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A decomposition theorem for singular spaces with trivial canonical class of dimension at most five

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Abstract In this paper we partly extend the Beauville–Bogomolov decomposition theorem to the singular setting. We show that any complex projective variety of dimension at most five with canonical singularities and numerically trivial canonical class admits a finite cover, étale in codimension one, that decomposes as a product of an Abelian variety, and singular analogues of irreducible Calabi–Yau and irreducible holomorphic symplectic varieties.

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Contents

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1 Introduction

The Beauville–Bogomolov decomposition theorem asserts that any compact Kähler manifold with numerically trivial canonical bundle admits an étale cover that decomposes into a product of a torus, and irreducible, simply-connected Calabi–Yau, and holomorphic symplectic manifolds (see [7]). Moreover, the decomposition of the simply-connected part corresponds to a decomposition of the tangent bundle into a direct sum whose summands are integrable and stable with respect to any polarization.

With the development of the minimal model program, it became clear that singularities arise as an inevitable part of higher dimensional life. If X is any complex projective manifold with Kodaira dimension $\kappa(X) = 0$, standard conjectures of the minimal model program predict the existence of a birational contraction $X \dashrightarrow X'$, where X' has terminal singularities and $K_{X'} \equiv 0$. This makes imperative to extend the Beauville–Bogomolov decomposition theorem to the singular setting.

Building on recent extension theorems for differential forms on singular spaces, Greb, Kebekus and Peternell prove a decomposition theorem for the tangent sheaf of complex projective varieties with canonical singularities and numerically trivial canonical class.

Theorem 1.1 ([25, Theorem 1.3]) *Let X be a normal complex projective variety with canonical singularities. Assume that $K_X \equiv 0$. Then there exists an abelian variety A as well as a projective variety $\widetilde{X}$ with canonical singularities, a finite cover $A \times \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$T_{\widetilde{X}} = \bigoplus_{i \in I} \mathscr{E}_i$$

such that the following holds.

1. *The $\mathscr{E}_i$ are integrable subsheaves of $T_{\widetilde{X}}$, with $\det(\mathscr{E}_i) \cong \mathscr{O}_{\widetilde{X}}$.*
   Further, if $g\colon \widehat{X} \to \widetilde{X}$ is any finite cover, étale in codimension one, then the following properties hold in addition.
2. *The sheaves $(g^*\mathscr{E}_i)^{**}$ are stable with respect to any polarization on $\widehat{X}$.*
3. *The irregularity $h^1(\widehat{X}, \mathscr{O}_{\widehat{X}})$ of $\widehat{X}$ is zero.*

Based on Theorem 1.1 above, they argue in [25, Sect. 8] that the natural building blocks for any structure theory of complex projective varieties with canonical singularities and numerically trivial canonical class are canonical varieties with *strongly stable* tangent sheaf (see Definition 2.4 for this notion). In dimension no more than five, they also show that canonical varieties with strongly stable tangent sheaf fall into two classes, which naturally generalize the notions of irreducible Calabi–Yau and irreducible holomorphic-symplectic manifolds, respectively.

The main result of our paper is the following decomposition theorem.

Theorem 1.2 *Let X be a normal complex projective variety of dimension at most 5, with klt singularities. Assume that $K_X \equiv 0$. Then there exists an abelian variety A as well as a projective variety $\widetilde{X}$ with canonical singularities, a finite cover $A \times \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$\widetilde{X} \cong \prod_{i \in I} Y_i \times \prod_{j \in J} Z_j$$

of $\widetilde{X}$ into normal projective varieties with trivial canonical class, such that the following holds.

1. *We have $h^0\big(\widetilde{Y}_i, \Omega^{[q]}_{\widetilde{Y}_i}\big) = 0$ for all numbers $0 < q < \dim Y_i$ and all finite covers $\widetilde{Y}_i \to Y_i$, étale in codimension one.*
2. *There exists a reflexive 2-form $\sigma \in H^0\big(Z_j, \Omega^{[2]}_{Z_j}\big)$ such that σ is everywhere non-degenerate on the smooth locus of Z_j, and such that for all finite covers $f : \widetilde{Z}_j \to Z_j$, étale in codimension one, the exterior algebra of global reflexive forms is generated by $f^{[*]}\sigma \in H^0\big(\widetilde{Z}_j, \Omega^{[2]}_{\widetilde{Z}_j}\big)$.*

Remark 1.3 The decomposition of $\widetilde{X}$ induces the decomposition of $T_{\widetilde{X}}$ given by Theorem 1.1 above up to permutation of the summands.

The proof of the Beauville–Bogomolov decomposition theorem heavily uses Kähler–Einstein metrics and the solution of the Calabi conjecture. But these results are not yet available in the singular setting. Instead, the proof of Theorem 1.2 relies on Theorem 1.1 and on sufficient criteria to guarantee that a given foliation has algebraic leaves. In [12], Bost proved an arithmetic algebraicity criterion for leaves of algebraic foliations defined over a number field. Building on his result, we obtain the following algebraicity criterion.

Theorem 1.4 *Let X be a normal complex projective variety of dimension n with terminal singularities, and let H be an ample Cartier divisor. Let*

$$T_X = \bigoplus_{i \in I} \mathscr{G}_i \oplus \mathscr{E}$$

be a decomposition of T_X into involutive subsheaves. Suppose that for any finite cover $g : \widehat{X} \to X$, étale in codimension one, the sheaf $(g^*\mathscr{G}_i)^{**}$ is g^*H-stable. Suppose furthermore that

$$\begin{aligned} c_1(\mathscr{G}_i) \cdot H^{n-1} &= 0, \quad \textit{and either} \\ c_1(\mathscr{G}_i)^2 \cdot H^{n-2} &\neq 0 \quad \textit{or} \\ c_2(\mathscr{G}_i) \cdot H^{n-2} &\neq 0 \end{aligned}$$

for each $i \in I$. Suppose finally that $\mathscr{E}$ is H-polystable with

$$\begin{aligned} c_1(\mathscr{E}) \cdot H^{n-1} &= c_1(\mathscr{E})^2 \cdot H^{n-2} \\ &= c_2(\mathscr{E}) \cdot H^{n-2} = 0. \end{aligned}$$

Then there exists an abelian variety A as well as a projective variety $\widetilde{X}$ with terminal singularities, and a finite cover $f\colon A \times \widetilde{X} \to X$, étale in codimension one, such that

$$(f^*\mathscr{E})^{**} = T_{A\times\widetilde{X}/\widetilde{X}}$$

as subsheaves of $T_{A\times\widetilde{X}}$.

Remark 1.5 In the setup of Theorem 1.4, there exists a finite cover $f\colon \widetilde{X} \to X$ étale in codimension one such that $(f^*\mathscr{E})^{**}$ is a locally free, flat sheaf on $\widetilde{X}$ ([26, Theorem 1.20]). In particular, if the étale fundamental group $\pi_1^{\text{ét}}(X_{\text{reg}})$ is finite, then the conclusion of Theorem 1.4 follows easily from the description of the Albanese map of mildly singular varieties whose canonical divisor is numerically trivial in [37, Proposition 8.3]. On the other hand, [25, Corollary 3.6] reduces the study of varieties with trivial canonical class to those with zero augmented irregularity (see Definition 4.1 for this notion), and it is expected that the étale fundamental group of their smooth locus is finite (see [25, Sect. 8] and [26, Theorem 1.5]). This is true if $\dim X \leqslant 4$ by [25, Corollary 8.25], providing an alternative proof of Theorem 1.4 in this case.

The geometric counterpart of Bost's arithmetic algebraicity criterion, independently obtained by Bogomolov and McQuillan [11], and very recently extended by Campana and Păun [15] leads to the following algebraicity criterion.

Theorem 1.6 *Let X be a normal complex projective variety of dimension n, and let H be an ample Cartier divisor. Suppose that X is smooth in codimension two. Let*

$$T_X = \mathscr{E} \oplus \mathscr{G}$$

be a decomposition of T_X into involutive subsheaves, where $\mathscr{E}$ is H-stable, $\det(\mathscr{E}) \cong \mathscr{O}_X$ and $c_2(\mathscr{E}) \cdot H^{n-2} \neq 0$. Suppose furthermore that $\mathscr{E}$ has rank at most 3. Then $\mathscr{E}$ has algebraic leaves.

Theorem 1.6 confirms a conjecture of Pereira and Touzet in some special cases (see [53, Remark 6.5]). It is one of the main technical contributions of this paper.

This paper is organized as follows.

In Sect. 3, we review basic definitions and results about foliations on normal varieties.

In Sect. 4, we show that algebraic integrability of direct summands in the decomposition of the tangent bundle given by Theorem 1.1 leads to a decomposition of the variety, perhaps after passing to a finite cover that is étale in codimension one (see Theorem 4.5 and Proposition 4.10). This solves [25, Problem 8.4].

Section 5 is devoted to the proof of Theorem 1.4.

In Sects. 6–8, we prove Theorem 1.6. In the setup of Theorem 1.6, we show that either $\mathscr{E}$ satisfies the Bost–Campana–Păun algebraicity criterion in Proposition 8.4, or $\mathscr{E}$ admits a holomorphic Riemannian metric. This is an immediate consequence of our study of stable reflexive sheaves of rank at most 3 with numerically trivial first Chern class and pseudo-effective tautological line bundle in Sect. 6. For a precise statement, see Theorem 6.1. If $\mathscr{E}$ admits a holomorphic metric, then it follows from Proposition 7.5 that $\mathscr{E}$ admits a holomorphic connection, yielding a contradiction.

In Sect. 9, we finally prove Theorem 1.2.

2 Notation, conventions, and basic facts

2.1 (Global Convention) Throughout the paper a variety is a reduced and irreducible scheme separated and of finite type over a field.

2.2 (Differentials, reflexive hull) Given a normal variety X, we denote the sheaf of Kähler differentials by Ω^1_X. If $0 \leqslant p \leqslant \dim X$ is any number, write $\Omega^{[p]}_X := (\Omega^p_X)^{**}$. The tangent sheaf will be denoted by $T_X := (\Omega^1_X)^*$.

Given a normal variety X, $m \in \mathbb{N}$, and coherent sheaves $\mathscr{E}$ and $\mathscr{G}$ on X, write $\mathscr{E}^{[m]} := (\mathscr{E}^{\otimes m})^{**}$, $S^{[m]}\mathscr{E} := (S^m\mathscr{E})^{**}$, $\det(\mathscr{E}) := (\Lambda^{\operatorname{rank}\mathscr{E}}\mathscr{E})^{**}$, and $\mathscr{E} \boxtimes \mathscr{G} := (\mathscr{E} \otimes \mathscr{G})^{**}$. Given any morphism $f\colon Y \to X$, write $f^{[*]}\mathscr{E} := (f^*\mathscr{E})^{**}$.

2.3 (Stability) The word “stable” will always mean “slope-stable with respect to a given polarization”. Ditto for semistability.

Definition 2.4 ([25, Definition 7.2]). Let X be a normal complex projective variety of dimension n, and let $\mathscr{G}$ be a coherent reflexive sheaf. We call $\mathscr{G}$ *strongly stable*, if for any finite morphism $f\colon \widetilde{X} \to X$ that is étale in codimension one, and for any choice of ample divisors $\widetilde{H}_1, \ldots, \widetilde{H}_{n-1}$ on $\widetilde{X}$, the reflexive pull-back $f^{[*]}\mathscr{G}$ is stable with respect to $(\widetilde{H}_1, \ldots, \widetilde{H}_{n-1})$.

2.5 (Nef and pseudo-effective cones) Let X be a complex projective variety and consider the finite dimensional dual $\mathbb{R}$-vector spaces

$$\mathrm{N}_1(X)_{\mathbb{R}} = \big(\{1 - \text{cycles}\}/\equiv\big) \otimes \mathbb{R} \quad \text{and} \quad \mathrm{N}^1(X)_{\mathbb{R}} = \big(\operatorname{Pic}(X)/\equiv\big) \otimes \mathbb{R},$$

where $\equiv$ denotes numerical equivalence. Set $\mathrm{N}^1(X)_{\mathbb{Q}} = \big(\mathrm{Pic}(X)/\equiv\big) \otimes \mathbb{Q}$. The *Mori cone* of X is the closure $\overline{\mathrm{NE}}(X) \subset \mathrm{N}_1(X)_{\mathbb{R}}$ of the cone $\mathrm{NE}(X)$ spanned by classes of effective curves. Its dual cone is the *nef cone* $\mathrm{Nef}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$, which by Kleiman's criterion is the closure of the cone spanned by ample classes. The closure of the cone spanned by effective classes in $\mathrm{N}^1(X)_{\mathbb{R}}$ is the *pseudo-effective cone* $\overline{\mathrm{Eff}}(X)$.

2.6 (Projective space bundles) If $\mathscr{E}$ is a locally free sheaf of finite rank on a variety X, we denote by $\mathbb{P}_X(\mathscr{E})$ the variety $\mathrm{Proj}_X\big(\mathrm{Sym}(\mathscr{E})\big)$, and by $\mathscr{O}_{\mathbb{P}_X(\mathscr{E})}(1)$ its tautological line bundle.

Lemma 2.7 *Let X be a complex projective variety, let H be an ample Cartier divisor, and let $\mathscr{E}$ be a locally free sheaf of finite rank. Then $[\mathscr{O}_{\mathbb{P}_X(\mathscr{E})}(1)] \in \mathrm{N}^1\big(\mathbb{P}_X(\mathscr{E})\big)_{\mathbb{R}}$ is not pseudo-effective if and only if there exists $c > 0$ such that $h^0\big(X, S^i\mathscr{E} \otimes \mathscr{O}_X(jH)\big) = 0$ for any positive integer j and any natural number i satisfying $i > cj$.*

Proof Set $Y := \mathbb{P}_X(\mathscr{E})$, denote by $\mathscr{O}_Y(1)$ the tautological line bundle on Y, and by $\pi : Y \to X$ the natural morphism. If $[\mathscr{O}_Y(1)] \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is not pseudo-effective, then $\mathscr{O}_Y(m) \otimes \pi^*\mathscr{O}_X(H)$ is not pseudo-effective either for a sufficiently large positive integer m. Let now i and j be positive integers such that $i > mj$. Then $\mathscr{O}_Y(i) \otimes \pi^*\mathscr{O}_X(jH)$ is not pseudo-effective as well, and hence

$$h^0\big(X, S^i\mathscr{E} \otimes \mathscr{O}_X(jH)\big) = h^0\big(Y, \mathscr{O}_Y(i) \otimes \pi^*\mathscr{O}_X(jH)\big) = 0$$

by the projection formula.

Conversely, suppose that $[\mathscr{O}_Y(1)] \in N^1(Y)$ is pseudo-effective. Pick $m_0 > 0$ such that $\mathscr{O}_Y(1) \otimes \pi^*\mathscr{O}_X(m_0H)$ is ample. Then, for each positive integer m, the line bundle $\mathscr{O}_Y(m+1) \otimes \pi^*\mathscr{O}_X(m_0H)$ is big, and hence there exists a positive integer k such that

$$h^0\big(X, S^{km+k}\mathscr{E} \otimes \mathscr{O}_X(km_0H)\big) = h^0\big(Y, \mathscr{O}_Y(km+k) \otimes \pi^*\mathscr{O}_X(km_0H)\big) \neq 0.$$

This completes the proof of the lemma. □

2.8 (Chern classes) We will need to consider intersection numbers of line bundles with Chern classes of reflexive sheaves on singular varieties. We use [21, Chapter 3] as our main reference for Chern classes on varieties over a field. Given a variety X, we denote by $\mathrm{A}_k(X)$ the group of k-dimensional cycles modulo rational equivalence.

Definition 2.9 Let X be a variety of dimension n, and let $\mathscr{G}$ be a coherent sheaf. Let $X^\circ \subset X_{\text{reg}}$ be the maximal open set where $\mathscr{G}$ is locally free. Assume that the complement of X° in X has codimension at least $k+1$ for some positive integer k. The k-th Chern class $c_k(\mathscr{G})$ of $\mathscr{G}$ is the image of $\mathbf{c}_k(\mathscr{G}_{|X^\circ}) \cap [X^\circ] \in \mathrm{A}_{n-k}(X^\circ)$ under the isomorphism $\mathrm{A}_{n-k}(X^\circ) \cong \mathrm{A}_{n-k}(X)$, where $\mathbf{c}_k(\mathscr{G}_{|X^\circ})\colon \mathrm{A}_\bullet(X^\circ) \to \mathrm{A}_{\bullet-k}(X^\circ)$ is the Chern class operation.

Remark 2.10 Complex varieties with terminal singularities are smooth in codimension two (see [40, Corollary 5.18]). So the first and second Chern classes of coherent sheaves are well-defined on these varieties.

We will need the following observation. Notice that much of the intersection theory developed in [21, Chapters 1–10] is valid for schemes, separated and of finite type, over a noetherian regular scheme (see [21, Chapter 20]). Given a scheme X, separated and of finite type over a noetherian regular scheme S, we denote by $\mathrm{A}_k(X/S)$ the group of *relative dimension k* cycles modulo rational equivalence.

Lemma 2.11 *Let T be an integral noetherian scheme of dimension m, let X be an integral scheme of dimension n, and let $\pi\colon X \to T$ be a dominant proper morphism. Let $\mathscr{G}$ be a coherent sheaf, and let H be a Cartier divisor on X. Given $t \in T$, we denote by X_t the fiber of π over the point t. Write $\mathscr{G}_t := \mathscr{G}_{|X_t}$ and $H_t := H_{|X_t}$. Let $X^\circ \subset X$ be the open set where $\mathscr{G}$ is locally free. Assume that the complement of X° in X has codimension at least $k+1$ for some positive integer k. Then there exists a dense open set $T^\circ \subset T$ such that the intersection number $c_k(\mathscr{G}_t) \cdot H_t^{n-m-k}$ is independent of $t \in T^\circ$.*

Remark 2.12 Let $t \in T$. The scheme X_t is viewed as a scheme over the residue field of t. If the complement of $X^\circ \cap X_t$ in X_t has codimension at least $k+1$, then $c_k(\mathscr{E}_t)$ is well-defined.

Proof of Lemma 2.11 Replacing T by a dense open set, we may assume that π is flat, and that, for any point $t \in T$, the complement of $X^\circ \cap X_t$ in X_t has codimension at least $k+1$. We will show that the intersection number $c_k(\mathscr{G}_t) \cdot H_t^{n-m-k}$ is independent of $t \in T$. In order to prove our claim, we may assume without loss of generality that $T = \operatorname{Spec} R$ for some discrete valuation ring R. Let η be the generic point of T, and let t be its closed point.

