has pure codimension 1 since X is \mathbb{Q} -factorial. We shall show that $B = B' = E$.

Let m be a positive integer such that mB is integral. Recall that $\mathcal{O}_U(K_{U/W}) \simeq \mathcal{O}_{\mathbb{P}_W(\mathcal{E})}(-r - 1) \otimes \pi^*\text{det}(\mathcal{E})$, and so $mB \in |\mathcal{O}_U(m) \otimes \pi^*\text{det}(\mathcal{E})^{\otimes -m}|$. Let C be a smooth complete curve, and $C \rightarrow W$ a non-constant morphism whose image is not contained in the image of any irreducible component of $\text{Supp}(B)$ distinct from B' . Let U_C be the normalization of $C \times_W U$, with induced morphisms $\pi_C : U_C \rightarrow C$ and $e_C : U_C \rightarrow X$, and set $\mathcal{E}_C := \pi_{C*}e_C^*\mathcal{L}$. Then \mathcal{E}_C is nef, $-(K_{U_C/C} + B|_{U_C}) \sim_{\mathbb{Q}} e_C^*(-K_{\mathcal{F}})$, and $mB|_{U_C} \in |\mathcal{O}_{U_C}(m) \otimes \pi_C^*\text{det}(\mathcal{E}_C)^{\otimes -m}|$. Suppose that

$B|_{U_C}$ is reducible, and write $B|_{U_C} = B'|_{U_C} + \sum_{1 \leq i \leq k} a_i F_i$, where the F_i 's are fibers of π_C and the a_i 's are positive rational numbers. Let $t \gg 0$ be such that $t a_i \in \mathbb{Z}$ for all $1 \leq i \leq k$. Then

$$mt B'|_{U_C} \in \left| \mathcal{O}_{U_C}(mt) \otimes \pi_C^* \det(\mathcal{E}_C)^{\otimes -mt} \otimes \mathcal{O}_{U_C}(-mt \sum_{1 \leq i \leq k} a_i F_i) \right|,$$

and thus

$$h^0(C, S^{mt} \mathcal{E}_C \otimes \det(\mathcal{E}_C)^{\otimes -mt} \otimes \mathcal{O}_C(-mt \sum_{1 \leq i \leq k} a_i c_i)) \neq 0,$$

where $c_i = \pi_C(F_i) \in C$. But this contradicts Lemma 4.11 below. We conclude that $B|_{U_C}$ is irreducible, and thus $B = B'$. To show that $E = B$, let $\tilde{C} \subset U_C$ be a curve such that $\pi_C(\tilde{C}) = C$, and suppose that e_C maps \tilde{C} to a point. Then $B \cdot \tilde{C} = -K_{U_C/C} \cdot \tilde{C} = -\pi_C^* \det(\mathcal{E}_C) \cdot \tilde{C}$. Since \mathcal{L} is ample, the later is < 0 again by Lemma 4.11. Hence, $\tilde{C} \subset \text{Supp}(B)$. We conclude that $E = B$. This argument also shows that $B \rightarrow W$ is a \mathbb{P}^{r-1} -bundle.

Next we observe that W is \mathbb{Q} -factorial with Picard number $\rho(W) = 1$. Indeed, since E is irreducible and X is \mathbb{Q} -factorial with Picard number $\rho(X) = 1$, we have $\dim(\text{Pic}(U) \otimes \mathbb{Q}) \leq \dim(\text{Cl}(U) \otimes \mathbb{Q}) = 2$. On the other hand, e is not finite, and thus $\dim(\text{Pic}(U) \otimes \mathbb{Q}) \geq 2$. Our claim follows.

Set $\mathcal{G} := (\pi|_B)_*(e^* \mathcal{L})|_B$, and notice that it is a nef vector bundle on W . The inclusion $B \subset U$ corresponds to $\mathcal{E} \rightarrow \mathcal{G}$. Since $B \in |\mathcal{O}_U(1) \otimes \pi^* \det(\mathcal{E})^*|$, we have an exact sequence

$$0 \rightarrow \det(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0,$$

and $\det(\mathcal{G}) \simeq \mathcal{O}_W$. We claim that $\mathcal{G} \simeq \mathcal{O}_W^{\oplus r}$. By replacing W with a resolution of singularities, and applying the projection formula, we may assume that W is smooth.

By Corollary 3.17, X is a \mathbb{Q} -Fano variety, and hence rationally connected by Theorem 2.4. Thus W is rationally connected, hence simply connected by [12, Corollary 4.18]. Our claim then follows from [13, Theorem 1.18].

Since W is a \mathbb{Q} -Fano variety with klt singularities and Picard number $\rho(W) = 1$, $\det(\mathcal{E})$ is ample, and $h^1(W, \det(\mathcal{E})) = 0$ (see [31, Theorem 1.2.5]). Hence $\mathcal{E} \simeq \det(\mathcal{E}) \oplus \mathcal{O}_W^{\oplus r}$, and X is a normal generalized cone over (W, \mathcal{M}) , where $\mathcal{M} = \det(\mathcal{E})$.

For the last statement, observe that the quotient $\mathcal{E} \rightarrow \mathcal{M}$ corresponds to a section σ of π , contained in $U \setminus E \simeq X \setminus e(E) \subset X$. Moreover, $\mathcal{L}|_{\sigma(W)} \simeq \mathcal{M}$ and $N_{\sigma(W)/X} \simeq \mathcal{M}^{\oplus r}$. Thus $\omega_{\sigma(W)} \simeq \omega_{X|\sigma(W)} \otimes \mathcal{M}^{\otimes r}$, and $i_W = k(i_X - r) \geq 1$ for some positive integer k . This completes the proof.

Suppose now that \mathcal{F} has rank $r = n - 1$. Then apply Proposition 4.4 and reduce to the previous case. □

Lemma 4.11 *Let C be a smooth complete curve, and \mathcal{G} a vector bundle of rank $r \geq 1$ on C .*

1. If \mathcal{G} is nef and $\text{deg}(\mathcal{G}) = 0$, then $h^0(C, \mathcal{G}) \leq r$.
2. If \mathcal{G} is nef, then $h^0(C, S^m(\mathcal{G}) \otimes \det(\mathcal{G})^{\otimes -m} \otimes \mathcal{M}^{\otimes -1}) = 0$ for any integer $m \geq 1$, and any line bundle \mathcal{M} on C with $\text{deg}(\mathcal{M}) > 0$.
3. Let $\mathcal{O}_{\mathbb{P}}(1)$ denote the tautological line bundle on $\mathbb{P}_C(\mathcal{G})$. If $\mathcal{O}_{\mathbb{P}}(1)$ is nef and big, then $\text{deg}(\mathcal{G}) > 0$.

Proof We prove (1) by induction on r . The result is clear if $r = 1$. So we assume that $r \geq 2$. We may assume that there exists a nonzero section $s \in H^0(C, \mathcal{G})$, which induces a nonzero map $\mathcal{O}_C \hookrightarrow \mathcal{G}$. Since \mathcal{G} is nef and $\text{deg}(\mathcal{G}) = 0$, \mathcal{O}_C is in fact a subbundle of \mathcal{G} . So $\mathcal{Q} = \mathcal{G}/\mathcal{O}_C$ is a nef vector bundle on C , and $\text{deg}(\mathcal{Q}) = 0$. By induction, $h^0(C, \mathcal{Q}) \leq r - 1$. Our claim follows.

To prove (2), we argue by contradiction. Suppose that $h^0(C, S^m(\mathcal{G}) \otimes \det(\mathcal{G})^{\otimes -m} \otimes \mathcal{M}^{\otimes -1}) \neq 0$ for some $m \geq 1$, and some line bundle \mathcal{M} on C with $\text{deg}(\mathcal{M}) > 0$. Then there is a nonzero map $\det(\mathcal{G})^{\otimes m} \otimes \mathcal{M} \hookrightarrow S^m(\mathcal{G})$. Since $\mu(\det(\mathcal{G})^{\otimes m} \otimes \mathcal{M}) = mr\mu(\mathcal{G}) + \mu(\mathcal{M}) > m\mu(\mathcal{G}) = \mu(S^m(\mathcal{G}))$, \mathcal{G} is not semistable, and there exists a subbundle $\mathcal{H} \subset \mathcal{G}$ of rank $k \geq 1$ such that $\mu(\mathcal{H}) \geq r\mu(\mathcal{G}) + \frac{1}{m}\mu(\mathcal{M})$. The quotient $\mathcal{Q} := \mathcal{G}/\mathcal{H}$ is a nef vector bundle, and thus

$$\begin{aligned} \text{deg}(\mathcal{G}) &= \text{deg}(\mathcal{H}) + \text{deg}(\mathcal{Q}) \geq \text{deg}(\mathcal{H}) \geq k \text{deg}(\mathcal{G}) + \frac{k}{m} \text{deg}(\mathcal{M}) \\ &\geq \text{deg}(\mathcal{G}) + \frac{k}{m} \text{deg}(\mathcal{M}), \end{aligned}$$

a contradiction.

Assertion (3) is an immediate consequence of (1). □

5 Codimension 1 del Pezzo foliations on varieties with mild singularities

In this section we prove Theorem 1.3. The proof naturally splits in two parts: the case of Picard number 1, and Picard number > 1 . First we treat the case of Picard number 1. We start by proving algebraic integrability.

Proposition 5.1 *Let X be a \mathbb{Q} -factorial klt projective variety such that $\rho(X) = 1$, and let $\mathcal{F} \subsetneq T_X$ be a del Pezzo foliation. Then either*

1. \mathcal{F} is algebraically integrable with rationally connected general leaves, or
2. X is a generalized cone over a projective normal variety Z with $\rho(Z) = 1$, and \mathcal{F} is the pullback by $X \dashrightarrow Z$ of a foliation induced by a nonzero global section of T_Z . Moreover, Z has \mathbb{Q} -factorial klt singularities, and $i_Z = k(i_X - r + 1) > 0$ for some positive integer k .