Let now $c_k(\mathscr{G})$ be the image of $\mathbf{c}_k(\mathscr{G}_{|X^\circ}) \cap [X^\circ] \in \mathrm{A}_{n-m-k}(X^\circ/S)$ under the isomorphism $\mathrm{A}_{n-m-k}(X^\circ/S) \cong \mathrm{A}_{n-m-k}(X/S)$, where $\mathbf{c}_k(\mathscr{G}_{|X^\circ})\colon \mathrm{A}_\bullet(X^\circ/S) \to \mathrm{A}_{\bullet-k}(X^\circ/S)$ is the Chern class operation. The inclusion $X_\eta \subset X$ induces a pull-back morphism

$$-_\eta\colon \mathrm{A}_\bullet(X/S) \to \mathrm{A}_\bullet(X_\eta),$$

and the regular embedding $X_t \subset X$ induces a Gysin homomorphism

$$-_t \colon \mathrm{A}_\bullet(X/S) \to \mathrm{A}_\bullet(X_t).$$

By [21, Chapter 20], there is a specialization map

$$s \colon \mathrm{A}_\bullet(X_\eta) \to \mathrm{A}_\bullet(X_t)$$

such that $s \circ -_\eta = -_t$.

The degree 0 component of s preserves degrees by [21, Proposition 20.3 a)] and, by [21, Example 20.3.3], we have $s(\alpha_\eta \cdot H) = s(\alpha_\eta) \cdot H_{|X_t}$ for any cycle $\alpha_\eta \in \mathrm{A}_\bullet(X_\eta)$ on the generic fiber. It follows that we have

$$c_k(\mathscr{G})_\eta \cdot H_{|X_\eta}^{n-m-k} = c_k(\mathscr{G})_t \cdot H_{|X_t}^{n-m-k}.$$

Notice that $[X]_t = [X_t]$ since X is flat over T and $\{t\} \subset T$ is a regular embedding (see [21, Theorem 6.2 b)]). Using functoriality of Gysin homomorphisms (see [21, Theorem 6.5]) together with [21, Proposition 6.3], one readily checks that the image of $c_k(\mathscr{G})_t$ under the isomorphism $\mathrm{A}_{n-k}(X_t) \cong \mathrm{A}_{n-k}(X^\circ \cap X_t)$ is $c_k(\mathscr{G}_{|X^\circ \cap X_t})$. This implies that $c_k(\mathscr{G})_t = c_k(\mathscr{G}_t) \in \mathrm{A}_{n-k}(X_t)$. Similarly, using functoriality of flat pull-backs together with [21, Theorem 3.2 d)], we see that $c_k(\mathscr{G})_\eta = c_k(\mathscr{G}_\eta) \in \mathrm{A}_{n-k}(X_\eta)$. This completes the proof of the lemma. □

2.13 (Singularities) We refer to [40, Sect. 2.3] for details. Let X be a normal complex projective variety. Suppose that K_X is $\mathbb{Q}$-Cartier, i.e., some nonzero multiple of it is a Cartier divisor. Let $\beta \colon \widehat{X} \to X$ be a resolution of singularities of X. This means that $\widehat{X}$ is a smooth projective variety, β is a birational projective morphism whose exceptional locus is the union of prime divisors E_i, and the divisor $\sum E_i$ has simple normal crossing support. There are uniquely defined rational numbers $a(E_i, X)$ such that

$$K_{\widehat{X}} \equiv \beta^* K_X + \sum a(E_i, X) E_i.$$

The numbers $a(E_i, X)$ do not depend on the resolution β, but only on the valuations associated to the divisors E_i. We say that X is *terminal* (respectively, *canonical*) if, for some resolution of singularities $\beta \colon \widehat{X} \to X$ of X, $a(E_i, X) > 0$ (respectively, $a(E_i, X) \geq 0$) for every β-exceptional prime divisor E_i. If these conditions hold for some log resolution of X, then they hold for every log resolution of X.

3 Foliations

We first recall basic facts concerning foliations.

Definition 3.1 A *foliation* on a normal variety X over a field k is a coherent subsheaf $\mathscr{G} \subseteq T_X$ such that

1. $\mathscr{G}$ is closed under the Lie bracket, and
2. $\mathscr{G}$ is saturated in T_X. In other words, the quotient $T_X/\mathscr{G}$ is torsion-free.

The *rank* r of $\mathscr{G}$ is the generic rank of $\mathscr{G}$. The *codimension* of $\mathscr{G}$ is defined as $q := \dim X - r$.

Suppose that $k = \mathbb{C}$.

Let $X^\circ \subset X_{\text{reg}}$ be the open set where $\mathscr{G}_{|X_{\text{reg}}}$ is a subbundle of $T_{X_{\text{reg}}}$. A *leaf* of $\mathscr{G}$ is a connected, locally closed holomorphic submanifold $L \subset X^\circ$ such that $T_L = \mathscr{G}_{|L}$. A leaf is called *algebraic* if it is open in its Zariski closure.

The foliation $\mathscr{G}$ is said to be *algebraically integrable* if its leaves are algebraic.

3.2 (Analytic graph of a regular foliation) Let X be a complex manifold, and let $\mathscr{G} \subseteq T_X$ be a regular foliation. Set $Z := X \times X$, and let $Y \subset Z$ be the diagonal embedding of $X =: Y$. Denote by $p_1, p_2 \colon Z = X \times X \to X$ the projections onto X. Applying Frobenius' Theorem to the regular foliation

$$p_1^*\mathscr{G} \subseteq p_1^*T_X \subset p_1^*T_X \oplus p_2^*T_X = T_Z$$

on Z, we see that there exists a smooth locally closed analytic submanifold $V \subset Z$ containing Y such that $p_{2|V}$ is smooth, and such that its fibers are analytic open subsets of the leaves of the foliation $p_1^*\mathscr{G} \subset T_Z$ passing through points of Y. Notice that $\mathscr{N}_{Y/V} \cong \mathscr{G}$. The *analytic graph* of the foliation $(X, \mathscr{G})$ is the analytic germ of V along Y (see also [12, Sect. 2.2.2]).

3.3 (Foliations defined by q-forms) Let $\mathscr{G}$ be a codimension q foliation on an n-dimensional normal variety X. The *normal sheaf* of $\mathscr{G}$ is $\mathscr{N} := (T_X/\mathscr{G})^{**}$. The q-th wedge product of the inclusion $\mathscr{N}^* \hookrightarrow \Omega_X^{[1]}$ gives rise to a non-zero global section $\omega \in H^0\big(X, \Omega_X^q \boxtimes \det(\mathscr{N})\big)$ whose zero locus has codimension at least two in X. Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{G}$ is closed under the Lie bracket.

Conversely, let $\mathscr{L}$ be a reflexive sheaf of rank 1 on X, and let $\omega \in H^0(X, \Omega_X^q \boxtimes \mathscr{L})$ be a global section whose zero locus has codimension at least two in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1} \boxtimes \mathscr{L}$ given by the contraction with ω defines a foliation of codimension q on X. These constructions are inverse of each other.

3.4 (Foliations described as pull-backs) Let X and Y be normal varieties, and let $\varphi\colon X \dashrightarrow Y$ be a dominant rational map that restricts to a morphism $\varphi^\circ\colon X^\circ \to Y^\circ$, where $X^\circ \subset X$ and $Y^\circ \subset Y$ are smooth open subsets.

Let $\mathscr{G}$ be a codimension q foliation on Y. Suppose that the restriction $\mathscr{G}^\circ$ of $\mathscr{G}$ to Y° is defined by a twisted q-form $\omega_{Y^\circ} \in H^0\big(Y^\circ, \Omega^q_{Y^\circ} \otimes \det(\mathscr{N}_{\mathscr{G}^\circ})\big)$. Then ω_{Y° induces a non-zero twisted q-form $\omega_{X^\circ} \in H^0\Big(X^\circ, \Omega^q_{X^\circ} \otimes (\varphi^\circ)^*\big(\det(\mathscr{N}_{\mathscr{G}})_{|Y^\circ}\big)\Big)$, which in turn defines a codimension q foliation $\mathscr{E}^\circ$ on X°. *The pull-back $\varphi^{-1}\mathscr{G}$ of $\mathscr{G}$ via φ* is the foliation on X whose restriction to X° is $\mathscr{E}^\circ$.

Definition 3.5 Let $\psi\colon X \to Y$ be an equidimensional dominant morphism of normal varieties, and let D be a Weil $\mathbb{Q}$-divisor on Y. The pull-back ψ^*D of D is defined as follows. We define ψ^*D to be the unique $\mathbb{Q}$-divisor on X whose restriction to $\psi^{-1}(Y_{\text{reg}})$ is $(\psi_{|\psi^{-1}(Y_{\text{reg}})})^*(D_{|Y_{\text{reg}}})$. This construction agrees with the usual pull-back if D is $\mathbb{Q}$-Cartier.

We will use the following notation.

Notation 3.6 Let $\psi\colon X \to Y$ be an equidimensional dominant morphism of normal varieties. Write $K_{X/Y} := K_X - \psi^*K_Y$. We refer to it as the *relative canonical divisor of X over Y*.

Notation 3.7 Let $\psi\colon X \to Y$ be an equidimensional dominant morphism of normal varieties. Set

$$R(\psi) = \sum_D \big(\psi^*D - (\psi^*D)_{\text{red}}\big)$$

where D runs through all prime divisors on Y. We refer to it as the *ramification divisor of ψ*.

Definition 3.8 Let $\mathscr{G}$ be a foliation on a normal projective variety X. The *canonical class $K_{\mathscr{G}}$* of $\mathscr{G}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{G}}) \cong \det(\mathscr{G})$.

Example 3.9 Let $\psi\colon X \to Y$ be an equidimensional dominant morphism of normal varieties, and let $\mathscr{G}$ be the foliation on X induced by ψ. A straightforward computation shows that

$$K_{\mathscr{G}} = K_{X/Y} - R(\psi).$$

3.10 (The family of leaves) Let X be a normal complex projective variety, and let $\mathscr{G}$ be an algebraically integrable foliation on X. We describe the *family of leaves* of $\mathscr{G}$ (see [1, Remark 3.12]).

There is a unique normal complex projective variety Y contained in the normalization of the Chow variety of X whose general point parametrizes the closure of a general leaf of $\mathscr{G}$ (viewed as a reduced and irreducible cycle in X). Let $Z \to Y \times X$ denotes the normalization of the universal cycle. It comes with morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\beta} & X \\ {\scriptstyle\psi}\downarrow & & \\ Y & & \end{array}$$

where $\beta\colon Z \to X$ is birational and, for a general point $y \in Y$, $\beta\big(\psi^{-1}(y)\big) \subset X$ is the closure of a leaf of $\mathscr{G}$. The variety Y is called the *family of leaves* of $\mathscr{G}$.

Suppose furthermore that $K_{\mathscr{G}}$ is **Q**-Cartier. There is a canonically defined effective Weil **Q**-divisor B on Z such that

$$K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}, \tag{3.1}$$

where $R(\psi)$ denotes the ramification divisor of ψ.

Remark 3.11 In the setup of 3.10, notice that B is β-exceptional. This is an immediate consequence of Example 3.9.

We will need the following easy observation.

Lemma 3.12 *Let X be a normal complex projective variety with $\mathbb{Q}$-factorial terminal singularities, and let $\mathscr{G}$ be an algebraically integrable foliation on X. Suppose that K_X is pseudo-effective and that $K_{\mathscr{G}} \sim_{\mathbb{Q}} 0$. Let $\psi\colon Z \to Y$ be the family of leaves, and let $\beta\colon Z \to X$ be the natural morphism. Then $\varphi := \psi \circ \beta^{-1}$ is an almost proper map, and $K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} 0$.*

Proof Notice that $\mathscr{G}$ is induced by $\varphi := \psi \circ \beta^{-1}\colon X \dashrightarrow Y$.

It follows from 3.10 that there is a canonically defined effective Weil **Q**-divisor B on Z such that

$$K_{\beta^{-1}\mathscr{G}} + B = K_{Z/Y} - R(\psi) + B \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}} \sim_{\mathbb{Q}} 0, \tag{3.2}$$

where $R(\psi)$ denotes the ramification divisor of ψ. Recall from Remark 3.11, that B is β-exceptional. Moreover, since X has $\mathbb{Q}$-factorial terminal singularities, there exists an effective $\mathbb{Q}$-divisor E on X such that

$$K_Z = \beta^* K_X + E \quad \text{and} \quad \operatorname{Supp}(E) = \operatorname{Exc}(\beta). \tag{3.3}$$

From Eqs. (3.2) and (3.3), we obtain

$$R(\psi) \sim_{\mathbb{Q}} \beta^* K_X - \psi^* K_Y + B + E. \tag{3.4}$$

Consider a general fiber F of ψ. Equation (3.4) then shows that

$$(\beta^* K_X + B + E)_{|F} \sim_{\mathbb{Q}} 0.$$

Since B and E are both effective divisors, and since K_X is pseudo-effective, we must have $E \cap F = \emptyset$. The equality $\mathrm{Exc}(\beta) = \mathrm{Supp}(E)$ then shows that φ is an almost proper map.

By the adjunction formula, $K_F \sim_{\mathbb{Z}} K_{Z|F}$, and thus K_F is pseudo-effective. Applying [16, Corollary 4.5] to ψ, we see that $K_{\beta^{-1}\mathscr{G}} = K_{Z/Y} - R(\psi)$ is pseudo-effective. Equation (3.2) then shows that

$$K_{\beta^{-1}\mathscr{G}} \sim_{\mathbb{Q}} 0 \quad \text{and} \quad B = 0.$$

This finishes the proof of the lemma. □

It is well-known that an algebraically integrable regular foliation on a complex projective manifold is induced by a morphism onto a normal projective variety (see [35, Proposition 2.5]). The next proposition extends this result to some foliations on mildly singular varieties.

Proposition 3.13 *Let X be a normal complex projective variety with $\mathbb{Q}$-factorial terminal singularities, and let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition of T_X into involutive subsheaves. Suppose that K_X is pseudo-effective and that $\det(\mathscr{E}) \cong \mathscr{O}_X$. Suppose furthermore that $\mathscr{E}$ is algebraically integrable. Then there exists an open subset $X^\circ \subset X$ with complement of codimension at least two and a projective morphism with irreducible fibers $\phi^\circ \colon X^\circ \to Y^\circ$ onto a smooth quasi-projective variety such that $\mathscr{E}$ is induced by ϕ°. Moreover, there exists a finite morphism $\gamma^\circ \colon \widetilde{Y}^\circ \to Y^\circ$ satisfying the following property. Let $\widetilde{X}^\circ$ be the normalization of $\widetilde{Y}^\circ \times_{Y^\circ} X^\circ$, and denote by $\widetilde{\phi}^\circ \colon \widetilde{X}^\circ \to \widetilde{Y}^\circ$ the natural morphism. Then $\widetilde{\phi}^\circ$ is a locally trivial analytic fibration for the analytic topology.*

Proof Let $\psi \colon Z \to Y$ be the family of leaves, and let $\beta \colon Z \to X$ be the natural morphism (see 3.10). By [16, Lemma 4.2], there exists a finite surjective morphism $\gamma \colon Y_1 \to Y$ with Y_1 normal and connected such that the following holds. Let Z_1 denotes the normalization of $Y_1 \times_Y Z$. Then the induced morphism $\psi_1 \colon Z_1 \to Y_1$ has reduced fibers over codimension one points in Y_1. Hence, we obtain a commutative diagram as follows,

$$\begin{array}{ccccc} Z_1 & \xrightarrow{\alpha,\ \text{finite}} & Z & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi} & \swarrow{\scriptstyle \phi} & \\ Y_1 & \xrightarrow[\gamma,\ \text{finite}]{} & Y. & & \end{array}$$

Claim 3.14 The tangent sheaf T_{Z_1} decomposes as a direct sum

$$T_{Z_1} = (\beta \circ \alpha)^{-1}\mathscr{E} \oplus (\beta \circ \alpha)^{-1}\mathscr{G}.$$

Proof of Claim 3.14 Set $q := \operatorname{rank} \mathscr{E}$, and let $\omega \in H^0\big(X, \Omega_X^{[q]}\big)$ a q-form defining $\mathscr{G}$. Then $\beta^*\omega_{|Z\setminus \mathrm{Exc}(\beta)}$ extends across $\mathrm{Exc}(\beta)$ and gives a q-form $\beta^*\omega \in H^0\big(Z, \Omega_Z^{[q]}\big)$ by [24, Theorem 1.5]. The q-form $\alpha^*(\beta^*\omega) \in H^0\big(Z_1, \Omega_{Z_1}^{[q]}\big)$ defines the foliation $(\beta\circ\alpha)^{-1}\mathscr{G}$ on a dense open set, and induces an $\mathscr{O}_{Z_1}$-linear map $(\Lambda^q T_{Z_1})^{**} \to \mathscr{O}_{Z_1}$ such that the composed morphism of reflexive sheaves of rank one

$$\sigma\colon \det\big((\beta \circ \alpha)^{-1}\mathscr{E}\big) \to (\Lambda^q T_{Z_1})^{**} \to \mathscr{O}_{Z_1}$$

is generically non-zero. By Lemma 3.12, we know that $K_{\beta^{-1}\mathscr{E}} \sim_{\mathbb{Q}} 0$. A straightforward computation then shows that

$$K_{(\beta\circ\alpha)^{-1}\mathscr{E}} = \alpha^* K_{\beta^{-1}\mathscr{E}} \sim_{\mathbb{Q}} 0,$$

and hence σ must be an isomorphism. This immediately implies that

$$T_{Z_1} = (\beta \circ \alpha)^{-1}\mathscr{E} \oplus (\beta \circ \alpha)^{-1}\mathscr{G},$$

proving our claim. □

Let $Z_1^\circ \subset \psi_1^{-1}(Y_{1\,\mathrm{reg}})$ be the open set where $\psi_{1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}$ is smooth. Notice that Z_1° has complement of codimension at least two since ψ_1 has reduced fibers over codimension one points in $Y_{1\,\mathrm{reg}}$. The restriction of the tangent map

$$T\psi_{1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}\colon T_{Z_1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})} \to \big(\psi_{1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}\big)^* T_{Y_{1\,\mathrm{reg}}}$$

to $(\beta \circ \alpha)^{-1}\mathscr{G}_{|\psi_1^{-1}(Y_{1\,\mathrm{reg}})} \subset T_{Z_1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}$ then induces an isomorphism $(\beta \circ \alpha)^{-1}\mathscr{G}_{|Z_1^\circ} \cong \big(\psi_{1|Z_1^\circ}\big)^* T_{Y_{1\,\mathrm{reg}}}$, and since $(\beta \circ \alpha)^{-1}\mathscr{G}_{|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}$ and $\big(\psi_{1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}\big)^* T_{Y_{1\,\mathrm{reg}}}$ are both reflexive sheaves, we finally obtain an isomorphism of sheaves of Lie algebras

$$\tau\colon (\beta \circ \alpha)^{-1}\mathscr{G}_{|\psi_1^{-1}(Y_{1\,\mathrm{reg}})} \cong \big(\psi_{1|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}\big)^* T_{Y_{1\,\mathrm{reg}}}.$$

Set $m := \dim Y = \dim Y_1$. Let $y \in Y_{1\,\mathrm{reg}}$, and let $U \ni y$ be an open neighborhood of y in $Y_{1\,\mathrm{reg}}$ with coordinates $y_1, \ldots, y_m$ on U. A classical result of complex analysis says that there exists a unique local $\mathbb{C}^m$-action on $\psi_1^{-1}(U)$

corresponding to the flat connection $(\beta\circ\alpha)^{-1}\mathscr{G}_{|\psi_1^{-1}(U)}$ on $\psi 1_{|\psi_1^{-1}(U)}$. The local $\mathbb{C}^m$-action on $\psi_1^{-1}(U)$ is given by a holomorphic map $\Phi\colon W\to\psi_1^{-1}(U)$, where W is an open neighborhood of the neutral section $\{0\}\times\psi_1^{-1}(U)$ in $\mathbb{C}^m\times\psi_1^{-1}(U)$ such that

1. For all $z\in\psi_1^{-1}(U)$, the subset $\{t\in\mathbb{C}^m\mid(t,z)\in W\}$ is connected,
2. Setting $t\top z:=\Phi(t,z)$, we have $0\top z=z$ for all $z\in\psi_1^{-1}(U)$, if $(t+t',z)\in W$, if $(t',z)\in W$ and $(t,t'\top z)\in W$, then $(t+t')\top z=t\top(t'\top z)$ holds.