Proof Let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} (r - 1)c_1(\mathcal{L})$. Then $\text{Pic}(X) = \mathbb{Z}[\mathcal{L}]$ by Lemma 2.5.

Suppose that \mathcal{F} is not algebraically integrable with rationally connected general leaves. By Lemma 4.7, there exists a foliation with algebraic leaves $\mathcal{G} \subsetneq \mathcal{F}$ of rank $s \leq r - 1$ and index $i_{\mathcal{G}} \geq r - 1$. By Proposition 4.5, we must have $s = i_{\mathcal{G}} = r - 1$. By Theorem 1.2, X is a generalized cone as described in (2). The fact that \mathcal{F} is the pullback by $X \dashrightarrow Z$ of a foliation induced by a nonzero global section of T_Z follows from [1, Lemma 6.7]. □

Theorem 5.2 *Let X be a factorial klt projective variety of dimension $n \geq 3$ and Picard number $\rho(X) = 1$. Suppose that X admits a codimension 1 del Pezzo foliation $\mathcal{F} \subsetneq T_X$. Then one of the following holds.*

1. \mathcal{F} is a degree 1 foliation on $X \simeq \mathbb{P}^n$.
2. X is isomorphic to a (possibly singular) quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$, and \mathcal{F} is a pencil of hyperplane sections of Q_n .
3. X is a normal generalized cone over a projective normal surface Z with $\rho(Z) = 1$, and \mathcal{F} is the pullback by $X \dashrightarrow Z$ of a foliation induced by a nonzero global section of T_Z . Moreover, Z is factorial and klt, $i_X > n - 2$ and $i_Z = k(i_X - n + 2)$ for some positive integer k .

The proof of Theorem 5.2 follows the line of argumentation of [37]. We will use the following results, which follow from [loc.cit.]. We provide proofs for the reader's convenience.

Lemma 5.3 ([37], Lemma 3.1 or [25]) *Let X be a smooth variety of dimension ≥ 3 , and $\mathcal{F} \subsetneq T_X$ a codimension one foliation on X . Suppose that there exists a nonzero \mathbb{Q} -divisor D on X such that $D \sim_{\mathbb{Q}} 0$ and such that $\text{Supp}(D)$ is invariant by \mathcal{F} . Then either \mathcal{F} is defined by a logarithmic 1-form with poles exactly along $\text{Supp}(D)$, or there exists a codimension 2 foliation $\mathcal{G} \subsetneq \mathcal{F}$ with $K_{\mathcal{G}} \leq K_{\mathcal{F}}$.*

Proof Let $\omega \in H^0(X, \Omega_X^1 \otimes N_{\mathcal{F}})$ be a twisted 1-form defining \mathcal{F} . We may assume that D is integral, and $D \sim_{\mathbb{Z}} 0$. Let $\{D_i\}_{i \in I}$ be the set of irreducible components of D , and write $D = \sum_{i \in I} d_i D_i$. There exist an affine open cover $(U_{\alpha})_{\alpha \in A}$ of X , a nonzero rational function f on X , and regular functions $(f_{i\alpha})_{i \in I, \alpha \in A}$ such that, over U_{α} , $f_{i\alpha}$ is a defining equation for D_i , and

$$\prod_{i \in I} f_{i\alpha}^{d_i} = u_{\alpha} f,$$

where u_{α} is a unit on U_{α} . Thus, for $\alpha, \beta \in A$,

$$\sum_{i \in I} d_i \left(\frac{df_{i\alpha}}{f_{i\alpha}} - \frac{df_{i\beta}}{f_{i\beta}} \right) = \frac{du_{\alpha}}{u_{\alpha}} - \frac{du_{\beta}}{u_{\beta}}$$

over $U_{\alpha\beta}$. For each $\alpha \in A$, set $\xi_{\alpha} = \sum_{i \in I} d_i \frac{df_{i\alpha}}{f_{i\alpha}} - \frac{du_{\alpha}}{u_{\alpha}}$. Then the ξ_{α} 's define a logarithmic 1-form ξ on X with poles exactly along $\text{Supp}(D)$, and $\omega \wedge \xi \in H^0(X, \Omega_X^2 \otimes N_{\mathcal{F}})$ since $\text{Supp}(D)$ is invariant by \mathcal{F} . If $\omega \wedge \xi = 0$ then \mathcal{F} is defined by the logarithmic 1-form ξ . If $\omega \wedge \xi \neq 0$, then $\omega \wedge \xi$ defines a codimension 2 foliation $\mathcal{G} \subsetneq \mathcal{F}$ such that $K_{\mathcal{G}} \leq K_{\mathcal{F}}$. This completes the proof. \square

Lemma 5.4 *Let X be a \mathbb{Q} -factorial normal projective variety with Picard number $\rho(X) = 1$, and $\mathcal{F} \subsetneq T_X$ an algebraically integrable codimension one foliation with a rational first integral $\pi : X \dashrightarrow C$. Then either any fiber of π is irreducible or there exists a codimension 2 foliation $\mathcal{G} \subsetneq \mathcal{F}$ on X with $K_{\mathcal{G}} \leq K_{\mathcal{F}}$.*