Moreover, the above local $\mathbb{C}^m$-action on $\psi_1^{-1}(U)$ extends the local $\mathbb{C}^m$-action on $U\subset\mathbb{C}^m$ given by $y_i(t\top y)=t_i+y_i(y)$ for any $t=(t_1,\dots,t_m)\in\mathbb{C}^m$ and $y\in U$ such that $t\top y\in U$. This immediately implies that $\psi 1_{|\psi_1^{-1}(Y_{1\,\mathrm{reg}})}$ is a locally trivial analytic fibration for the analytic topology.

By Lemma 3.12, ψ maps any irreducible component of $\mathrm{Exc}(\beta)$ to a codimension one hypersurface. It follows that

$$\mathrm{Exc}(\beta)=\psi^{-1}\Big(\psi\big(\mathrm{Exc}(\beta)\big)\Big).$$

Set $Y^\circ:=Y_{\mathrm{reg}}\setminus\Big(\psi\big(\mathrm{Exc}(\beta)\big)\cup\gamma\big(Y_1\setminus Y_{1\,\mathrm{reg}}\big)\Big)$, $X^\circ:=\beta\big(\psi^{-1}(Y^\circ)\big)$, $\phi^\circ:=\phi_{|X^\circ}$, and consider $\gamma^\circ:=\gamma_{|\gamma^{-1}(Y^\circ)}\colon\gamma^{-1}(Y^\circ)=:\widetilde{Y}^\circ\to Y^\circ$. One readily checks that ϕ°, γ°, and $\widetilde{\phi}^\circ$ satisfy the conclusions of Proposition 3.13. □

4 Towards a decomposition theorem

The main results of this section assert that algebraic integrability of direct summands in the infinitesimal analogue of the Beauville–Bogomolov decomposition theorem (Theorem 1.1) leads to a decomposition of the variety, perhaps after passing to a finite cover that is étale in codimension one (see Theorem 4.5 and Proposition 4.10). This solves [25, Problem 8.4].

First, we recall structure results for complex varieties with numerically trivial canonical divisor. The following invariant is relevant in their investigation (see [25, Definition 3.1]).

Definition 4.1 Let X be a normal complex projective variety. We denote the irregularity of X by $q(X):=h^1(X,\mathscr{O}_X)$ and define the *augmented irregularity* as

$$\widetilde{q}(X):=\max\big\{q(\widetilde{X})\mid\widetilde{X}\to X\text{ a finite cover, étale in codimension one}\big\}\\ \in\mathbb{N}\cup\{\infty\}.$$

Remark 4.2 By a result of Elkik [19], canonical singularities are rational. It follows that the irregularity is a birational invariant of complex projective varieties with canonical singularities.

Remark 4.3 If X is a complex projective variety with canonical singularities and numerically trivial canonical class, [37, Proposition 8.3] implies that $q(X) \leq \dim X$. If $\widetilde{X} \to X$ is any finite cover, étale in codimension one, then $\widetilde{X}$ will likewise have canonical singularities (see [43, Proposition 3.16]), and numerically trivial canonical class. In summary, we see that $\widetilde{q}(X) \leq \dim X$. The augmented irregularity of canonical varieties with numerically trivial canonical class is therefore finite.

We will need the following easy observation.

Lemma 4.4 *Let X and Y be normal complex projective varieties with canonical singularities, and let $\beta: Y \to X$ be a birational morphism. Suppose that $K_Y \equiv 0$. Then $\widetilde{q}(X) \geq \widetilde{q}(Y)$.*

Proof Notice first that $\widetilde{q}(Y)$ is finite by Remark 4.3 above. Let $g: Y_1 \to Y$ be a finite cover, étale in codimension one, such that $h^1(Y_1, \mathscr{O}_{Y_1}) = \widetilde{q}(Y)$. Let $f: X_1 \to X$ be the Stein factorization of the composed map $Y_1 \to Y \to X$. Then f is obviously étale in codimension one. From [43, Proposition 3.16], we see that X_1 has canonical singularities, and hence $h^1(X_1, \mathscr{O}_{X_1}) = h^1(Y_1, \mathscr{O}_{Y_1})$ by Remark 4.2. This finishes the proof of the lemma. □

The following result often reduces the study of varieties with trivial canonical class to those with $\widetilde{q}(X) = 0$ (see also [37, Proposition 8.3]).

Theorem 4.5 [25, Corollary 3.6] *Let X be a normal complex projective variety with canonical singularities. Assume that K_X is numerically trivial. Then there exist projective varieties A, $\widetilde{X}$ and a morphism $f: A \times \widetilde{X} \to X$ such that the following holds.*

1. *The variety A is Abelian.*
2. *The variety $\widetilde{X}$ is normal and has canonical singularities.*
3. *The canonical class of $\widetilde{X}$ is trivial, $\omega_{\widetilde{X}} \cong \mathscr{O}_{\widetilde{X}}$.*
4. *The augmented irregularity of $\widetilde{X}$ is zero, $\widetilde{q}(\widetilde{X}) = 0$.*
5. *The morphism f is finite, surjective and étale in codimension one.*

Before we give the proof of Proposition 4.10, we need the following auxiliary results. The author would like to thank Cinzia Casagrande who explained Lemma 4.6 to him.

Lemma 4.6 *Let X_1, X_2 and Y be complex normal projective varieties. Suppose that there exists a surjective morphism with connected fibers $\beta: X_1 \times X_2 \to Y$. Suppose furthermore that $q(X_1) = 0$. Then Y decomposes as a product $Y \cong Y_1 \times Y_2$, and there exist surjective morphisms with connected fibers $\beta_1: X_1 \to Y_1$ and $\beta_2: X_2 \to Y_2$ such that $\beta = \beta_1 \times \beta_2$.*

Proof Let H be an ample Cartier divisor on Y. Note that $\mathrm{Pic}(X_1 \times X_2) \cong \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$ since $q(X_1) = 0$. Thus there exist Cartier divisors G_1 and G_2 on X_1 and X_2 respectively such that $\beta^* H \sim_{\mathbb{Z}} \pi_1^* G_1 + \pi_2^* G_2$, where π_i is the projection onto X_i. Let $\beta_i : X_i \to Y_i$ be the morphism corresponding to the semiample divisor G_i, so that $m_i G_i \sim_{\mathbb{Z}} \beta_i^* H_i$ for some ample Cartier divisor H_i on Y_i and some positive integer m_i.

Let $C \subset X_1 \times X_2$ be a complete curve contracted by $\beta_1 \times \beta_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$. Then $(\pi_i^* G_i) \cdot C = 0$, and hence $\beta^* H \cdot C = 0$. This implies that C is contracted by β, and hence β factors through $\beta_1 \times \beta_2$ by the rigidity lemma. Thus, there exists a morphism $\gamma \colon Y_1 \times Y_2 \to Y$ such that $\beta = \gamma \circ (\beta_1 \times \beta_2)$. Denote by $p_i \colon Y_1 \times Y_2 \to Y_i$ the projection onto Y_i. We obtain a commutative diagram as follows,

$$\begin{array}{ccccc} X_1 \times X_2 & \xrightarrow{\beta_1 \times \beta_2} & Y_1 \times Y_2 & \xrightarrow{\gamma} & Y \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle p_i} & & \\ X_i & \xrightarrow[\beta_i]{} & Y_i. & & \end{array} \qquad (\beta = \gamma \circ (\beta_1 \times \beta_2) \colon X_1 \times X_2 \to Y)$$

Then

$$(\beta_1 \times \beta_2)^*(m_2 p_1^* H_1 + m_1 p_2^* H_2) \sim_{\mathbb{Z}} m_1 m_2 (\pi_1^* G_1 + \pi_2^* G_2) \sim_{\mathbb{Z}} m_1 m_2 \beta^* H$$
$$= (\beta_1 \times \beta_2)^*(m_1 m_2 \gamma^* H).$$

This implies that

$$m_2 p_1^* H_1 + m_1 p_2^* H_2 \sim_{\mathbb{Z}} m_1 m_2 \gamma^* H$$

since the map $\beta_1 \times \beta_2$ is surjective with connected fibers. Because $m_2 p_1^* H_1 + m_1 p_2^* H_2$ is ample, we conclude that γ is a finite morphism, and hence an isomorphism since β is surjective with connected fibers. This completes the proof of the lemma. □

4.7 (Terminalization) Let X be a normal complex projective variety with canonical singularities. Recall that a $\mathbb{Q}$-*factorial terminalization* of X is a birational crepant morphism $\beta \colon \widehat{X} \to X$ where $\widehat{X}$ is a $\mathbb{Q}$-factorial projective variety with terminal singularities. The existence of β is established in [5, Corollary 1.4.3].

Proposition 4.8 *Let X_1, X_2 and Y be complex projective varieties with canonical singularities such that K_{X_1}, K_{X_2} and K_Y are nef, and let $\varphi \colon X_1 \times X_2 \dashrightarrow Y$ be a birational map. Suppose that $q(X_1) = 0$. Then Y decomposes as a product $Y \cong Y_1 \times Y_2$, and there exist birational maps $\varphi_1 \colon X_1 \dashrightarrow Y_1$ and $\varphi_2 \colon X_2 \dashrightarrow Y_2$ such that $\varphi = \varphi_1 \times \varphi_2$.*

Proof Let $\beta_1 \colon \widehat{X}_1 \to X_1$, $\beta_2 \colon \widehat{X}_2 \to X_2$ and $\gamma \colon \widehat{Y} \to Y$ be $\mathbb{Q}$-factorial terminalizations of X_1, X_2, and Y respectively. Notice that $K_{\widehat{X}_1 \times \widehat{X}_2}$ and $K_{\widehat{Y}}$ are nef. Set $\widehat{\varphi} := \gamma^{-1} \circ \varphi \circ (\beta_1 \times \beta_2) \colon \widehat{X}_1 \times \widehat{X}_2 \dashrightarrow \widehat{Y}$. Hence, we obtain a commutative diagram as follows,

$$\begin{array}{ccc} \widehat{X}_1 \times \widehat{X}_2 & \overset{\widehat{\varphi}}{\dashrightarrow} & \widehat{Y} \\ {\scriptstyle \beta_1 \times \beta_2} \downarrow & & \downarrow {\scriptstyle \gamma} \\ X_1 \times X_2 & \underset{\varphi}{\dashrightarrow} & Y. \end{array}$$

Recall from [8, Théorème 6.5] that the product of complex $\mathbb{Q}$-factorial algebraic varieties is $\mathbb{Q}$-factorial. In particular, $\widehat{X}_1 \times \widehat{X}_2$ is $\mathbb{Q}$-factorial.

It follows from Lemma 4.6 applied to γ and Remark 4.2 that it suffices to prove Proposition 4.8 for $\widehat{\varphi}$.

Now, by [38, Theorem 1], $\widehat{\varphi}$ decomposes into a sequence of flops, and therefore, using repeatedly Lemma 4.6, it suffices to prove Proposition 4.8 for a flop. Thus, we may assume that there exists a commutative diagram

$$\begin{array}{ccc} \widehat{X}_1 \times \widehat{X}_2 & \overset{\varphi}{\dashrightarrow} & \widehat{Y} \\ & {\scriptstyle \alpha} \searrow \quad \swarrow {\scriptstyle \alpha^+} & \\ & Z & \end{array}$$

where α and α^+ are small elementary birational contractions, $K_{\widehat{X}_1 \times \widehat{X}_2}$ is numerically α-trivial, and $K_{\widehat{Y}}$ is numerically α^+-trivial. By Lemma 4.6 applied to α, Z decomposes as a product $Z \cong Z_1 \times Z_2$, and there exist birational morphisms $\alpha_1 \colon \widehat{X}_1 \to Z_1$ and $\alpha_2 \colon \widehat{X}_2 \to Z_2$ such that $\alpha = \alpha_1 \times \alpha_2$. Note that α_1 or α_2 is an isomorphism since $\rho(\widehat{X}_1 \times \widehat{X}_2/Z) = 1$. We may therefore assume without loss of generality that α_2 is an isomorphism. Let $\alpha_1^+ \colon \widehat{X}_1^+ \to Z_1$ be the flop of α_1 whose existence is established in [5, Corollary 1.4.1]. Then $\widehat{X}_1^+ \times \widehat{X}_2 \to Z_1 \times Z_2$ is the flop of α, proving the proposition. □

We end the preparation for the proof of Proposition 4.10 with the following lemma. It reduces the study of varieties with canonical singularities and trivial canonical class to those with terminal singularities.

Lemma 4.9 *Let X be a normal complex projective variety with canonical singularities, and let $\beta \colon \widehat{X} \to X$ be a $\mathbb{Q}$-factorial terminalization of X. Let*

$$T_X = \bigoplus_{i \in I} \mathscr{E}_i$$

be a decomposition of T_X into involutive subsheaves with $\det(\mathscr{E}_i) \cong \mathscr{O}_X$. *Then there is a decomposition*

$$T_{\widehat{X}} = \bigoplus_{i \in I} \widehat{\mathscr{E}}_i$$

of $T_{\widehat{X}}$ *into involutive subsheaves with* $\det(\widehat{\mathscr{E}_i}) \cong \mathscr{O}_{\widehat{X}}$ *such that* $\mathscr{E}_i \cong (\beta_*\widehat{\mathscr{E}_i})^{**}$.

Proof Notice that $\omega_{\widehat{X}} \cong \mathscr{O}_{\widehat{X}}$. Denote by q_i the codimension of $\mathscr{E}_i$, and consider $\omega_i \in H^0\big(X, \Omega_X^{[q_i]}\big)$ a q_i-form defining $\mathscr{E}_i$. By [24, Theorem 1.5], ω_i extends to a q_i-form $\widehat{\omega}_i \in H^0\big(\widehat{X}, \Omega_{\widehat{X}}^{[q_i]}\big)$. Then $\widehat{\omega}_i$ defines a foliation $\widehat{\mathscr{E}_i} \subseteq T_{\widehat{X}}$ with $\det(\widehat{\mathscr{E}_i}) \cong \mathscr{O}_{\widehat{X}}(E_i)$ where E_i is the maximal effective divisor on $\widehat{X}$ such that $\widehat{\omega}_i \in H^0\big(\widehat{X}, \Omega_{\widehat{X}}^{q_i} \boxtimes \mathscr{O}_{\widehat{X}}(-E_i)\big)$. The natural map $\bigoplus_{i\in I} \widehat{\mathscr{E}_i} \to T_{\widehat{X}}$ being generically injective, we obtain

$$\mathscr{O}_{\widehat{X}}\left(\sum_{i\in I} E_i + E\right) \cong \det(T_{\widehat{X}}) \cong \mathscr{O}_{\widehat{X}}$$

for some effective divisor E on $\widehat{X}$. It follows that $E_i = 0$ for every $i \in I$, and that $T_{\widehat{X}}$ decomposes as a direct sum

$$T_{\widehat{X}} = \bigoplus_{i\in I} \widehat{\mathscr{E}_i}$$

of involutive subsheaves with trivial determinants. The sheaves $\mathscr{E}_i$ and $(\beta_*\widehat{\mathscr{E}_i})^{**}$ agree outside of the β-exceptional set, and since both are reflexive, we obtain an isomorphism $\mathscr{E}_i \cong (\beta_*\widehat{\mathscr{E}_i})^{**}$. This finishes the proof of Lemma 4.9. □

The following result together with Theorem 4.5 can be seen as a first step towards a decomposition theorem.

Proposition 4.10 *Let* X *be a normal complex projective variety with canonical singularities, and let*

$$T_X = \bigoplus_{i\in I} \mathscr{E}_i$$

be a decomposition of T_X *into involutive subsheaves. Suppose that* $\widetilde{q}(X) = 0$. *Suppose furthermore that the* $\mathscr{E}_i$ *are algebraically integrable with* $\det(\mathscr{E}_i) \cong \mathscr{O}_X$. *Then there exist a projective variety* $\widetilde{X}$ *with canonical singularities, a finite cover* $f\colon \widetilde{X} \to X$, *étale in codimension one, and a decomposition*

$$\widetilde{X} \cong \prod_{i\in I} Y_i$$

such that the induced decomposition of $T_{\widetilde{X}}$ *agree with the decomposition*

$$T_{\widetilde{X}} = \bigoplus_{i\in I} f^{[*]}\mathscr{E}_i.$$

Proof For the reader's convenience, the proof is subdivided into a number of relatively independent steps.