Proof Suppose that π has a reducible fibre. Let D_1 be an irreducible component of a reducible fiber, and let D_2 be a general fiber. Since X is \mathbb{Q} -factorial with $\rho(X) = 1$, $D_1 \sim_{\mathbb{Q}} d D_2$ for some nonzero rational number d . Set $D := D_1 - d D_2 \sim_{\mathbb{Q}} 0$. By Lemma 5.3 above applied to the smooth locus of X , either \mathcal{F} is defined by a logarithmic 1-form with poles exactly along $\text{Supp}(D)$, or there exists a codimension 2 foliation $\mathcal{G} \subsetneq \mathcal{F}$ with $K_{\mathcal{G}} \leq K_{\mathcal{F}}$.

Suppose that \mathcal{F} is defined by a logarithmic 1-form ξ with poles exactly along $\text{Supp}(D)$. Since a general fiber of π is irreducible, $\xi = \pi^*\gamma$, where γ is a rational 1-form on C . This implies that ξ has poles along fibers, a contradiction. \square

Proof of Theorem 5.2 Let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} (n - 2)c_1(\mathcal{L})$. Then $\text{Pic}(X) = \mathbb{Z}[\mathcal{L}]$ by Lemma 2.5.

If \mathcal{F} is not algebraically integrable, then the result follows from Proposition 5.1.

So from now on we assume that \mathcal{F} is algebraically integrable, and consider a rational first integral $\pi : X \dashrightarrow W$. By Corollary 3.17, X is a \mathbb{Q} -Fano variety, and hence rationally connected by Theorem 2.4. So we must have $W \simeq \mathbb{P}^1$.

By Lemma 5.4, either any fiber of π is irreducible, or there exists a codimension 2 foliation $\mathcal{G} \subsetneq \mathcal{F}$ on X with $K_{\mathcal{G}} \leq K_{\mathcal{F}}$. In the latter case, \mathcal{G} has index $i_{\mathcal{G}} \geq n - 2$, and so, by Theorem 1.2, we are in case (3).

From now on we assume that any fiber of π is irreducible. We denote by $s \geq 0$ the number of multiple fibers of π . When $s \geq 1$, we denote by $m_1 F_1, \dots, m_s F_s$ the non reduced fibers of π , with F_i reduced and $m_1 \geq \dots \geq m_s \geq 2$. For notational convenience, we let F_{s+1} and F_{s+2} be general fibers of π , and set $m_{s+1} = m_{s+2} = 1$. For $1 \leq i \leq s + 2$, set $p_i := \pi(F_i) \in \mathbb{P}^1$, and let k_i be the positive integer such that $\mathcal{O}_X(F_i) \simeq \mathcal{L}^{\otimes k_i}$. Notice that

$$k_1 m_1 = \dots = k_s m_s = k_{s+1} m_{s+1} = k_{s+2} m_{s+2}. \tag{5.1}$$

In particular, $k_1 \leq \dots \leq k_{s+1} = k_{s+2}$.

Let $R(\pi) = \sum_{1 \leq i \leq s} \frac{m_i - 1}{m_i} \pi^* \mathcal{O}_{\mathbb{P}^1}(p_i)$ be the ramification divisor of π . By [14, Lemma 4.4], we have

$$\begin{aligned} \det(N_{\mathcal{F}}) &\simeq \pi^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_X(-R(\pi)) \\ &\simeq \mathcal{O}_X(F_1 + F_2 - \sum_{3 \leq i \leq s} (m_i - 1)F_i) \\ &\simeq \mathcal{L}^{\otimes (k_1 + k_2 - \sum_{3 \leq i \leq s} k_i(m_i - 1))}. \end{aligned}$$

On the other hand,

$$\mathcal{O}_X(K_X) \simeq \mathcal{O}_X(K_{\mathcal{F}}) \otimes \det(N_{\mathcal{F}}^*).$$

Thus the index i_X of X satisfies

$$i_X = n - 2 + k_1 + k_2 - \sum_{3 \leq i \leq s} k_i(m_i - 1). \tag{5.2}$$

By Corollary 3.17, we must have $i_X \geq n - 1$. If $i_X \geq n + 1$, then, by Theorem 1.1, $i = n + 1$, $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and \mathcal{F} is degree one foliation on \mathbb{P}^n . So from now on we assume that $n - 1 \leq i_X \leq n$.

We claim that there exists a finite cover $c : C \rightarrow \mathbb{P}^1$, unramified over $\mathbb{P}^1 \setminus \{p_1, \dots, p_s\}$, and having ramification index m_i at each point of $c^{-1}(p_i)$. If $s \leq 2$, then $i_X = n, k_1 = k_2 = 1$, and $m_1 = m_2$ by (5.2) and (5.1). In particular, either $s = 0$ (when $m_1 = m_2 = 1$) or $s = 2$ (when $m_1 = m_2 \geq 2$). In either case the existence of $c : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is clear. If $s > 2$, the existence of $c : C \rightarrow \mathbb{P}^1$ follows from [32, Lemma 6.1]. Let X' be the normalization of X in the function field of $X \times_{\mathbb{P}^1} C$, with induced morphism $c' : X' \rightarrow X$, and set $\mathcal{L}' := c'^* \mathcal{L}$. An easy calculation shows that c' is étale in codimension one. Hence c' is étale over $X \setminus \text{Sing}(X)$ by purity of the branch locus. In particular, $X \setminus \text{Sing}(X)$ is not simply connected if $s > 0$.