Step 1 Reduction to X $\mathbb{Q}$-factorial and terminal. Let $\beta\colon Z \to X$ be a $\mathbb{Q}$-factorial terminalization of X. By Lemma 4.9, the tangent sheaf T_Z decomposes as a direct sum

$$T_Z = \bigoplus_{i \in I} \mathscr{G}_i$$

of involutive subsheaves with trivial determinants such that $\mathscr{E}_i \cong (\beta_* \mathscr{G}_i)^{**}$. Notice that $\widetilde{q}(Z) = 0$ by Lemma 4.4.

Suppose that there exists a finite cover $g\colon \widetilde{Z} \to Z$, étale in codimension one, such that $\widetilde{Z}$ decomposes as a product

$$\widetilde{Z} \cong \prod_{i \in I} T_i$$

such that the induced decomposition of $T_{\widetilde{Z}}$ agree with the decomposition

$$T_{\widetilde{Z}} = \bigoplus_{i \in I} g^{[*]} \mathscr{G}_i.$$

From the Künneth formula (see [29, Theorem 6.7.8]), we see that $q(T_i) = 0$ for any $i \in I$. Let $f\colon \widetilde{X} \to X$ be the Stein factorization of the composed map $\widetilde{Z} \to Z \to X$. Then f is étale in codimension one, and thus $\widetilde{X}$ has canonical singularities by [43, Proposition 3.16]. Applying Lemma 4.6 to $\widetilde{Z} \to \widetilde{X}$, we see that $\widetilde{X}$ decomposes as a product

$$\widetilde{X} \cong \prod_{i \in I} Y_i$$

such that the induced decomposition of $T_{\widetilde{X}}$ agree with the decomposition

$$T_{\widetilde{X}} = \bigoplus_{i \in I} f^{[*]} \mathscr{G}_i.$$

We can therefore assume without loss of generality that the following holds.

Assumption 4.11 The variety X has $\mathbb{Q}$-factorial terminal singularities.

To prove Proposition 4.10, it is obviously enough to consider the case where $I = \{1, 2\}$. Set $\tau(i) = 3 - i$ for each $i \in I$.

Step 2 For $i \in I$, let $\psi_i\colon Z_i \to Y_i$ be the family of leaves, and let $\beta_i\colon Z_i \to X$ be the natural morphism (see 3.10). Notice that $\mathscr{E}_i$ is induced by $\varphi_i :=$

$\psi_i \circ \beta_i^{-1} : X \dashrightarrow Y_i$. By Lemma 3.12, the rational map φ_i is almost proper. Moreover, it induces a regular map $X_{\text{reg}} \to Y_i$ since $\mathscr{E}_i$ is a regular foliation on X_{reg}.

Let F_i be a general fiber of $\varphi_{\tau(i)}$. Then F_i is a normal projective variety with terminal singularities, and $K_{F_i} \sim_{\mathbb{Q}} 0$. Set $F_i^\circ := F_i \cap X_{\text{reg}}$, and denote by $F_i^\circ \widetilde{\times_{Y_i}} X_{\text{reg}}$ the normalization of $F_i^\circ \times_{Y_i} X_{\text{reg}}$. Next, we will prove the following.

Claim 4.12 The natural map $F_i^\circ \widetilde{\times_{Y_i}} X_{\text{reg}} \to X_{\text{reg}}$ is finite and étale over an open subset contained in X_{reg} with complement of codimension at least two.

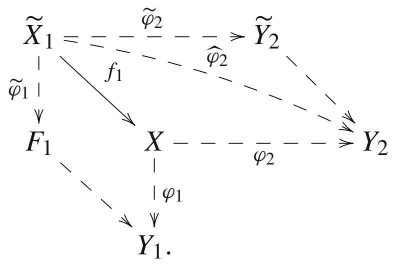
Proof of Claim 4.12 By Lemma 3.12 and Proposition 3.13, there exists an open subset $Y_i^\circ \subset Y_{i\,\text{reg}}$ and a dense open subset $X_i^\circ \subset X$ with complement of codimension at least two in X such that $\varphi_{i\,|X_i^\circ} : X_i^\circ \to Y_i^\circ$ is a projective morphism with irreducible fibers. Let P be a prime divisor on Y_i°, and write $\varphi_{i\,|X_i^\circ}^{\;*} P = t\,Q$ for some positive integer t and some prime divisor Q on X. Set $n := \dim X$, and $m_i := \dim Y_i$. Notice that $F_i \cap Q \neq \emptyset$. Since $\mathscr{E}_1$ and $\mathscr{E}_2$ are regular foliations at a general point x in Q and $T_X = \mathscr{E}_1 \oplus \mathscr{E}_2$, there exist local analytic coordinates centered at x and $y := \phi_i(x)$ respectively such that ϕ_i is given by $(x_1, x_2, \ldots, x_n) \mapsto (x_1^t, x_2 \ldots, x_{m_i})$, and such that F_i is given by equations $x_{m_i+1} = \cdots = x_n = 0$. The claim then follows from a straightforward local computation. □

Step 3 End of proof. Let $\widetilde{X}_1$ denotes the normalization of X in the function field of $F_1^\circ \widetilde{\times_{Y_1}} X_{\text{reg}}$. It comes with a finite morphism $f_1 : \widetilde{X}_1 \to X$ which is étale in codimension one by Claim 4.12. Let $\widetilde{\varphi}_1 : \widetilde{X}_1 \dashrightarrow F_1$ be the almost proper map induced by the first projection $F_1^\circ \times_{Y_1} X_{\text{reg}} \to F_1^\circ$, and let $G_1 \subset \widetilde{X}_1$ be the Zariski closure of the rational section of $\widetilde{\varphi}_1$ given by $F_1^\circ \to Y_1$. Finally, let $\widehat{\phi}_2$ denotes the composed map $\widetilde{X}_1 \to X \dashrightarrow Y_2$, and set $y_2 := \widehat{\varphi}_2(G_1) = \phi_2(F_1)$. Notice that $\widehat{\phi}_2$ is an almost proper map.

Claim 4.13 The following holds.

1. The variety G_1 is a fiber of the Stein factorization $\widetilde{\varphi}_2 : \widetilde{X}_1 \dashrightarrow \widetilde{Y}_2$ of $\widehat{\phi}_2$.
2. The fiber $\widehat{\varphi}_2^{-1}(y_2) = f_1^{-1}(F_1)$ is reduced along G_1.

Proof of Claim 4.13 We have a commutative diagram as follows,



Applying [43, Proposition 3.16], we see that $\widetilde{X}_1$ has terminal singularities. In particular, $\widetilde{X}_1$ is Cohen–Macaulay, and hence so is $\widehat{\phi}_2^{-1}(y_2)$ by [17, Proposition 18.13]. By the Nagata–Zariski purity theorem, f_1 branches only over the singular set of X. This immediately implies that $f_1^{-1}(F_1)$ is smooth in codimension one. Then (2) follows easily. By Hartshorne's connectedness theorem (see [17, Theorem 18.12]), we see that irreducible components of $\widehat{\varphi}_2^{-1}(y_2)$ are disjoint, proving (1). □

Let $\widetilde{F}_2$ be a general fiber of $\widetilde{\varphi}_1$. Then $\widetilde{F}_2$ intersects G_1 transversely in a point. From Claim 4.13 (1), it follows that $\widetilde{F}_2$ intersects a general fiber $\widetilde{F}_1$ of $\widetilde{\varphi}_2$ transversely in a point. This immediately implies that the map $\widetilde{\varphi}_1 \times \widetilde{\varphi}_2 \colon \widetilde{X}_1 \dashrightarrow F_1 \times \widetilde{Y}_2$ is birational. But it also implies that F_1 and $\widetilde{Y}_2$ are birationally equivalent to $\widetilde{F}_1$ and $\widetilde{F}_2$ respectively, and we conclude that there exists a birational map $\widetilde{X}_1 \dashrightarrow \widetilde{F}_1 \times \widetilde{F}_2$. From the Künneth formula (see [29, Theorem 6.7.8]) together with Remark 4.2, we see that $q(\widetilde{F}_i) = 0$ for any $i \in \{1, 2\}$. The conclusion then follows from Proposition 4.8. This finishes the proof of Proposition 4.10. □

5 Algebraicity of leaves, I

In this section we prove Theorem 1.4. The proof relies on an algebraicity criterion for leaves of algebraic foliations proved in [12, Theorem 2.1], which we recall now.

5.1 Let X be an algebraic variety over some field k of positive characteristic p, and let $\mathscr{G} \subset T_X$ be a subsheaf. We will denote by $\mathrm{F}_{\mathrm{abs}} \colon X \to X$ the absolute Frobenius morphism of X.

The sheaf of derivations $\mathrm{Der}_k(\mathscr{O}_X) \cong T_X$ is endowed with the p-th power operation, which maps any local k-derivation D of $\mathscr{O}_X$ to its p-th iterate $D^{[p]}$. When $\mathscr{G}$ is involutive, the map $\mathrm{F}_{\mathrm{abs}}^*\mathscr{G} \to T_X/\mathscr{G}$ which sends D to the class in $T_X/\mathscr{G}$ of $D^{[p]}$ is $\mathscr{O}_X$-linear. The sheaf $\mathscr{G}$ is said to be *closed under p-th powers* if the map $\mathrm{F}_{\mathrm{abs}}^*\mathscr{G} \to T_X/\mathscr{G}$ vanishes.

A connected complex manifold M satisfies the *Liouville property* when every plurisubharmonic function on M bounded from above is constant (see [12, Sect. 2.1.2]). Examples of complex manifolds satisfying the Liouville property are provided by the affine space $\mathbb{C}^n$ and connected compact complex manifolds.

We will use the following notation.

Notation **5.2** If K is a number field, its ring of integers will be denoted by $\mathscr{O}_K$. For any non-zero prime ideal $\mathfrak{p}$ of $\mathscr{O}_K$, we let $k(\mathfrak{p})$ be the finite field $\mathscr{O}_K/\mathfrak{p}$. We denote by $\bar{\mathfrak{p}} \colon \mathrm{Spec}\, k(\bar{\mathfrak{p}}) \to S$ a geometric point of $S := \mathrm{Spec}\, \mathscr{O}_K$ lying over $\mathfrak{p}$ with $k(\bar{\mathfrak{p}})$ an algebraic closure of $k(\mathfrak{p})$. Given a scheme X over S, we let $X_K := X \otimes K$, $X_{\mathfrak{p}} := X \otimes k(\mathfrak{p})$, and $X_{\bar{\mathfrak{p}}} := X \otimes k(\bar{\mathfrak{p}})$. Given a sheaf $\mathscr{G}$ on X, we let $\mathscr{G}_K := \mathscr{G} \otimes K$, $\mathscr{G}_{\mathfrak{p}} := \mathscr{G} \otimes k(\mathfrak{p})$, and $\mathscr{G}_{\bar{\mathfrak{p}}} := \mathscr{G} \otimes k(\bar{\mathfrak{p}})$.

Theorem 5.3 ([12, Theorem 2.2]) *Let X be a smooth geometrically connected algebraic variety over a number field K, and let $\mathscr{G}$ be an involutive subbundle of the tangent bundle T_X of X (defined over K). For some sufficiently divisible integer N, let $\mathbf{X}$ (resp. $\boldsymbol{\mathscr{G}}$) be a smooth model of X over $\mathbf{S} := \operatorname{Spec} \mathscr{O}_K[1/N]$ (resp. a sub-vector bundle of the relative tangent bundle $T_{\mathbf{X}/\mathbf{S}}$ such that $\boldsymbol{\mathscr{G}}_K$ coincides with $\mathscr{G}$). Assume that the following two conditions are satisfied.*

1) *For almost every non-zero prime ideal $\mathfrak{p}$ of $\mathscr{O}_K[1/N]$, the subbundle $\boldsymbol{\mathscr{G}}_{\mathfrak{p}}$ of $T_{\mathbf{X}_{\mathfrak{p}}}$ is stable by p-th power, where p denotes the characteristic of $k(\mathfrak{p})$.*
2) *There exist a complex manifold M satisfying the Liouville property as well as an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, a holomorphic embedding $i\colon X_\sigma(\mathbb{C}) \to M$ and a holomorphic map $j\colon M \to X_\sigma(\mathbb{C}) \times X_\sigma(\mathbb{C})$ such that $j \circ i$ coincide with the diagonal embedding $X_\sigma(\mathbb{C}) \hookrightarrow X_\sigma(\mathbb{C}) \times X_\sigma(\mathbb{C})$ and j restricts to an isomorphism from the analytic germ of M along $i\big(X_\sigma(\mathbb{C})\big)$ onto the analytic graph of $\big(X_\sigma, \mathscr{G}_\sigma\big)$.*

Then $\mathscr{G}_\sigma$ is algebraically integrable.

It is well-known that in positive characteristic, there exist semistable vector bundles such that their pull-back under the absolute Frobenius morphism is no longer semistable. The next result says that this phenomenon does not occur on projective varieties whose tangent bundle is semistable with zero slope. It partly extends [50, Theorem 2.1] to the setting where polarizations are given by big semiample divisors. The proof of Proposition 5.4 is similar to that of [50, Theorem 2.1].

Proposition 5.4 *Let X be a smooth projective variety over an algebraically closed field k of positive characteristic p, and let H be a big semiample divisor on X. Suppose that T_X is H-semistable and that $\mu_H(T_X) \geq 0$. Let $\mathscr{E}$ be a coherent locally free sheaf on X. Suppose furthermore that $p \geq \operatorname{rank} \mathscr{E} + \dim X$. If $\mathscr{E}$ is H-semistable, then so is $\mathrm{F}^*_{\mathrm{abs}}\mathscr{E}$.*

Proof Suppose that $\mathrm{F}^*_{\mathrm{abs}}\mathscr{E}$ is not H-semistable, and let

$$\{0\} = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r := \mathrm{F}^*_{\mathrm{abs}}\mathscr{E}$$

be the Harder–Narasimhan filtration of $\mathrm{F}^*_{\mathrm{abs}}\mathscr{E}$. By [55, Proposition 1^p] (see also [46, Corollary 2.4]), the canonical connection on $\mathrm{F}^*_{\mathrm{abs}}\mathscr{E}$ induces a non-zero $\mathscr{O}_X$-linear map

$$\mathscr{E}_{r-1} \to \mathscr{E}_r/\mathscr{E}_{r-1} \otimes \Omega^1_X.$$

Let C be a smooth complete intersection curve of elements of $|mH|$ for some sufficiently large integer m. By [46, Corollary 5.4], the sheaves $\mathscr{E}_i/\mathscr{E}_{i-1|C}$ and $\Omega^1_{X|C}$ are semistable. This implies that the sheaves $(\mathscr{E}_i/\mathscr{E}_{i-1})_{|C} \otimes \Omega^1_{X|C}$ are semistable as well by [36, Remark 3.4] using the assumption that $p \geq \operatorname{rank} \mathscr{E} + \dim X$. It follows that

$$\mu_H^{\max}\big(\mathscr{E}_r/\mathscr{E}_{r-1} \otimes \Omega^1_X\big) \leqslant \mu_H^{\max}\big(\mathscr{E}_r/\mathscr{E}_{r-1}\big)$$

using the assumption that $\mu_H(\Omega^1_X) \leqslant 0$. The inequality

$$\mu_H^{\min}\big(\mathscr{E}_{r-1}\big) = \mu_H\big(\mathscr{E}_{r-1}/\mathscr{E}_{r-2}\big) > \mu_H\big(\mathscr{E}_r/\mathscr{E}_{r-1}\big) = \mu_H^{\max}\big(\mathscr{E}_r/\mathscr{E}_{r-1}\big)$$

then shows that the map $\mathscr{E}_{r-1} \to \mathscr{E}_r/\mathscr{E}_{r-1} \otimes \Omega^1_X$ must vanish, yielding a contradiction. □

Remark 5.5 The condition "T_X H-semistable with $\mu_H(T_X) \geq 0$" in Proposition 5.4 can be weakened to "$\mu_H^{\min}(T_X) \geq 0$", but we will not need this stronger statement.

Remark 5.6 We will use Proposition 5.4 together with [47, Proposition 5.1] to conclude that, in the setup of Proposition 5.4, a H-semistable vector bundle $\mathscr{E}$ is numerically flat if and only if

$$\begin{aligned} c_1(\mathscr{E}) \cdot H^{n-1} &= c_1(\mathscr{E})^2 \cdot H^{n-2} \\ &= c_2(\mathscr{E}) \cdot H^{n-2} = 0, \end{aligned}$$

where $n := \dim X$.

A coherent locally free sheaf $\mathscr{E}$ on a smooth projective variety X over an algebraically closed field k is said to be *étale trivializable* if there exists a finite étale cover of X on which $\mathscr{E}$ becomes trivial. We will need the following observation.

Lemma 5.7 *Let X be a connected smooth projective variety over an algebraically closed field k, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let K be any algebraically closed extension of k. If $\mathscr{E}_K$ is étale trivializable, then so is $\mathscr{E}$.*

Proof Set $r := \operatorname{rank} \mathscr{E}$, and let $g \colon Y \to X_K$ be a finite étale cover such that $g^*\mathscr{E}_K \cong \mathscr{O}_Y^{\oplus r}$. By [54, Exposé X, Corollaire 1.8], there exists a finite étale cover $f \colon Z \to X$ such that $g \cong f_K$. Since $g^*\mathscr{E}_K \cong \mathscr{O}_Y^{\oplus r}$, we conclude easily that $f^*\mathscr{E}_K \cong \mathscr{O}_Z^{\oplus r}$. □

Examples of étale trivializable vector bundles are provided by the following lemma. We use ideas from the proof of [53, Theorem A].

Lemma 5.8 *Let X be a complex projective manifold, and let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition of T_X into involutive subsheaves. Suppose that $\mathscr{E}$ is polystable with respect to some polarization and that $c_1(\mathscr{E}) \equiv 0$ and $c_2(\mathscr{E}) \equiv 0$. Suppose furthermore that $\mathscr{E}$ is algebraically integrable. Then $\mathscr{E}$ is étale trivializable.*

Proof By [56, Corollary 3.10], $\mathscr{E}$ is flat. By a result of Uhlenbeck and Yau (see [57]), we conclude that $\mathscr{E}$ admits a flat hermitian metric. Let $x \in X$, and let $\rho\colon \pi_1(X, x) \to \mathbb{U}(\mathscr{E}_x)$ be the corresponding unitary representation. We will show that the image of ρ is finite.