Suppose that $i_X = n$. Then, by Theorem 1.1, $(X, \mathcal{L}) \simeq (Q_n, \mathcal{O}_{Q_n}(1))$ where Q_n is a possibly singular quadric hypersurface in \mathbb{P}^{n+1} . Moreover, since $X \setminus \text{Sing}(X)$ is simply connected, $s = 0$ and $k_1 = k_2 = 1$ by (5.2). In other words, \mathcal{F} is a pencil of hyperplane sections of Q_n .

Suppose now that $i_X = n - 1$. In particular, (X, \mathcal{L}) is a (normal) del Pezzo variety. Then, by (5.2) and (5.1), we must have $s = 3$ and either $(k_1, k_2, k_3, m_1, m_2, m_3) = (1, k, k, 2k, 2, 2)$ for some integer $k \geq 1$, or $(k_1, k_2, k_3, m_1, m_2, m_3) \in \{(2, 2, 3, 3, 3, 2), (3, 4, 6, 4, 3, 2), (6, 10, 15, 5, 3, 2)\}$.

Note that $K_{X'} = c'^* K_X$ since $c' : X' \rightarrow X$ is étale in codimension one. This implies that X' has Gorenstein klt singularities (see [34, Proposition 5.20]). Thus (X', \mathcal{L}') is a (normal) del Pezzo variety, and hence rationally connected by Theorem 2.4. So $C \simeq \mathbb{P}^1$. By Hurwitz formula, either $\text{deg}(c) = 4k$ for some integer $k \geq 1$, or $\text{deg}(c) \in \{12, 24, 60\}$, respectively. Since $\mathcal{L}^n \geq 3$ by Lemma 2.8, we have $\mathcal{L}^m = \text{deg}(c) \cdot \mathcal{L}^n \geq 12$. On the other hand, Proposition 2.7 tells us that $\mathcal{L}^m \leq 9$, yielding a contradiction and completing the proof. \square

Now we consider the case of Picard number $\rho(X) \geq 2$.

Theorem 5.5 *Let X be a factorial klt projective variety of dimension $n \geq 3$ and Picard number $\rho(X) \geq 2$. Suppose that X admits a codimension 1 del Pezzo foliation $\mathcal{F} \subsetneq T_X$. Then there is an exact sequence of vector bundles*

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{E} \rightarrow \mathcal{V} \rightarrow 0$$

on \mathbb{P}^1 such that $X \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, and \mathcal{F} is the pullback via the relative linear projection $X \dashrightarrow Z = \mathbb{P}_{\mathbb{P}^1}(\mathcal{H})$ of a foliation on Z induced by a nonzero global section of $T_Z \otimes q^* \det(\mathcal{V})^*$. Here $q : Z \rightarrow \mathbb{P}^1$ denotes the natural projection. Moreover, one of the following holds.

1. $(\mathcal{E}, \mathcal{H}) \simeq (\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ for some positive integer a .
2. $(\mathcal{E}, \mathcal{H}) \simeq (\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus 2})$ for some positive integer a .
3. $(\mathcal{E}, \mathcal{H}) \simeq (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ for distinct positive integers a and b .

Proof Note that X has Gorenstein rational singularities. Denote by n the dimension of X , and let \mathcal{L} be an ample line bundle on X such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} (n - 2)c_1(\mathcal{L})$. By

Theorem 1.4, $K_X + (n - 2)c_1(\mathcal{L})$ is not nef. By [21, Theorem 3], one of the following holds:

- (A) X is the blowup of Y at a smooth point (see also [3, Theorem 3.1]).
- (B) There exists a surjective morphism $p : X \rightarrow Y$ onto a normal projective variety Y with Picard number $\rho(Y) = \rho(X) - 1$, with general fiber F , satisfying one of the following conditions.
 - (1) $\dim(Y) = 1$ and $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
 - (2) $\dim(Y) = 1$ and $(F, \mathcal{L}|_F) \simeq (Q_{n-1}, \mathcal{O}_{Q_{n-1}}(1))$, where Q_{n-1} is a (possibly singular) quadric hypersurface in \mathbb{P}^n .
 - (3) $\dim(Y) = 1$, $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$, and there exists an ample vector bundle \mathcal{G} on Y such that $X \simeq \mathbb{P}_Y(\mathcal{G})$ as varieties over Y , with tautological line bundle $\mathcal{O}_X(1) \simeq \mathcal{L}$.
 - (4) $\dim(Y) = 2$ and $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{n-2}, \mathcal{O}_{\mathbb{P}^{n-2}}(1))$.