By [35, Proposition 2.5], $\mathscr{E}$ is induced by a morphism $\varphi\colon X \to Y$ onto a normal projective variety. Let $Y^\circ \subset Y$ be a dense open subset such that φ restricts to a smooth morphism on $X^\circ := \varphi^{-1}(Y^\circ)$, and set $y = \varphi(x)$. Suppose that $y \in Y^\circ$. As a classical consequence of Yau's theorem on the existence of Kähler–Einstein metrics, the geometric generic fiber of φ is covered by an abelian variety (see [42, Chap. IV Corollary 4.15]). Thus, replacing Y° by a dense open subset and X° by a finite étale cover, if necessary, we may assume that all fibers of $\varphi_{|X^\circ}$ are abelian varieties. Let F be a general fiber of φ. Note that the induced metric on $T_F = \mathscr{E}_{|F}$ is constant. This implies that the representation

$$\pi_1(X^\circ, x) \to \pi_1(X, x) \to \mathbb{U}(\mathscr{E}_x)$$

factors through $\varphi_{|X^\circ}$.

Because $\varphi_{|X^\circ}$ admits a holomorphic flat connection, $H^0(X_y, \mathscr{E}^*_{|X_y}) \subset H^1(X_y, \mathbb{C})$ is invariant under the monodromy representation $\pi_1(Y^\circ, y) \to \mathrm{GL}\big(H^1(X_y, \mathbb{C})\big)$. Moreover, the induced representation

$$\pi_1(Y^\circ, y) \to \mathrm{GL}\big(H^0(X_y, \mathscr{E}^*_{|X_y})\big) \cong \mathrm{GL}(\mathscr{E}^*_x)$$

coincides with ρ since $\mathscr{G}$ is transversely flat hermitian (see [53, Sect. 2.4]).

By Lemma 5.9 below, we see that the monodromy representation $\pi_1(Y^\circ) \to \mathrm{GL}\big(H^1(X_y, \mathbb{C})\big)$ is finite, and hence so is ρ. This shows that $\mathscr{E}$ is étale trivializable. □

The following result is probably well-known. We include a full proof here for the reader's convenience.

Lemma 5.9 *Let $\varphi\colon X \to Y$ be a smooth projective morphism of quasi-projective complex manifolds with fibers isomorphic to abelian varieties. Suppose that φ is a locally trivial analytic fibration for the analytic topology. Then there exists an abelian variety as well as a finite étale cover $\widetilde{Y} \to Y$ such that $X \times_Y \widetilde{Y} \cong A \times \widetilde{Y}$ as varieties over $\widetilde{Y}$.*

Proof Let $y \in Y$, and denote by X_y the fiber $\varphi^{-1}(y)$. Let $\mathrm{Aut}^\circ(X_y) \cong X_y$ denotes the neutral component of the automorphism group $\mathrm{Aut}(X_y)$ of X_y. Recall from [28, Exposé $\mathrm{VI_B}$, Théorème 3.10] that the algebraic groups $\mathrm{Aut}^\circ(X_y)$ fit together to form an abelian scheme $\mathscr{A}$ over Y. Since $\mathscr{A}$ is locally trivial, there exist an abelian variety A, and a finite étale cover $Y_1 \to Y$ such that $\mathscr{A} \times_Y Y_1 \cong A \times Y_1$ as group schemes over Y_1. This follows from the fact that there is a fine moduli scheme for polarized abelian varieties of dimension g, with level N structure and polarization of degree d provided that N is large enough. In particular, A acts faithfully on $X_1 := X \times_Y Y_1$. By [14, Theorem 2], there exist a finite étale cover $\widetilde{X}$ of X_1 equipped with a faithful action of A, and an A-isomorphism $\widetilde{X} \cong A \times \widetilde{Y}$ for some quasi-projective manifold $\widetilde{Y}$, where A acts trivially on $\widetilde{Y}$ and diagonally on $A \times \widetilde{Y}$. One readily checks that the natural morphism $\widetilde{Y} \cong \{0_A\} \times \widetilde{Y} \to Y_1$ is étale, and that $\widetilde{X} \cong X_1 \times_{Y_1} \widetilde{Y}$ as varieties over $\widetilde{Y}$. The lemma then follows easily. □

We end the preparation for the proof of Theorem 1.4 with the following lemma.

Lemma 5.10 *Let X be a normal complex projective variety with canonical singularities, and let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition of T_X into sheaves. Suppose that $\mathscr{E}$ is locally free. There exists a resolution of singularities $\beta\colon \widehat{X} \to X$ such that the following holds.*

1 *The morphism β induces an isomorphism over the smooth locus X_{reg} of X.*
2 *The tangent sheaf $T_{\widehat{X}}$ decomposes as a direct sum $T_{\widehat{X}} \cong \beta^*\mathscr{E} \oplus \widehat{\mathscr{G}}$ of locally free sheaves. Moreover, we have $(\beta_*\widehat{\mathscr{G}})^{**} \cong \mathscr{G}$.*
3 *If X, $\mathscr{E}$, and $\mathscr{G}$ are defined over a subfield $k \subseteq \mathbb{C}$, then $\widehat{X}$, β, and $\widehat{\mathscr{G}}$ are defined over k as well.*

Proof Let $\beta\colon \widehat{X} \to X$ be a resolution of singularities of X such that $\beta_* T_{\widehat{X}} \cong T_X$, and such that β induces an isomorphism over X_{reg}. The existence of β is established in [23, Corollary 4.7]. It relies on the existence of functorial resolutions of singularities (see [44, Theorem3.36]). Consider the generically injective morphism of locally free sheaves

$$\beta^*\mathscr{E} \to \beta^* T_X \cong \beta^*\big(\beta_* T_{\widehat{X}}\big) \to T_{\widehat{X}},$$

where $\beta^*\big(\beta_* T_{\widehat{X}}\big) \to T_{\widehat{X}}$ is the evaluation map. By [24, Theorem 1.5], the projection morphism $T_{X_{\mathrm{reg}}} \to \mathscr{E}_{|X_{\mathrm{reg}}}$ extends to a morphism

$$T_{\widehat{X}} \to \beta^*\mathscr{E}.$$

The composed morphism $\beta^*\mathscr{E} \to T_{\widehat{X}} \to \beta^*\mathscr{E}$ must be the identity map, and thus $T_{\widehat{X}}$ decomposes as a direct sum $T_{\widehat{X}} \cong \beta^*\mathscr{E} \oplus \widehat{\mathscr{G}}$, where $\widehat{\mathscr{G}}$ is the kernel of the map $T_{\widehat{X}} \to \beta^*\mathscr{E}$. The sheaves $(\beta_*\widehat{\mathscr{G}})^{**}$ and $\mathscr{G}$ agree on X_{reg}, and since both are reflexive, we obtain an isomorphism $(\beta_*\widehat{\mathscr{G}})^{**} \cong \mathscr{G}$.

Suppose that X, $\mathscr{E}$, and $\mathscr{G}$ are defined over a subfield $k \subseteq \mathbb{C}$. Then $\widehat{X}$ and β are defined over k as well by [44, Theorem 3.36]. This implies that $\widehat{\mathscr{G}}$ is also defined over k, completing the proof of the lemma. □

Before proving Theorem 1.4 below, we note the following immediate corollary.

Corollary 5.11 *Let X be a normal complex projective variety of dimension n with terminal singularities, and let*

$$T_X = \bigoplus_{i \in I} \mathscr{G}_i \oplus \mathscr{E}$$

be a decomposition of T_X into involutive subsheaves with trivial determinants. Suppose that $\mathscr{G}_i$ is stable with respect to some ample Cartier divisor H, and that $c_2(\mathscr{G}_i) \cdot H^{n-2} \neq 0$. Suppose also that for any finite cover $g\colon \widehat{X} \to X$, étale in codimension one, the sheaf $g^{[*]}\mathscr{G}_i$ is g^*H-stable. Suppose furthermore that $\mathscr{E}$ is H-polystable, and that $c_2(\mathscr{E}) \cdot H^{n-2} = 0$. Then $\widetilde{q}(X) = \operatorname{rank} \mathscr{E}$.

Proof of Theorem 1.4 We maintain notation and assumptions of Theorem 1.4. For the reader's convenience, the proof is subdivided into a number of relatively independent steps. □

Step 1. Reduction step. Set $\mathscr{G} := \bigoplus_{i \in I} \mathscr{G}_i$. By [26, Theorem 1.20], there exists a finite cover $f_1\colon X_1 \to X$ that is étale in codimension one such that $f_1^{[*]}\mathscr{E}$ is a locally free, flat sheaf on X_1. From [43, Proposition 3.16], we see that X_1 has terminal singularities. The sheaves T_{X_1} and $f_1^{[*]}\mathscr{G} \oplus f_1^{[*]}\mathscr{E}$ agree on $f_1^{-1}(X_{\text{reg}})$, and since both are reflexive, we obtain a decomposition

$$T_{X_1} \cong f_1^{[*]}\mathscr{G} \oplus f_1^{[*]}\mathscr{E}$$

of T_{X_1} into involutive subsheaves with trivial determinants. The same argument also shows that

$$f_1^{[*]}\mathscr{G} \cong \bigoplus_{i \in I} f_1^{[*]}\mathscr{G}_i.$$

Notice that f_1 branches only over the singular set of X. It follows that $c_1(f_1^{[*]}\mathscr{G}_i) \cdot (f_1^*H)^{n-1} = 0$ and

$$\text{either} \quad c_1(f_1^{[*]}\mathscr{G}_i)^2 \cdot (f_1^*H)^{n-2} \neq 0 \quad \text{or} \quad c_2(f_1^{[*]}\mathscr{G}_i) \cdot (f_1^*H)^{n-2} \neq 0$$

for each $i \in I$. Moreover, $f_1^{[*]}\mathscr{E}$ is f_1^*H-polystable by [34, Lemma 3.2.3]. Therefore, we may assume without loss of generality that the following holds.

Assumption 5.12 The sheaf $\mathscr{E}$ is a locally free, flat sheaf.

By Lemma 5.10, there exists a resolution of singularities $\beta\colon \widehat{X} \to X$ such that $T_{\widehat{X}}$ decomposes as

$$T_{\widehat{X}} \cong \beta^*\mathscr{E} \oplus \widehat{\mathscr{G}}.$$

Moreover, we may assume that β induces an isomorphism over X_{reg}. Set $\widehat{\mathscr{E}} := \beta^*\mathscr{E}$, and $\widehat{H} := \beta^*H$. Notice that $\widehat{\mathscr{E}}$ and $\beta^{[*]}T_X$ are respectively polystable and semistable with respect to $\widehat{H}$. Moreover, $T_{\widehat{X}}$ is also $\widehat{H}$-semistable since $T_{\widehat{X}}$ and $\beta^{[*]}T_X$ agree away from the β-exceptional set.

Step 2. Algebraic integrability over number fields. Suppose that X, H, $\mathscr{E}$, and the sheaves $\mathscr{G}_i$ are defined over a number field K. We will show that $\mathscr{E}$ has algebraic leaves. Recall from Lemma 5.10 that $\widehat{X}$, β, and $\widehat{\mathscr{G}}$ are also defined over K.

For some sufficiently divisible integer N, let $\mathbf{X}$ be a flat projective model of X over $\mathbf{T} := \operatorname{Spec} \mathscr{O}_K[1/N]$ with normal (see [30, Théorème IV.12.2.4]) and regular in codimension 2 geometric fibers. Let $\boldsymbol{\mathscr{E}}$ (resp. $\boldsymbol{\mathscr{G}}_i$) be a locally free (resp. a coherent) subsheaf of $T_{\mathbf{X}/\mathbf{T}}$ such that $\boldsymbol{\mathscr{E}}_{\mathbb{C}}$ (resp. $\boldsymbol{\mathscr{G}}_{i\,\mathbb{C}}$) coincides with $\mathscr{E}$ (resp. $\mathscr{G}_i$). Suppose moreover that the sheaves $\boldsymbol{\mathscr{G}}_i$ are flat over $\mathbf{T}$, and that

$$T_{\mathbf{X}/\mathbf{T}} = \bigoplus_{i\in I} \boldsymbol{\mathscr{G}}_i \oplus \boldsymbol{\mathscr{E}}.$$

Let $\mathbf{H}$ be an ample Cartier divisor on $\mathbf{X}$ such that $\mathbf{H}_{\mathbb{C}} \sim H$.

Since semistability and geometric stability with respect to an ample divisor are open conditions in flat families of sheaves (see proof of [34, Proposition 2.3.1]), we may assume that the sheaves $\boldsymbol{\mathscr{G}}_{i\,\bar{\mathfrak{p}}}$ are stable with respect to $\mathbf{H}_{\bar{\mathfrak{p}}}$, and that $\boldsymbol{\mathscr{E}}_{\bar{\mathfrak{p}}}$ is $\mathbf{H}_{\bar{\mathfrak{p}}}$-polystable for every non-zero prime ideal $\mathfrak{p}$ of $\mathscr{O}_K[1/N]$. Suppose furthermore that $\boldsymbol{\mathscr{E}}_{\bar{\mathfrak{p}}}$ is involutive. By Lemma 2.11, we may also assume without loss of generality that the following holds:

1. $c_1(\boldsymbol{\mathscr{E}}_{\bar{\mathfrak{p}}}) \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-1} = c_1(\boldsymbol{\mathscr{E}}_{\bar{\mathfrak{p}}})^2 \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-2} = c_2(\boldsymbol{\mathscr{E}}_{\bar{\mathfrak{p}}}) \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-2} = 0$;
2. $c_1(\boldsymbol{\mathscr{G}}_{i\,\bar{\mathfrak{p}}}) \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-1} = 0$, and either $c_1(\boldsymbol{\mathscr{G}}_{i\,\bar{\mathfrak{p}}})^2 \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-2} \neq 0$ or $c_2(\boldsymbol{\mathscr{G}}_{i\,\bar{\mathfrak{p}}}) \cdot \mathbf{H}_{\bar{\mathfrak{p}}}^{n-2} \neq 0$.

Let $\widehat{\mathbf{X}}$ be a smooth projective model of $\widehat{X}$ over $\mathbf{T}$, and let $\boldsymbol{\beta}\colon \widehat{\mathbf{X}} \to \mathbf{X}$ be a projective birational morphism such that $\boldsymbol{\beta}_{\mathbb{C}}$ coincides with β. Suppose moreover that $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}\colon \widehat{\mathbf{X}}_{\bar{\mathfrak{p}}} \to \mathbf{X}_{\bar{\mathfrak{p}}}$ is birational, and that the $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}$-exceptional set maps to a closed subset of codimension at least three in $\mathbf{X}_{\bar{\mathfrak{p}}}$. Set $\widehat{\boldsymbol{\mathscr{E}}} := \boldsymbol{\beta}^*\boldsymbol{\mathscr{E}}$ and $\widehat{\mathbf{H}} := \boldsymbol{\beta}^*\mathbf{H}$. Notice that $\widehat{\boldsymbol{\mathscr{E}}}_{\bar{\mathfrak{p}}}$ and $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}^{[*]}T_{\mathbf{X}_{\bar{\mathfrak{p}}}}$ are semistable with respect to $\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}$.

Moreover, $T_{\widehat{\mathbf{X}}_{\bar{\mathfrak{p}}}}$ is also $\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}$-semistable since $T_{\widehat{\mathbf{X}}_{\bar{\mathfrak{p}}}}$ and $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}^{[*]}T_{\mathbf{X}_{\bar{\mathfrak{p}}}}$ agree away from the $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}$-exceptional set.

Let $\mathfrak{p}$ be non-zero prime ideal of $\mathscr{O}_K[1/N]$, and denote by p the characteristic $k(\mathfrak{p})$. We claim the following.

Claim 5.13 The involutive sub-vector bundle $\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}$ of $T_{\widehat{\mathbf{X}}_{\bar{\mathfrak{p}}}}$ is closed under p-th power.

Proof We argue by contradiction and assume that $\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}$ is not closed under p-th power, and so neither is $\mathscr{E}_{\bar{\mathfrak{p}}}$. Thus, the map $\mathrm{F}^*_{\mathrm{abs},\bar{\mathfrak{p}}}\mathscr{E}_{\bar{\mathfrak{p}}} \to T_{\mathbf{X}_{\bar{\mathfrak{p}}}}/\mathscr{E}_{\bar{\mathfrak{p}}}$ induced by the p-th power operation does not vanish identically. By Proposition 5.4 above, the locally free sheaf $\mathrm{F}^*_{\mathrm{abs},\bar{\mathfrak{p}}}\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}$ is semistable with respect to $\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}$. This implies that $\mathrm{F}^*_{\mathrm{abs},\bar{\mathfrak{p}}}\mathscr{E}_{\bar{\mathfrak{p}}}$ is semistable with respect to $\mathbf{H}_{\bar{\mathfrak{p}}}$. Since $c_1(\mathscr{E}_{\bar{\mathfrak{p}}})\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-1}=0$, and since the sheaves $\mathscr{G}_{i\,\bar{\mathfrak{p}}}$ are stable with $c_1(\mathscr{G}_{i\,\bar{\mathfrak{p}}})\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-1}=0$ as well, there exists $i_0\in I$ such that the induced morphism $\mathrm{F}^*_{\mathrm{abs},\bar{\mathfrak{p}}}\mathscr{E}_{\bar{\mathfrak{p}}}\to\mathscr{G}_{i_0\,\bar{\mathfrak{p}}}$ is surjective in codimension one.

Let $\mathbf{S}_{\bar{\mathfrak{p}}}$ be a smooth two dimensional complete intersection of general elements of $|m\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}| = \boldsymbol{\beta}_{\bar{\mathfrak{p}}}^*|m\mathbf{H}_{\bar{\mathfrak{p}}}|$ for some positive integer m. Since the $\boldsymbol{\beta}_{\bar{\mathfrak{p}}}$-exceptional set maps to a closed subset of codimension at least three in $\mathbf{X}_{\bar{\mathfrak{p}}}$, $\mathbf{S}_{\bar{\mathfrak{p}}}$ is contained in $\widehat{\mathbf{X}}_{\bar{\mathfrak{p}}}\setminus\mathrm{Exc}(\boldsymbol{\beta}_{\bar{\mathfrak{p}}})\cong\mathbf{X}_{\bar{\mathfrak{p}}}\setminus(\boldsymbol{\beta}_{\bar{\mathfrak{p}}}\big(\mathrm{Exc}(\boldsymbol{\beta}_{\bar{\mathfrak{p}}})\big)$, and thus the restriction of $\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}$ to $\mathbf{S}_{\bar{\mathfrak{p}}}$ is ample.

By Proposition 5.4 again, the locally free sheaves $(\mathrm{F}^{\circ k}_{\mathrm{abs},\bar{\mathfrak{p}}})^*\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}$ are semistable with respect to $\widehat{\mathbf{H}}_{\bar{\mathfrak{p}}}$. Applying [46, Corollary 5.4] to the locally free sheaves $(\mathrm{F}^{\circ k}_{\mathrm{abs},\bar{\mathfrak{p}}})^*\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}$, we see that there exists a positive integer m (that does not depend on $k\geq 1$) such that the restrictions $(\mathrm{F}^{\circ k}_{\mathrm{abs},\bar{\mathfrak{p}}})^*\widehat{\mathscr{E}_{\bar{\mathfrak{p}}}}_{|\mathbf{S}_{\bar{\mathfrak{p}}}} \cong (\mathrm{F}^{\circ k}_{\mathrm{abs},\bar{\mathfrak{p}}})^*\mathscr{E}_{\bar{\mathfrak{p}}\,|\mathbf{S}_{\bar{\mathfrak{p}}}}$ are semistable with respect to $\mathbf{H}_{\bar{\mathfrak{p}}\,|\mathbf{S}_{\bar{\mathfrak{p}}}}$ for any positive integer k. From [47, Proposition 5.1], we conclude that $\mathscr{E}_{\bar{\mathfrak{p}}\,|\mathbf{S}_{\bar{\mathfrak{p}}}}$ is nef using the fact that

$$c_1(\mathscr{E}_{\bar{\mathfrak{p}}})\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-1}=c_1(\mathscr{E}_{\bar{\mathfrak{p}}})^2\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-2}=c_2(\mathscr{E}_{\bar{\mathfrak{p}}})\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-2}=0.$$

We view $\mathbf{S}_{\bar{\mathfrak{p}}}$ as a surface contained in $\mathbf{X}_{\bar{\mathfrak{p}}}$. Observe that $\mathscr{G}_{i_0\,\bar{\mathfrak{p}}}$ is locally free along $\mathbf{S}_{\bar{\mathfrak{p}}}$, and that the restriction of $\mathrm{F}^*_{\mathrm{abs},\bar{\mathfrak{p}}}\mathscr{E}_{\bar{\mathfrak{p}}}\to\mathscr{G}_{i_0\,\bar{\mathfrak{p}}}$ to $\mathbf{S}_{\bar{\mathfrak{p}}}$ is surjective in codimension one by choice of i_0. Since $\mathscr{E}_{\bar{\mathfrak{p}}\,|\mathbf{S}_{\bar{\mathfrak{p}}}}$ is nef, we obtain that $\mathscr{G}_{i_0\,\bar{\mathfrak{p}}\,|\mathbf{S}_{\bar{\mathfrak{p}}}}$ if nef. This implies that

$$c_1(\mathscr{G}_{i_0})^2\cdot H^{n-2}=c_1(\mathscr{G}_{i_0\,\bar{\mathfrak{p}}})^2\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-2}=0\quad\text{and}$$
$$c_2(\mathscr{G}_{i_0})\cdot H^{n-2}=c_2(\mathscr{G}_{i_0\,\bar{\mathfrak{p}}})\cdot\mathbf{H}_{\bar{\mathfrak{p}}}^{n-2}=0$$

by [47, Proposition 5.1] and Lemma 2.11, yielding a contradiction. This completes the proof of our claim.

Set $r := \operatorname{rank} \mathscr{E}$. Recall that $\widehat{\mathscr{E}}$ is polystable with respect to the nef and big divisor $\widehat{H}$. By [27, Theorem3.3], $\widehat{\mathscr{E}}$ is also polystable with respect to some ample divisor on $\widehat{X}$, and hence unitary flat. By [53, Theorem B], the universal covering space $\widetilde{X}$ of $\widehat{X}$ is a product $\mathbb{C}^r \times B$ in such a way that the decomposition

$$T_{\widehat{X}} = \widehat{\mathscr{E}} \oplus \widehat{\mathscr{G}}$$

lifts to the decomposition

$$T_{\mathbb{C}^r \times B} = T_{\mathbb{C}^r} \oplus T_B.$$

In particular, the group $\pi_1(\widehat{X})$ acts diagonally on $\mathbb{C}^r \times B$. Notice that the analytic graph of the foliation induced by $\mathscr{E}$ on $\widetilde{X}$ is the germ of $\widetilde{X} \times_B \widetilde{X}$ along the diagonal embedding $\widetilde{X} \hookrightarrow \widetilde{X} \times_B \widetilde{X}$.

Set $M := (\mathbb{C}^r \times \widetilde{X})/\pi_1(\widehat{X})$, and denote by p and q the projections of $\widetilde{X} \cong \mathbb{C}^r \times B$ onto $\mathbb{C}^r$ and B respectively. Let $i\colon \widehat{X} \cong \widetilde{X}/\pi_1(\widehat{X}) \to M$ be the embedding induced by

$$\mathbb{C}^r \times B \ni (a, b) \mapsto (a, a, b) \in \mathbb{C}^r \times \mathbb{C}^r \times B$$

and let $j\colon M \to \widehat{X} \times \widehat{X}$ be the holomorphic map induced by

$$\mathbb{C}^r \times \mathbb{C}^r \times B \ni (a', a, b) \mapsto (a', b, a, b) \in \mathbb{C}^r \times B \times \mathbb{C}^r \times B.$$

One readily checks that $j \circ i$ coincide with the diagonal embedding $\widehat{X} \hookrightarrow \widehat{X} \times \widehat{X}$ and that j restricts to an isomorphism from the analytic germ of M along $i(\widehat{X})$ onto the analytic graph of $(\widehat{X}, \widehat{\mathscr{E}})$.

We finally show that M satisfies the Liouville property. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^r \times \widetilde{X} & \longrightarrow & (\mathbb{C}^r \times \widetilde{X})/\pi_1(\widehat{X}) = M \\ \downarrow & & \downarrow \\ \widetilde{X} & \longrightarrow & \widetilde{X}/\pi_1(\widehat{X}) \cong \widehat{X} \end{array}$$

where the vertical maps are induced by the second projection $\mathbb{C}^r \times \widetilde{X} \to \widetilde{X}$, and where the horizontal maps are the quotient maps. Let ψ be a plurisubharmonic function on M, bounded from above. The restriction to any fiber of $\mathbb{C}^r \times \widetilde{X} \to \widetilde{X}$ of the pull-back $\widetilde{\psi}$ of ψ to $\mathbb{C}^r \times \widetilde{X}$ is either $-\infty$ or a plurisubharmonic function bounded from above. In either case, it is constant, and hence, $\widetilde{\psi}$ is the pull-back of a $\pi_1(\widetilde{X})$-invariant function on $\widetilde{X}$. The latter is then induced by a plurisubharmonic function on the compact complex manifold $\widehat{X}$. This implies that ψ is constant, proving our claim.

By Theorem 5.3, we conclude that $\widehat{\mathscr{E}}$ and hence $\mathscr{E}$ have algebraic leaves.

Step 3. End of proof. To show Theorem 1.4, let R be a subring of $\mathbb{C}$, finitely generated over $\mathbb{Q}$, and let $\mathbf{X}$ be a flat projective model of X over $\mathbf{T} := \operatorname{Spec} R$ with normal geometric fibers. We may also assume that the geometric fibers of $\mathbf{X} \times_{\mathbf{T}} \mathbf{T}_{\mathbb{C}}$ over $\mathbf{T}_{\mathbb{C}} = \operatorname{Spec} R \otimes \mathbb{C}$ have terminal singularities. Let $\boldsymbol{\mathscr{E}}$ (resp. $\boldsymbol{\mathscr{G}}_i$) be a locally free (resp. a coherent) subsheaf of $T_{\mathbf{X}/\mathbf{T}}$ such that $\boldsymbol{\mathscr{E}}_{\mathbb{C}}$ (resp. $\boldsymbol{\mathscr{G}}_{i\,\mathbb{C}}$) coincides with $\mathscr{E}$ (resp. $\mathscr{G}_i$). Suppose moreover that the sheaves $\boldsymbol{\mathscr{G}}_i$ are flat over $\mathbf{T}$, and that

$$T_{\mathbf{X}/\mathbf{T}} = \bigoplus_{i \in I} \boldsymbol{\mathscr{G}}_i \oplus \boldsymbol{\mathscr{E}}.$$

Let $\mathbf{H}$ be an ample Cartier divisor on $\mathbf{X}$ such that $\mathbf{H}_{\mathbb{C}} \sim H$. As above, we may assume that the sheaves $\boldsymbol{\mathscr{G}}_{i\,\bar{t}}$ are stable with respect to $\mathbf{H}_{\bar{t}}$, and that $\boldsymbol{\mathscr{E}}_{\bar{t}}$ is polystable, for any geometric point $\bar{t}$ of $\mathbf{T}$. Suppose furthermore that $\boldsymbol{\mathscr{E}}_{\bar{t}}$ and the sheaves $\boldsymbol{\mathscr{G}}_{i\,\bar{t}}$ are involutive. By Lemma 2.11 again, we may also assume without loss of generality that

$$\begin{aligned} c_1(\boldsymbol{\mathscr{E}}_{\bar{t}}) \cdot \mathbf{H}_{\bar{t}}^{n-1} &= c_1(\boldsymbol{\mathscr{E}}_{\bar{t}})^2 \cdot \mathbf{H}_{\bar{t}}^{n-2} = c_2(\boldsymbol{\mathscr{E}}_{\bar{t}}) \cdot \mathbf{H}_{\bar{t}}^{n-2} = 0, \\ c_1(\boldsymbol{\mathscr{G}}_{i\,\bar{t}}) \cdot \mathbf{H}_{\bar{t}}^{n-1} &= 0 \quad \text{and either} \\ c_1(\boldsymbol{\mathscr{G}}_{i\,\bar{t}})^2 \cdot \mathbf{H}_{\bar{t}}^{n-2} &\neq 0 \quad \text{or} \quad c_2(\boldsymbol{\mathscr{G}}_{i\,\bar{t}}) \cdot \mathbf{H}_{\bar{t}}^{n-2} \neq 0. \end{aligned}$$

Recall that R is finitely generated over $\mathbb{Q}$. When $t \in \mathbf{T} = \operatorname{Spec} R$ is a closed point, its residue field is an algebraic number field, and hence $\boldsymbol{\mathscr{E}}_{\bar{t}}$ has algebraic leaves by the previous step.

Let also $\widehat{\mathbf{X}}$ be a smooth projective model of $\widehat{X}$ over $\mathbf{T}$, and let $\boldsymbol{\beta}\colon \widehat{\mathbf{X}} \to \mathbf{X}$ be a projective birational morphism such that $\boldsymbol{\beta}_{\mathbb{C}}$ coincides with β. Suppose moreover that $\boldsymbol{\beta}_{\bar{t}}\colon \widehat{\mathbf{X}}_{\bar{t}} \to \mathbf{X}_{\bar{t}}$ is birational. Set $\widehat{\boldsymbol{\mathscr{E}}} := \boldsymbol{\beta}^*\boldsymbol{\mathscr{E}}$. Let $\widehat{\boldsymbol{\mathscr{G}}}$ be a locally free subsheaf of $T_{\widehat{\mathbf{X}}/\mathbf{T}}$ such that $\boldsymbol{\mathscr{G}}_{\mathbb{C}}$ coincides with $\widehat{\mathscr{G}}$. Suppose moreover that

$$T_{\widehat{\mathbf{X}}/\mathbf{T}} = \widehat{\boldsymbol{\mathscr{E}}} \oplus \widehat{\boldsymbol{\mathscr{G}}}$$

and that $\widehat{\boldsymbol{\mathscr{G}}}$ is involutive. By [56, Corollary 3.10], $\widehat{\boldsymbol{\mathscr{E}}}_{\bar{t}}$ is a flat vector bundle, and hence $c_1(\widehat{\boldsymbol{\mathscr{E}}}_{\bar{t}}) \equiv 0$ and $c_2(\widehat{\boldsymbol{\mathscr{E}}}_{\bar{t}}) \equiv 0$. Moreover, the vector bundle $\widehat{\boldsymbol{\mathscr{E}}}_{\bar{t}}$ is polystable with respect to the nef and big divisor $\boldsymbol{\beta}_{\bar{t}}^*\mathbf{H}_{\bar{t}}$, and hence polystable with respect to some ample divisor by [27, Theorem 3.3]. Applying Lemma 5.8 and Lemma 5.7, we see that $\widehat{\mathscr{E}}_{\bar{t}}$ is étale trivializable for any closed point $t \in T$. From [48, Theorem 7.9] (see also Theorem 5.14 below), we conclude that $\widehat{\mathscr{E}}$ is étale trivializable. Thus, replacing X by a further cover that is étale in codimension one, if necessary, we may assume that $\mathscr{E} \cong \mathscr{O}_X^{\oplus r}$.

We claim that the neutral component $\mathrm{Aut}^\circ(X)$ of the automorphism group of X is an abelian variety. Suppose otherwise. Then $\mathrm{Aut}^\circ(X)$ contains a positive dimensional affine subgroup by Chevalley's structure theorem. Hence, it contains an algebraic subgroup G isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$. Let $x \in X$, and let $y \in \overline{G \cdot x} \setminus G \cdot x$. Then y is a fixed point of G. On the other hand, $\dim \mathrm{Aut}^\circ(X) = h^0(X, T_X) = r$. It follows that $\mathrm{Lie}(G)$ is generated by a nowhere vanishing global section of T_X, yielding a contradiction. Set $A := \mathrm{Aut}^\circ(X)$. By [14, Theorem 2], there exists a normal projective variety $\widetilde{X}$, and a finite étale cover $f : A \times \widetilde{X} \to X$. The decomposition

$$T_{A\times\widetilde{X}} \cong \bigoplus_{i\in I} f^{[*]}\mathscr{G}_i \oplus f^{[*]}\mathscr{E}$$

of $T_{A\times\widetilde{X}}$ together with the assumption that $f^{[*]}\mathscr{G}_i$ is f^*H-stable then imply easily that

$$f^{[*]}\mathscr{E} = T_{A\times\widetilde{X}/\widetilde{X}}.$$

This finishes the proof of the theorem. □

Theorem 7.9 in [48] is stated without an actual proof, as being an application of the methods in some earlier work of André and Esnault–Langer. We include a proof here of Theorem 5.14 for the reader's convenience. This special case of [48, Theorem 7.9] is enough to complete the proof of Theorem 1.4. Note also that the proof is very similar to that of [2, Théorème 7.2.2] and [18, Theorem 5.1].

Theorem 5.14 *Let $\mathbf{T}$ be a variety over $\bar{\mathbb{Q}} \subset \mathbb{C}$, and let $\mathbf{X} \to \mathbf{T}$ be a smooth projective morphism with connected fibers. Denote by η the generic point of $\mathbf{T}$. Let $\mathscr{E}$ be a vector bundle on $\mathbf{X}$ such that $\mathscr{E}_{\bar{\eta}}$ is polystable with respect to some ample divisor on $\mathbf{X}_{\bar{\eta}}$. Suppose furthermore that $\mathscr{E}_{\bar{t}}$ is étale trivializable for every closed point $t \in \mathbf{T}$. Then $\mathscr{E}_{\bar{\eta}}$ is étale trivializable.*

Proof We may assume without loss of generality that $\mathbf{X} \to \mathbf{T}$ has a section. The proof of [2, Lemma 10.1.1] shows that, up to replacing $\mathbf{X}$ with a finite étale cover, we may also assume that $\mathscr{E}_{\bar{t}}$ is a direct sum of torsion line bundles for every closed point $t \in \mathbf{T}$.

Let $\mathbf{H}$ be a relatively ample Cartier divisor on $\mathbf{X}$ such that $\mathscr{E}_{\bar{\eta}}$ is polystable with respect to $\mathbf{H}_{\bar{\eta}}$, and let $\Phi \in \mathbb{Q}[z]$ be the Hilbert polynomial of $\mathscr{E}_{\bar{\eta}}$ with respect to $\mathbf{H}_{\bar{\eta}}$. We denote by $\mathrm{M}^{\Phi}_{\mathbf{H}}(\mathbf{X}/\mathbf{T})$ the coarse moduli space of Gieseker semistable sheaves on fibers of $\mathbf{X} \to \mathbf{T}$ with Hilbert polynomial Φ, whose existence is guaranteed by [34, Theorem 4.3.7]. Recall that $\mathrm{M}^{\Phi}_{\mathbf{H}}(\mathbf{X}/\mathbf{T})$ is projective over $\mathbf{T}$.

Since Gieseker semistability is an open condition in flat families of sheaves (see for instance [34, Proposition 2.3.1]) and since any slope polystable vector bundle is Gieseker semistable, we conclude that $\mathscr{E}_{\bar{t}}$ is Gieseker semistable for any point $t \in T$. Hence $\mathscr{E}$ induces a section σ of $\mathrm{M}_{\mathbf{H}}^{\Phi}(\mathbf{X}/\mathbf{T}) \to \mathbf{T}$.

Consider the relative Picard scheme $\mathrm{Pic}(\mathbf{X}/\mathbf{T})$ whose existence is guaranteed by [31, Théorème 3.1]. By a theorem of Deligne, $h^1(\mathbf{X}_t, \mathscr{O}_{\mathbf{X}_t})$ is independent of $t \in \mathbf{T}$. This implies that the group scheme $\mathrm{Pic}(\mathbf{X}/\mathbf{T})$ is smooth over $\mathbf{T}$ ([28, Exposé VI_{B}, Proposition 1.6]). Recall now from [28, Exposé VI_{B}, Théorème 3.10] that the algebraic groups $\mathrm{Pic}^{\circ}(\mathbf{X}_t)$ fit together to form a group scheme $\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})$ over $\mathbf{T}$, and that $\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T}) \subset \mathrm{Pic}(\mathbf{X}/\mathbf{T})$ is an open subscheme. From [10, Chapter 8, Theorem 5], we see that $\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})$ is quasi-projective. Using [30, Corollaire 15.7.11], we conclude that it is projective over $\mathbf{T}$. This shows that $\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})$ is an abelian scheme.