First we consider case (B). We claim that $T_{X/Y} \not\subseteq \mathcal{F}$. Since F is log terminal, $\mathcal{F} \neq T_{X/Y}$ by Proposition 3.14. Hence, if $T_{X/Y} \subset \mathcal{F}$, we must be in case (B4). In this case, by [1, Lemma 6.7], \mathcal{F} is the pullback by p of a rank 1 foliation on Y , and thus $\mathcal{F}|_\ell \simeq T_{F|\ell} \oplus \mathcal{O}_{\mathbb{P}^1}$, where ℓ is a general line on F . But this contradicts the assumptions that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} (n - 2)c_1(\mathcal{L})$, and proves the claim.

By considering the possible splitting types of the restriction of \mathcal{F} to a general line on F , and using the fact that $T_{X/Y} \not\subseteq \mathcal{F}$, we see that cases (B1) and (B4) do not occur.

Next we show that case (B2) cannot occur either. Let $C \subset Q_{n-1} \setminus \text{Sing}(Q_{n-1})$ be a general conic. Then X is smooth in a neighborhood of C and $\mathcal{F}|_C \subset T_{X|C} \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Since $\mathcal{F} \neq T_{X/Y}$, the foliation $\mathcal{G} := \mathcal{F} \cap T_{X/Y}$ has rank $n - 2$, and $\mathcal{G}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus n-2}$. It induces a foliation on $F \simeq Q_{n-1}$ defined by a twisted 1-form in $H^0(F, \Omega_F^{[1]} \otimes \mathcal{L}|_F)$. But $h^0(F, \Omega_F^{[1]} \otimes \mathcal{L}|_F) = 0$, a contradiction.

Suppose we are in case (B3). The short exact sequence

$$0 \rightarrow T_{X/Y} \rightarrow T_X \rightarrow p^*T_Y \rightarrow 0$$

yields an exact sequence

$$0 \rightarrow \wedge^{n-1}T_{X/Y} \rightarrow \wedge^{n-1}T_X \rightarrow \wedge^{n-2}T_{X/Y} \otimes p^*T_Y \rightarrow 0.$$

Since $\mathcal{F} \not\subseteq T_{X/Y}$, $h^0(X, \wedge^{n-2}T_{X/Y} \otimes p^*T_Y \otimes \mathcal{L}^{\otimes -(n-2)}) \neq 0$. By [1, Proposition 9.11 (4)], $Y \simeq \mathbb{P}^1$. The description of X and \mathcal{F} now follows from [1, Theorem 9.6].

Finally, we show that case (A) does not occur. Let $p : X \rightarrow Y$ be the blowup of Y at a smooth point $y \in Y$, with exceptional divisor E . Then $\mathcal{L} + E \simeq p^*\mathcal{M}$ for some ample line bundle \mathcal{M} on Y , and \mathcal{F} induces a codimension 1 del Pezzo foliation \mathcal{G} on Y with $\det(\mathcal{G}) \simeq \mathcal{M}^{\otimes n-2}$.

By replacing $(X, \mathcal{F}, \mathcal{L})$ with $(Y, \mathcal{G}, \mathcal{M})$ successively, we may assume that either $\rho(Y) = 1$ or Y satisfies (B3). In either case there is an unsplit covering family H of rational curves on Y such that $\mathcal{M} \cdot H = 1$ (when $\rho(Y) = 1$, this follows from the classification in Theorem 5.2). Let C be a curve from the family H passing through y , and \tilde{C} its strict transform in X . Then

$$\mathcal{M} \cdot H = \mathcal{M} \cdot C = \pi^* \mathcal{M} \cdot \tilde{C} = \mathcal{L} \cdot \tilde{C} + E \cdot \tilde{C} \geq 2,$$

contradicting the choice of H . This completes the proof. □

Remark 5.6 Under the assumptions of Theorem 5.5, we have shown in particular that X is smooth. So \mathcal{F} is algebraically integrable, and [1, Remark 7.11] provides complete description of the general log leaf.

Proof of Theorem 1.3 Apply Theorems 5.2 and 5.5. □

6 Regular foliations

In this section, we discuss the singular locus of \mathbb{Q} -Fano foliations on varieties with mild singularities. Let $\mathcal{F} \subsetneq T_X$ be a regular foliation on a complex projective manifold X . If \mathcal{F} is algebraically integrable, then $-K_{\mathcal{F}}$ is not ample by Proposition 3.14. We start by generalizing this result to arbitrary regular foliations on complex projective manifolds with Picard number one.