Next, consider the natural morphism

$$\varpi : \underbrace{\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T}) \times_{\mathbf{T}} \cdots \times_{\mathbf{T}} \mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})}_{r \text{ factors}} \to \mathrm{M}_{\mathbf{H}}^{\Phi}(\mathbf{X}/\mathbf{T})$$

which maps $([\mathscr{L}_1], \ldots, [\mathscr{L}_r])$ to $[\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r]$, where r denotes the rank of $\mathscr{E}$ and $\mathscr{L}_1, \ldots, \mathscr{L}_r$ are topologically trivial line bundles on $\mathbf{X}_{\bar{t}}$ for some closed point $t \in T$. Since

$$\sigma(t) \in \varpi\big(\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T}) \times_{\mathbf{T}} \cdots \times_{\mathbf{T}} \mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})\big)$$

for any closed point $t \in \mathbf{T}$ and since ϖ is closed, we must have

$$\sigma(\mathbf{T}) \subset \varpi\big(\mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T}) \times_{\mathbf{T}} \cdots \times_{\mathbf{T}} \mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})\big).$$

Since $\mathscr{E}_{\bar{\eta}}$ is polystable with respect to $\mathbf{H}_{\bar{\eta}}$, we conclude that, up to shrinking $\mathbf{T}$ if necessary, there exist line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_r$ on $\mathbf{X}$ with $[\mathscr{L}_i] \in \mathrm{Pic}^{\circ}(\mathbf{X}/\mathbf{T})(\mathbf{T})$ such that $\mathscr{E}_{\bar{\eta}} \cong \mathscr{L}_{1\,\bar{\eta}} \oplus \cdots \oplus \mathscr{L}_{r\,\bar{\eta}}$. Moreover, for any closed point $t \in T$, $\mathscr{L}_{i\,\bar{t}}$ is a torsion point. From [45, Chapter 9, Corollary 6.3], we see that $\mathscr{L}_{\bar{\eta}}$ is a torsion point. This shows that $\mathscr{E}_{\bar{\eta}}$ is étale trivializable, completing the proof of the theorem. □

6 Stable reflexive sheaves with pseudo-effective tautological line bundle

In this section we provide a technical tool for the proof of Theorem 1.6: we study stable reflexive sheaves with numerically trivial first Chern class and pseudo-effective tautological line bundle.

In [51], Nakayama studies semistable vector bundle $\mathscr{E}$ of rank two on complex projective manifold with $c_1(\mathscr{E}) \equiv 0$ and pseudo-effective tautological class (see [51, Theorem IV.4.8] for a precise statement). Our strategy of proof for Theorem 6.1 partly follows his line of reasoning.

Theorem 6.1 *Let X be a normal complex projective variety of dimension n, let H be an ample Cartier divisor, and let $\mathscr{E}$ be a reflexive sheaf of rank $r \in \{1, 2, 3\}$ on X. Suppose that X is smooth in codimension two and that $\mathscr{E}$ is H-stable with $c_1(\mathscr{E}) \cdot H^{n-1} = 0$. Suppose furthermore that, for any finite morphism $f : \widetilde{X} \to X$ that is étale in codimension one, the reflexive pull-back $f^{[*]}\mathscr{E}$ is stable with respect to f^*H. Then one of the following holds:*

1. *either there exists $c > 0$ such that $h^0\big(X, S^{[i]}\mathscr{E} \otimes \mathscr{O}_X(jH)\big) = 0$ for any positive integer j and any natural number i satisfying $i > cj$,*
2. *or $c_1(\mathscr{E})^2 \cdot H^{n-2} = c_2(\mathscr{E}) \cdot H^{n-2} = 0$,*
3. *or $r = 3$, and there exists a finite morphism $f : \widetilde{X} \to X$ that is étale in codimension one, and a rank 1 reflexive sheaf $\mathscr{L}$ on $\widetilde{X}$ with $c_1(\mathscr{L}) \cdot (f^*H)^{n-1} = 0$ such that $h^0\big(\widetilde{X}, (S^2 f^*\mathscr{E}) \boxtimes \mathscr{L}\big) \neq 0$.*

Remark 6.2 The condition on the dimension of the singular locus of X posed in Theorem 6.1 allows to define the Chern class $c_2(\mathscr{E})$.

Remark 6.3 The condition (1) in the statement of Theorem 6.1 is a way of saying that the tautological line bundle is not pseudo-effective on singular spaces (see Lemma 2.7).

Remark 6.4 In the setup of Theorem 6.1, suppose furthermore that X is smooth, and that $\mathscr{E}$ is locally free. If $c_1(\mathscr{E})^2 \cdot H^{n-2} = c_2(\mathscr{E}) \cdot H^{n-2} = 0$, then $\mathscr{E}$ is flat by a result of Uhlenbeck and Yau (see [57]). In particular, the tautological line bundle is nef.

The following consequence of Theorem 6.1 improves [6, Theorem 7.7]. The conclusion also holds for K3-surfaces by [51, Theorem IV.4.15].

Corollary 6.5 *Let X be a Calabi–Yau complex projective manifold of dimension 3. Then the tautological line bundle on $\mathbb{P}_X(T_X)$ is not pseudo-effective.*

Proof It is well-known that the tangent sheaf T_X is stable with respect to any polarization H, and that $c_2(X) \cdot H^2 \neq 0$. We argue by contradiction and assume that the tautological line bundle on $\mathbb{P}_X(T_X)$ is pseudo-effective. By Theorem 6.1, there exists a line bundle $\mathscr{L}$ with $\mu_H(\mathscr{L}) = 0$ such that $h^0\big(X, (S^2 T_X) \otimes \mathscr{L}\big) \neq 0$. This implies that $\Omega^1_X \cong T_X \otimes \mathscr{L}$. Taking determinants, we obtain $\mathscr{L}^{\otimes 2} \cong \mathscr{O}_X$. Since X is simply connected, we must have $\mathscr{L} \cong \mathscr{O}_X$. On the other hand, by [41, Corollary 8], we have $h^0(X, S^2 T_X) = 0$, yielding a contradiction. □

We collect several examples which illustrate to what extend our result is sharp.

Example 6.6 Let X be a projective K3-surface. The tautological line bundle on $\mathbb{P}_X(T_X)$ is not pseudo-effective by [51, Theorem IV.4.15]. Thus $\mathscr{E}$ satisfies (1) in the statement of Theorem 6.1 by Lemma 2.7.

Example 6.7 Let X be a complex projective manifold, and let $\mathscr{L} \in \mathrm{Pic}^0(X)$. Then $\mathscr{L}$ obviously satisfies (2) in the statement of Theorem 6.1.

Example 6.8 Let C be a complete curve of genus $g \geq 2$. We construct a rank two vector bundle $\mathscr{E}$ on C of degree 0 such that, for any étale cover $f : \widetilde{C} \to C$, the pull-back $f^*\mathscr{E}$ is stable. The vector bundle $\mathscr{E}$ satisfies (2) in the statement of Theorem 6.1.

The construction is very similar to that of Hartshorne in [32, Theorem I.10.5], and so we leave some easy details to the reader. Pick $c \in C$. By a result of Narasimhan and Seshadri [52], we must construct a unitary representation $\rho\colon \pi_1(C, c) \to \mathbb{U}(2)$ such that, for any normal subgroup $H \lhd \pi_1(C, c)$ of finite index, the induced representation $H \to \mathbb{U}(2)$ is irreducible.

It is well-known that $\pi_1(C, c)$ is generated by elements $a_1, b_1, \ldots, a_g, b_g$ satisfying the relation

$$[a_1, b_1] \cdot \cdots \cdot [a_g, b_g] = 1.$$

If we have chosen any two unitary matrices $A_1, B_1 \in \mathbb{U}(2)$, then we can find further unitary matrices $A_2, B_2, \ldots, A_g, B_g$ satisfying the relation above. Let $A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $|\lambda_i| = 1$, and $\lambda_1\lambda_2^{-1}$ is not a root of unity. Let B_1 be a very general unitary matrix. Then all the entries of all the matrices B_1^m ($m \geq 1$) are non-zero. Let $H \lhd \pi_1(C, c)$ be a normal subgroup of finite index. Let m be a positive integer such that $A_1^m, B_1^m \in H$. The only invariant subspaces of A_1^m are the subspaces spanned by some subset of the standard basis. This implies that the representation $H \to \mathbb{U}(2)$ is irreducible. Indeed, in order for B_1^m to have as fixed subspace a subspace generated by a subset of the standard basis, it would have to have some entries zero.

Example 6.9 Let X be a projective K3-surface, and consider $\mathscr{E} := S^2T_X$. A straightforward computation shows that $c_2(\mathscr{E}) = 4 \cdot 24$. It is known that $\mathscr{E}$ is stable with respect to any polarization. On the other hand,

$$S^2(S^2T_X) \cong S^4T_X \oplus \det(T_X)^{\otimes 2} \cong S^4T_X \oplus \mathscr{O}_X,$$

and hence $h^0(X, S^2\mathscr{E}) \neq 0$. Thus $\mathscr{E}$ satisfies (3) in the statement of Theorem 6.1 above.

We have divided the proof of Theorem 6.1 into a sequence of steps, each formulated as a separate result. Some of these statements might indeed be of independent interest. The proof of Theorem 6.1 then follows quickly from these preliminary steps.

6.10 (Divisorial Zariski decomposition) We briefly recall the definition of the divisorial Zariski decomposition from [51]. Let X be a complex projective manifold, let D be a big $\mathbb{R}$-divisor on X, and let P be a prime divisor. The asymptotic order of vanishing of D along P is

$$\sigma_P(D) = \inf_G \left\{\mathrm{mult}_P(G)\right\},$$

where the infimum is over all effective $\mathbb{R}$-divisor G with $G \sim_{\mathbb{R}} D$.

Let now D be a pseudo-effective $\mathbb{R}$-divisor, and let A be an ample $\mathbb{R}$-divisor on X. Let

$$\sigma_P(D) = \lim_{\varepsilon \downarrow 0} \sigma_P(D + \varepsilon A).$$

Then $\sigma_P(D)$ exists and is independent of the choice of A. There are only finitely many prime divisors P such that $\sigma_P(D) > 0$, and the $\mathbb{R}$-divisor $N_\sigma(D) := \sum_P \sigma_P(D)P$ is determined by the numerical equivalence class of D. Set $P_\sigma(D) := D - N_\sigma(D)$.

6.11 (Diminished base locus) Let D be a $\mathbb{Q}$-divisor on a smooth projective manifold X. Let k be a positive integer such that kD is integral. The *stable base locus* of D is

$$\mathbf{B}(D) := \bigcap_{m \geq 1} \mathrm{Bs}(mkD).$$

It is independent of the choice of k.

The *diminished base locus* of an $\mathbb{R}$-divisor D is

$$\mathbf{B}_-(D) = \bigcup_A \mathbf{B}(D + A)$$

where the union is taken over all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor (see [20, Definition 1.12]). The diminished base locus of a divisor is a countable union of Zariski closed subsets of X by [20, Proposition 1.19]. Notice that $\mathbf{B}_-(D) \subsetneq X$ if and only if D is pseudo-effective, and that $\mathbf{B}_-(D) = \emptyset$ if and only if D nef. By [51, Proposition III 1.14], $N_\sigma(D) = 0$ if and only if D is movable.

We will need the following observation.

Lemma 6.12 *Let X be a complex projective manifold, and let D be a pseudo-effective $\mathbb{R}$-divisor on X. Let B be an irreducible component of $\mathbf{B}_-(D)$. Let*

$\beta_1\colon Y_1 \to X$ be the blow-up of X along B, and let $\beta_2\colon Y \to Y_1$ be a resolution of singularities. Suppose that β_2 induces an isomorphism over the smooth locus of Y_1. Set $\beta := \beta_1 \circ \beta_2$, and let E be the unique β-exceptional divisor on Y with center B in X. Then $\sigma_E(\beta^* D) > 0$.

Proof Let A be an ample $\mathbb{R}$-divisor on X. By [20, Lemma 3.3] applied to the divisorial valuation mult_E of the function field of X given by the order of vanishing at the generic point of B,

$$\sigma_E\big(\beta^*(D+A)\big) = \inf_G \left\{\mathrm{mult}_E(\beta^* G)\right\}$$

where the infimum is over all effective $\mathbb{R}$-divisor G with $G \equiv D+A$. If $D+A$ is a $\mathbb{Q}$-divisor, then by [5, Lemma 3.5.3],

$$\mathbf{B}(D+A) = \bigcap_F \mathrm{Supp}(F)$$

where the intersection is over all effective $\mathbb{R}$-divisor F with $F \equiv D+A$. The assertion now follows from [51, Lemmata V.1.9 and III.2.3]. □

The proof of Proposition 6.22 below makes use of the following lemma.

Lemma 6.13 *Let Y be a complex projective manifold of dimension $n \geq 2$, and let D be an $\mathbb{R}$-divisor. Suppose that D is movable, and suppose furthermore that there is an irreducible component B of $\mathbf{B}_-(D)$ of codimension two. Let $|H|$ be a base-point-free linear system on Y, let $0 \leqslant k \leqslant n-2$ be an integer, and let S be a complete intersection of k very general elements in $|H|$. There exists a real number $a > 0$ such that*

$$(D_{|S}^2 - aB \cap S) \cdot h_3 \cdot \dots \cdot h_{n-k} \geq 0$$

for arbitrary codimension one nef classes $h_3, \dots, h_{n-k}$ on S.

Proof Let $\beta_1\colon Z_1 \to Y$ be an embedded resolution of B, and let $\beta_2\colon Z \to Z_1$ be the blow-up of Z_1 along the strict transform B_1 of B in Z_1. Set $\beta := \beta_1 \circ \beta_2$, and let $E_1, \dots, E_r$ be the β-exceptional divisors. Suppose that $E_1 = \mathrm{Exc}(\beta_2)$.

Set $S_1 := \beta_1^{-1}(S)$, and $T := \beta^{-1}(S)$. Notice that S_1 is an embedded resolution of $B \cap S$, and that $T \to S_1$ is the blow-up of the strict transform $B_1 \cap S_1$ of $B \cap S$ in S_1. Write $F_i := E_i \cap T$, and denote by $\mu_1\colon S_1 \to S$, $\mu_2\colon T \to S_1$, and $\mu\colon T \to S$ the natural morphisms. Set $D_S := D_{|S}$, and let $h_3, \dots, h_{n-k}$ be codimension one nef classes on S.

Write $a_i := \sigma_{E_i}(\beta^* D) \in \mathbb{R}_{\geq 0}$. The $\mathbb{R}$-divisor $\beta^* D - \sum_{1 \leqslant i \leqslant r} a_i E_i$ is then movable. Notice that $a_1 > 0$ by Lemma 6.12 above.

By Lemma 6.14 below, the restriction of $\beta^* D - \sum_{1\leqslant i\leqslant r} a_i E_i$ to T is also movable, and therefore

$$\Bigg(\mu^* D_S - \sum_{1\leqslant i\leqslant r} a_i F_i\Bigg)^2 \cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k} \geq 0.$$

But $F_i \cdot \mu^* D_S \cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k} = 0$ since F_i is μ-exceptional, and $F_i \cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k} \equiv 0$ when $i \geq 2$ since $\mu(F_i) \subsetneq B \cap S$ for each $i \geq 2$. Thus

$$\begin{aligned}\Bigg(\mu^* D_S - \sum_{1\leqslant i\leqslant r} a_i F_i\Bigg)^2 &\cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k}\\ &= \big(\mu^*(D_S^2) + a_1^2 F_1^2\big) \cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k}.\end{aligned}$$

The projection formula gives

$$\big(\mu^*(D_S^2) + a_1^2 F_1^2\big) \cdot \mu^* h_3 \cdot \dots \cdot \mu^* h_{n-k} = \big(D_S^2 - a_1^2 B \cap S\big) \cdot h_3 \cdot \dots \cdot h_{n-k},$$

using the fact that $\mu_{2*} F_1^2 = -B_1 \cap S_1$. This proves the lemma. □

Lemma 6.14 *Let Y be a complex projective manifold, let $V \subset |H|$ be a not necessarily complete base-point-free linear system on Y, and let D be an $\mathbb{R}$-divisor. If D is pseudo-effective (resp. movable), then its restriction to a very general element in V is pseudo-effective (resp. movable) as well.*

Proof Suppose that D is pseudo-effective (resp. movable). There is a sequence of effective (resp. effective movable) integral divisors M_i on Y and a sequence λ_i of non-negative real numbers such that $\lambda_i[M_i] \to [D]$ in $\overline{\mathrm{Eff}}(Y)$ as $i \to +\infty$. Now, the restriction of an effective (resp. effective movable) divisor to a general element in V is effective (resp. effective movable). So, if H' is a very general element in V, then for each i, $M_{i\,|H'}$ is effective (resp. movable), and hence so is $D_{|H'}$. □

The proofs of Lemma 6.15 and Proposition 6.16 follow arguments that go back at least as far as [51, Theorem IV 4.8].

Lemma 6.15 *Let X be a complex projective manifold, and let $\mathscr{E}$ be a coherent locally free sheaf on X. Suppose that $\mathscr{E}$ is semistable with respect to any polarization and that $c_1(\mathscr{E}) \equiv 0$. Set $Y := \mathbb{P}_X(\mathscr{E})$, and denote by $\mathscr{O}_Y(1)$ the tautological line bundle. If $\xi := [\mathscr{O}_Y(1)] \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is pseudo-effective, then it generates an extremal ray of the cone of pseudo-effective classes $\overline{\mathrm{Eff}}(Y)$.*

Proof Let ξ_1 and ξ_2 be pseudo-effective classes on Y such that $\xi = \xi_1 + \xi_2$. Write $\xi_i = a_i\xi + \pi^*\gamma_i$ for some real number a_i and some class γ_i on X. Notice that $a_i \geq 0$, $a_1 + a_2 = 1$, and $\gamma_1 + \gamma_2 = 0$.