Theorem 6.1 *Let $\mathcal{F} \subsetneq T_X$ be a regular foliation on a complex projective manifold X with Picard number $\rho(X) = 1$. Then $-K_{\mathcal{F}}$ is not ample.*

Proof Suppose to the contrary that $-K_{\mathcal{F}}$ is ample. Set $\mathcal{Q} := T_X/\mathcal{F}$, $\mathcal{L} := \det(\mathcal{F})$, and denote by r the rank of \mathcal{F} . Then $\det(\mathcal{Q}) \simeq \mathcal{O}_X(-K_X) \otimes \mathcal{L}^{\otimes -1}$. By [6, Corollary 3.4], $\det(\mathcal{Q})^{\dim(X)} = 0$. Since $\rho(X) = 1$, we must have $\det(\mathcal{Q}) \equiv 0$. This implies that X is a Fano manifold, and, by Lemma 2.5, $\det(\mathcal{Q}) \simeq \mathcal{O}_X$. So $h^0(X, \Omega_X^{\dim(X)-r}) \neq 0$. On the other hand, by Hodge symmetry, $h^0(X, \Omega_X^{\dim(X)-r}) = h^{\dim(X)-r}(X, \mathcal{O}_X)$, and the latter vanishes by Kodaira vanishing theorem, a contradiction. □

6.2 (The Atiyah class of a locally free sheaf) *Let X be a smooth variety, and \mathcal{E} a locally free sheaf of rank $r \geq 1$ on X . Let $J_X^1(\mathcal{E})$ be the sheaf of 1-jets of \mathcal{E} . I.e., as a sheaf of abelian groups on X , $J_X^1(\mathcal{E}) \simeq \mathcal{E} \oplus (\Omega_X^1 \otimes \mathcal{E})$, and the \mathcal{O}_X -module structure is given by $f(e, \alpha) = (fe, f\alpha - df \otimes e)$, where f, e and α are local sections of $\mathcal{O}_X, \mathcal{E}$ and $\Omega_X^1 \otimes \mathcal{E}$, respectively. The Atiyah class of \mathcal{E} is defined to be the element $at(\mathcal{E}) \in H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$ corresponding to the Atiyah extension*

$$0 \rightarrow \Omega_X^1 \otimes \mathcal{E} \rightarrow J_X^1(\mathcal{E}) \rightarrow \mathcal{E} \rightarrow 0.$$

It can be explicitly described as follows. Choose an affine open cover $(U_i)_{i \in I}$ of X such that \mathcal{E} admits a frame $f_i : \mathcal{O}_{U_i}^r \xrightarrow{\sim} \mathcal{E}|_{U_i}$ for each U_i . For $i, j \in I$, define $f_{ij} := f_j^{-1}|_{U_{ij}} \circ f_i|_{U_{ij}}$. Then

$$at(\mathcal{E}) = [(-f_j|_{U_{ij}} \circ df_{ij}|_{U_{ij}} \circ f_i^{-1}|_{U_{ij}})_{i,j}] \in H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$$

(see [4, Proof of Theorem 5]).

The next result follows from the proof of [6, Proposition 3.3] and [6, Corollary 3.4]. We sketch the proof for the reader's convenience.

Lemma 6.3 *Let X be a smooth variety, and $\mathcal{F} \subsetneq T_X$ a regular foliation. Set $\mathcal{Q} = T_X/\mathcal{F}$. Then $at(\mathcal{Q}) \in H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q}))$ is in the image of the natural map*

$$H^1(X, \mathcal{Q}^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})) \rightarrow H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})).$$

Proof Since \mathcal{F} is regular, there is an exact sequence

$$0 \rightarrow \mathcal{Q}^* \rightarrow \Omega_X^1 \rightarrow \mathcal{F}^* \rightarrow 0.$$

Taking cohomology yields

$$\begin{aligned} H^1(X, \mathcal{Q}^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})) &\rightarrow H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})) \\ &\xrightarrow{\delta} H^1(X, \mathcal{F}^* \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})). \end{aligned}$$

We must show that $\delta(at(\mathcal{Q})) = 0$.

Denote by q the rank of \mathcal{Q} . Choose an affine open cover $(U_i)_{i \in I}$ of X such that, over each U_i , \mathcal{Q} admits a frame $\alpha_i : \mathcal{O}_{U_i}^q \xrightarrow{\sim} \mathcal{Q}^*|_{U_i}$. By assumption, \mathcal{F} is stable under the Lie bracket. This is equivalent to saying that $d\mathcal{Q}^* \subset \mathcal{Q}^* \wedge \Omega_X^1$. Thus, viewing α_i as a line vector whose entries are local sections of $\mathcal{Q}^* \subset \Omega_X^1$ over U_i , we get a matrix β_i , whose entries are local sections of $\mathcal{Q}^* \subset \Omega_X^1$ over U_i , such that $d\alpha_i = \alpha_i \wedge \beta_i$.

For $i, j \in I$, set $f_{ij} := \alpha_j^{-1}|_{U_{ij}} \circ \alpha_i|_{U_{ij}}$. Then

$$at(\mathcal{Q}) = [(-\alpha_j|_{U_{ij}} \circ df_{ij}|_{U_{ij}} \circ \alpha_i^{-1}|_{U_{ij}})_{i,j}] \in H^1(X, \Omega_X^1 \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{Q})).$$

Since $\alpha_i = \alpha_j \cdot f_{ij}$ on U_{ij} ,

$$d\alpha_i = d\alpha_j \cdot f_{ij} + \alpha_j \wedge df_{ij},$$

and

$$\alpha_i \wedge \beta_i = \alpha_j \cdot f_{ij} \wedge \beta_i = \alpha_j \wedge \beta_j \cdot f_{ij} + \alpha_j \wedge df_{ij}.$$

Given $\mathbf{v} \in H^0(U_{ij}, \mathcal{F}|_{U_{ij}})$,

$$\begin{aligned} \alpha_j \cdot f_{ij} \cdot \beta_i(\mathbf{v}) &= i_{\mathbf{v}}(\alpha_j \cdot f_{ij} \wedge \beta_i) \\ &= i_{\mathbf{v}}(\alpha_j \wedge \beta_j \cdot f_{ij} + \alpha_j \wedge df_{ij}). \\ &= \alpha_j \cdot \beta_j(\mathbf{v}) \cdot f_{ij} + \alpha_j \cdot df_{ij}(\mathbf{v}). \end{aligned}$$

This implies that

$$\delta(at(\mathcal{Q})) = [(\beta_j|_{U_{ij}} - \beta_i|_{U_{ij}})_{i,j}],$$

and thus

$$\delta(at(\mathcal{Q})) = 0.$$

□

The following result is certainly well known. We include a proof for lack of adequate reference. We remark that a locally free sheaf of finite rank on a Cohen–Macaulay locally noetherian scheme satisfies Serre’s condition S_k for any integer $k \geq 0$.

Lemma 6.4 *Let X be a noetherian scheme, and \mathcal{G} a coherent sheaf of \mathcal{O}_X -modules on X . Suppose that \mathcal{G} satisfies Serre’s condition S_k for some integer $k \geq 1$. Let $U \subset X$ be an open subset such that $\text{codim}_X(X \setminus U) \geq k$. Then $H^i(X, \mathcal{G}) \simeq H^i(U, \mathcal{G}|_U)$ for $0 \leq i \leq k - 1$.*

Proof Set $Y := X \setminus U$. Denote by $\mathcal{H}_Y^i(\mathcal{G})$ the local cohomology sheaf. By assumption, $\mathcal{H}_Y^i(\mathcal{G}) = 0$ for $0 \leq i \leq k$. The spectral sequence of local cohomology groups and sheaves implies that $H_Y^i(X, \mathcal{G}) = 0$. The long exact sequence

$$\dots \rightarrow H_Y^i(X, \mathcal{G}) \rightarrow H^i(X, \mathcal{G}) \rightarrow H^i(U, \mathcal{G}|_U) \rightarrow H_Y^{i+1}(X, \mathcal{G}) \rightarrow \dots$$

then yields $H^i(X, \mathcal{G}) \simeq H^i(U, \mathcal{G}|_U)$ for $0 \leq i \leq k - 1$, as claimed. □

We can now prove our result in the mildly singular setting.

Proof of Theorem 1.5 Let $\mathcal{F} \subsetneq T_X$ be a codimension 1 foliation on a klt projective variety X , and suppose that X and \mathcal{F} are both regular in codimension 2. We want to show that $-K_{\mathcal{F}}$ is not ample.

Assume to the contrary that $-K_{\mathcal{F}}$ is ample. Set $\mathcal{Q} := T_X/\mathcal{F}$ and $\mathcal{L} := \det(\mathcal{F})$. Then $\mathcal{Q}^{**} \simeq \mathcal{O}_X(-K_X) \otimes \mathcal{L}^{\otimes -1}$. Let $U \subset X$ be a smooth open subset such that $\text{codim}(X \setminus U) \geq 3$, and \mathcal{F} is regular on U . By Lemma 6.4, $h^1(U, \mathcal{Q}|_U) = h^1(X, \mathcal{Q}^*) = h^1(X, \mathcal{O}_X(K_X) \otimes \mathcal{L})$, and the latter vanishes by the Kawamata–Viehweg Vanishing Theorem (see [31, Theorem 1.2.5]). By Lemma 6.3, $0 = at(\mathcal{Q}|_U) \in H^1(U, \Omega_U^1)$. This implies that, for any smooth complete curve $C \subset U$, $(K_X + c_1(\mathcal{L})) \cdot C = 0$. Hence $K_X + c_1(\mathcal{L}) \equiv 0$. Thus, X is a \mathbb{Q} -Fano variety. Let k be a positive integer such that $k(K_X + c_1(\mathcal{L}))$ is Cartier. By Lemma 2.5, $\mathcal{O}_X(k(K_{\mathcal{F}} + c_1(\mathcal{L}))) \simeq \mathcal{O}_X$. This contradicts Lemma 2.10 since $h^0(X, \Omega_X^q[\otimes] \mathcal{O}_X(-K_{\mathcal{F}} - c_1(\mathcal{L}))) \neq 0$. □

Acknowledgments C. Araujo was partially supported by CNPq and Faperj Research Fellowships. S. Druel was partially supported by the CLASS project of the A.N.R. Much of this work was developed during the authors’ visits to IMPA and Institut Fourier. We would like to thank both institutions for their support and hospitality. We would also like to thank the referee for helpful comments.

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