Denote by $\pi : Y \to X$ the natural projection. Let H be an ample divisor, and let $C \subset X$ be a general complete intersection curve of elements in $|mH|$ where m is a sufficiently large positive integer. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{|C}$ is stable with $\deg(\mathscr{E}_{|C}) = 0$. In particular, $\mathscr{E}_{|C}$ is nef. Set $Z := \pi^{-1}(C)$. By [22, Lemma 2.2],

$$\mathrm{Nef}(Z) = \overline{\mathrm{Eff}}(Z) = \langle \xi_{|Z}, f\rangle$$

where f denotes the numerical class of a fiber of the projection morphism $Z \to C$. This implies that $\deg(\gamma_{i\,|C}) \geq 0$, and hence $\deg(\gamma_{i\,|C}) = 0$ since $\gamma_1 + \gamma_2 = 0$. Since H is arbitrary, we conclude that $\gamma_1 = \gamma_2 = 0$. This completes the proof of the lemma. □

Proposition 6.16 *Let X be a complex projective manifold, and let $\mathscr{E}$ be a locally free sheaf on X. Suppose that $\mathscr{E}$ is semistable with respect to an ample divisor H, and that $\mu_H(\mathscr{E}) = 0$. Set $Y := \mathbb{P}_X(\mathscr{E})$. Suppose that the tautological line bundle $\mathscr{O}_Y(1)$ is pseudo-effective. If $\mathscr{O}_Y(1)$ is not movable, then there exists a line bundle $\mathscr{L}$ with $\mu_H(\mathscr{L}) = 0$ and a positive integer m such that $h^0\big(X, (S^m\mathscr{E}) \otimes \mathscr{L}\big) \neq 0$.*

Proof Set $n := \dim X$. Denote by Ξ a tautological divisor on Y, and by $\pi : Y \to X$ the natural morphism. Set $P := P_\sigma(\Xi)$ and $N := N_\sigma(\Xi)$. Write $N = \sum_{i\in I} \sigma_i N_i$, where N_i is a prime exceptional divisor, and $\sigma_i \in \mathbb{R}_{>0}$. We have $N_i \sim_{\mathbb{Z}} m_i\Xi + \pi^*\Gamma_i$ for some divisor Γ_i on X and some non-negative integer m_i.

Let $C \subset X$ be a complete intersection curve of very general elements in $|mH|$ where m is a sufficiently large positive integer. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{|C}$ is stable with $\deg(\mathscr{E}_{|C}) = 0$. Set $Z := \pi^{-1}(C)$. Notice that $\Xi_{|Z}$, $P_{|Z}$, and $N_{i\,|Z}$ are pseudo-effective by Lemma 6.14. Applying Lemma 6.15, we see that $[\pi^*\Gamma_{i\,|Z}] \in \mathbb{R}[\Xi_{|Z}]$ for each $i \in I$, and thus $\Gamma_i \cdot H^{n-1} = 0$ since $\Xi_{|Z}$ is relatively ample over Z. Pick $i \in I$. If $m_i = 0$, then $h^0\big(X, \mathscr{O}_X(\Gamma_i)\big) \neq 0$, and hence $\Gamma_i \sim_{\mathbb{Z}} 0$ since $\Gamma_i \cdot H^{n-1} = 0$. This implies that $N_i = 0$, yielding a contradiction. Therefore, $m_i > 0$ and $h^0\big(X, (S^{m_i}\mathscr{E}) \otimes \mathscr{O}_X(\Gamma_i)\big) \neq 0$. This proves the proposition. □

The proof of the next result follows the line of argument given in [6, Theorem 7.6].

Lemma 6.17 *Let S be a smooth complex projective surface, and let $\mathscr{E}$ be a locally free sheaf of rank $r \geq 2$ on S. Suppose that $\mathscr{E}$ is semistable with respect to an ample divisor H, and that $c_1(\mathscr{E}) \cdot H = 0$. Let $\xi \in \mathrm{N}^1(Y)_{\mathbb{R}}$ be the class* *of the tautological line bundle* $\mathscr{O}_Y(1)$. *If any irreducible component of* $\mathbf{B}_-(\xi)$ *has dimension at most* 1, *then* $c_1(\mathscr{E})^2 = c_2(\mathscr{E}) = 0$.

Proof Denote by $\pi : Y \to S$ the natural morphism, and denote by $h \in \mathrm{N}^1(S)_{\mathbb{R}}$ the class of H. Let $G \subset Y$ be a very general hyperplane section. Suppose that $[G] \equiv m(\xi + t\pi^*h)$ for some positive integers m and t. Notice that G does not contain any irreducible component of $\mathbf{B}_-(\xi)$. It follows that $\xi_{|G}$ is nef, and hence

$$\xi^r \cdot G \geq 0.$$

The equation

$$\xi^r \equiv \pi^*c_1(\mathscr{E}) \cdot \xi^{r-1} - \pi^*c_2(\mathscr{E}) \cdot \xi^{r-2},$$

yields

$$\begin{aligned} \xi^r \cdot G &= m\big(\pi^*c_1(\mathscr{E}) \cdot \xi^{r-1} - \pi^*c_2(\mathscr{E}) \cdot \xi^{r-2}\big) \cdot (\xi + t\pi^*h) \\ &= m\big(c_1(\mathscr{E})^2 - c_2(\mathscr{E})\big), \end{aligned}$$

and hence

$$c_1(\mathscr{E})^2 - c_2(\mathscr{E}) \geq 0.$$

On the other hand, we have

$$2rc_2(\mathscr{E}) - (r-1)c_1(\mathscr{E})^2 \geq 0$$

by Bogomolov's inequality (see [34, Theorem 3.4.1]), and thus $c_1(\mathscr{E})^2 \geq 0$. Finally, the Hodge index theorem implies that $c_1(\mathscr{E})^2 \leqslant 0$, and hence we must have $c_1(\mathscr{E})^2 = c_2(\mathscr{E}) = 0$. This proves the lemma. □

6.18 (The holonomy group of a stable reflexive sheaf) Let X be a normal complex projective variety, and let $\mathscr{E}$ be a reflexive sheaf on X. Suppose that $\mathscr{E}$ is stable with respect to an ample Cartier divisor H with slope $\mu_H(\mathscr{E}) = 0$. For a sufficiently large positive integer m, let $C \subset X$ be a general complete intersection curve of elements in $|mH|$. Let $x \in C$. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{|C}$ is stable with $\deg(\mathscr{E}_{|C}) = 0$, and hence it corresponds to a unique unitary representation $\rho\colon \pi_1(C, x) \to \mathbb{U}(\mathscr{E}_x)$ by a result of Narasimhan and Seshadri [52]. The *holonomy group* $\mathrm{Hol}_x(\mathscr{E})$ of $\mathscr{E}$ is the Zariski closure of $\rho\big(\pi_1(C, x)\big)$ in $\mathrm{GL}(\mathscr{E}_x)$. It does not depend on $C \ni x$ provided that m is large enough. Moreover, the fiber map $\mathscr{E} \to \mathscr{E}_x$ induces a one-to-one correspondence between direct summands of

$\mathscr{E}^{\otimes r} \boxtimes (\mathscr{E}^*)^{\otimes s}$ and $\mathrm{Hol}_x(\mathscr{E})$-invariant subspaces of $\mathscr{E}_x^{\otimes r} \otimes (\mathscr{E}_x^*)^{\otimes s}$, where r and s are non-negative integers (see [9, Theorem 1]).

The proof of Theorem 6.1 makes use of the following lemma. Example 6.20 below shows that the statement of [9, Lemma 40] is slightly incorrect. An extra assumption is needed to guarantee that the holonomy groups are well-defined.

Lemma 6.19 [9, Lemma 40] *Let X be a normal complex projective variety, let $x \in X$ be a general point, and let $\mathscr{E}$ be a reflexive sheaf on X. Suppose that $\mathscr{E}$ is stable with respect to an ample divisor H, and that $\mu_H(\mathscr{E}) = 0$. Suppose furthermore that, for any finite morphism $f : \widetilde{X} \to X$ that is étale in codimension one, the reflexive pull-back $f^{[*]}\mathscr{E}$ is stable with respect to f^*H. Then there exists a finite morphism $f : \widehat{X} \to X$, étale in codimension one, such that $\mathrm{Hol}_{\widehat{x}}(f^{[*]}\mathscr{E})$ is connected, where $\widehat{x}$ is a point on $\widehat{X}$ such that $f(\widehat{x}) = x$.*

Example 6.20 (see [25, Example 8.6]) Let Z be a complex projective K3-surface, let $\widetilde{X} := Z \times Z$, and let $\iota \in \mathrm{Aut}(\widetilde{X})$ be the automorphism which interchanges the two factors. The quotient $X := \widetilde{X}/\iota$ is then a normal projective variety, and the quotient map $\pi : \widetilde{X} \to X$ is finite and étale in codimension one. The tangent sheaf T_X of X is stable with respect to any ample polarization on X (see [25, Example 8.6]). Let x is a general point on X. Then $\mathrm{Hol}_x(T_X)^\circ = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, and $\mathrm{Hol}_x(T_X)/\mathrm{Hol}_x(T_X)^\circ \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, the morphism π is the map given by [9, Lemma 40]. But, the reflexive pull-back $\pi^{[*]}T_X = T_{\widetilde{X}} = T_Z \boxplus T_Z$ is obviously not stable.

Lemma 6.21 *Let X be a complex projective manifold, let $x \in X$, and let $\mathscr{E}$ be a coherent locally free sheaf on X. Suppose that $\mathscr{E}$ is stable with respect to an ample divisor H, and that $\mu_H(\mathscr{E}) = 0$. Suppose furthermore that its holonomy group $\mathrm{Hol}_x(\mathscr{E})$ is connected. Then, for any finite cover $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth and projective, the pull-back $f^*\mathscr{E}$ is stable with respect to f^*H.*

Proof Let $\widetilde{X}$ be a complex projective manifold, and let $f : \widetilde{X} \to X$ be a finite cover. By [39, Theorem 1], the locally free sheaf $f^*\mathscr{E}$ is polystable with respect to f^*H. For a sufficiently large positive integer m, let $C \subset X$ (resp. $\widetilde{C}$) be a general complete intersection curve of elements in $|mH|$ (resp. $|mf^*H|$), and let $x \in C$ (resp. $\widetilde{x} \in f^{-1}(x)$). By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{|C}$ is stable with $\deg(\mathscr{E}_{|C}) = 0$, and hence it corresponds to a unique unitary irreducible representation $\rho : \pi_1(C, x) \to \mathbb{U}(\mathscr{E}_x)$ by [52]. Notice that $f^*\mathscr{E}_{|\widetilde{C}}$ is then induced by the representation

$$\widetilde{\rho} := \rho \circ \pi_1(f_{|\widetilde{C}}) : \pi_1(\widetilde{C}, \widetilde{x}) \to \pi_1(C, x) \to \mathbb{U}(\mathscr{E}_x) \cong \mathbb{U}\big((f^*\mathscr{E})_{\widetilde{x}}\big).$$

We argue by contradiction and assume that $f^*\mathscr{E}$ is not stable with respect to f^*H. So let $\mathscr{G}$ be a f^*H-stable direct summand of $f^*\mathscr{E}$. Then $\mathscr{G}_{\widetilde{x}}$ is $\widetilde{\rho}$-invariant, and since the image of

$$\pi_1(f_{|\widetilde{C}})\colon \pi_1(\widetilde{C}, \widetilde{x}) \to \pi_1(C, x)$$

has finite index, the orbit $\pi_1(C, x) \cdot \mathscr{G}_{\widetilde{x}}$ of $\mathscr{G}_{\widetilde{x}}$ is a finite union of proper linear subspaces, where we view $\mathscr{G}_{\widetilde{x}}$ as a linear subspace of $\mathscr{E}_x \cong (f^*\mathscr{E})_{\widetilde{x}}$. It follows that $\pi_1(C, x) \cdot \mathscr{G}_{\widetilde{x}}$ is also $\mathrm{Hol}_x(\mathscr{E})$-invariant. Now, since $\mathrm{Hol}_x(\mathscr{E})$ is a connected algebraic group, we conclude that $\mathscr{G}_{\widetilde{x}}$ is $\pi_1(C, x)$-invariant. Therefore $\mathscr{E}$ is not H-stable, yielding a contradiction. This completes the proof of the lemma. □

We now provide another technical tool for the proof of Theorem 6.1.

Proposition 6.22 *Let X be a complex projective manifold, and let $\mathscr{E}$ be a rank 3 locally free sheaf on X. Suppose that $\mathscr{E}$ is stable with respect to an ample divisor H, and that $\mu_H(\mathscr{E}) = 0$. Suppose furthermore that $\mathrm{Hol}_x(\mathscr{E})$ is connected for $x \in X$. Set $Y := \mathbb{P}_X(\mathscr{E})$, and denote by $\mathscr{O}_Y(1)$ the tautological line bundle. Suppose that $\xi := [\mathscr{O}_Y(1)] \in \mathrm{N}^1(Y)_{\mathbb{R}}$ is pseudo-effective. If ξ is movable, then any irreducible component of $\mathbf{B}_-(\xi)$ has codimension at least three.*

Proof Denote by $\pi : Y \to X$ the natural projection. We argue by contradiction and assume that ξ is movable, and that there exists an irreducible component B of $\mathbf{B}_-(\xi)$ of codimension two.

For a sufficiently large positive integer m, let $C \subset X$ be a complete intersection curve of very general elements in $|mH|$. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{|C}$ is stable with $\deg(\mathscr{E}_{|C}) = 0$. Moreover, if $x \in C$, then $\mathrm{Hol}_x(\mathscr{E}_{|C}) = \mathrm{Hol}_x(\mathscr{E})$ is connected. Set $Z := \pi^{-1}(C)$, and $\xi_Z := \xi_{|Z}$. Notice that $\dim Z = 3$. Then ξ_Z is nef, and $\xi_Z^3 = 0$. By Lemma 6.13, there exists a real number $a > 0$ such that

$$0 \leqslant a(B \cap Z) \cdot \xi_Z \leqslant \xi_Z^3 = 0,$$

and hence $(B \cap Z) \cdot \xi_Z = 0$. In particular, since ξ_Z is relatively ample over C, any irreducible component of $B \cap Z$ maps onto C. Let $\widetilde{C}$ be the normalization of $B \cap Z$, and denote by $f : \widetilde{C} \to C$ the induced morphism. The natural morphism $g\colon \widetilde{C} \to Z$ induces a surjective morphism $f^*(\mathscr{E}_{|C}) \twoheadrightarrow g^*(\mathscr{O}_Y(1)_{|Z})$. Since

$$\deg g^*(\mathscr{O}_Y(1)_{|Z}) = (B \cap Z) \cdot \xi_Z = 0,$$

the locally free sheaf $f^*(\mathscr{E}_{|C})$ is not stable. But this contradicts Lemma 6.21, completing the proof of the proposition. □

Proof of Theorem 6.1 We maintain notation and assumptions of Theorem 6.1. We claim that we may assume without loss of generality that the algebraic group $\mathrm{Hol}_x(\mathscr{E})$ is connected. Indeed, by Lemma 6.19, there exists a finite cover $f\colon \widetilde{X} \to X$, étale in codimension one, such that $\mathrm{Hol}_{\widetilde{x}}(f^{[*]}\mathscr{E})$ is connected, where $\widetilde{x}$ is a point on $\widetilde{X}$ such that $f(\widetilde{x}) = x$. By the Nagata–Zariski purity theorem, f is étale in codimension two, and hence $\widetilde{X}$ is smooth in codimension two as well. Suppose now that the conclusion of Theorem 6.1 holds for $f^{[*]}\mathscr{E}$. The sheaf $S^{[i]}\mathscr{E} \otimes \mathscr{O}_X(jH)$ is a direct summand of $f_{[*]}\big(S^i(f^*\mathscr{E}) \otimes f^*\mathscr{O}_X(jH)\big)$ for any non-negative integers i and j. Thus, if $f^{[*]}\mathscr{E}$ satisfies condition (1) in the statement of Theorem 6.1, then the same holds for $\mathscr{E}$. Let $X^\circ \subset X_{\text{reg}}$ be the open set where $\mathscr{E}$ is locally free. Then X° has codimension at least 3 in X by [33, Corollary 1.4] using the fact that X is smooth in codimension two. Since

$$
\begin{aligned}
c_1\big(f^*_{|f^{-1}(X^\circ)}\mathscr{E}_{|X^\circ}\big) &= f^*_{|f^{-1}(X^\circ)}c_1(\mathscr{E}_{|X^\circ}) \quad \text{and}\\
c_2\big(f^*_{|f^{-1}(X^\circ)}\mathscr{E}_{|X^\circ}\big) &= f^*_{|f^{-1}(X^\circ)}c_2(\mathscr{E}_{|X^\circ}),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
c_1(f^{[*]}\mathscr{E})^2 \cdot (f^*H)^{n-2} &= \deg(f)c_1(\mathscr{E})^2 \cdot H^{n-2} \quad \text{and}\\
c_2(f^{[*]}\mathscr{E}) \cdot (f^*H)^{n-2} &= \deg(f)c_2(\mathscr{E}) \cdot H^{n-2}.
\end{aligned}
$$

This implies that if $f^{[*]}\mathscr{E}$ satisfies condition (2) in the statement of Theorem 6.1, then the same holds for $\mathscr{E}$. Finally, if $f^{[*]}\mathscr{E}$ satisfies condition (3) in the statement of Theorem 6.1, then the same obviously holds for $\mathscr{E}$. Thus, by replacing X with $\widetilde{X}$, we may assume that $\mathrm{Hol}_x(\mathscr{E})$ is connected. This proves our claim.

Suppose from now on that $h^0\big(X, S^{[i]}\mathscr{E} \otimes \mathscr{O}_X(jH)\big) \neq 0$ for infinitely many $(i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 1}$ with $i/j \to +\infty$. Let S be a smooth two dimensional complete intersection of very general elements in $|mH|$ for a sufficiently large positive integer m. Observe that S is contained in the smooth locus X_{reg} of X, and that $\mathscr{E}$ is locally free along S (see [33, Corollary 1.4]). We may also obviously assume that $x \in S$. Set $\mathscr{E}_S := \mathscr{E}_{|S}$, and $H_S := H_{|S}$. By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_S$ is stable with respect to H_S, and $\mu_{H_S}(\mathscr{E}_S) = 0$. Moreover, the algebraic group $\mathrm{Hol}_x(\mathscr{E}_S) = \mathrm{Hol}_x(\mathscr{E})$ is connected. Set $Y := \mathbb{P}_S(\mathscr{E}_S)$ with natural morphism $\pi : Y \to S$. Denote by $\xi \in \mathrm{N}^1(Y)_{\mathbb{R}}$ the numerical class of the tautological line bundle $\mathscr{O}_Y(1)$. Notice that $h^0\big(Y, S^i\mathscr{E}_S \otimes \mathscr{O}_S(jH_S)\big) \neq 0$ for infinitely many $(i, j) \in \mathbb{N} \times \mathbb{N}_{\geq 1}$ with $i/j \to +\infty$. This implies that $\xi \in \overline{\mathrm{Eff}}(Y)$. If $r = 1$, then obviously we must have $\xi = 0$. Suppose from now on that $r \in \{2, 3\}$.

Case 1: ξ *is movable.* If $r = 3$, then any irreducible component of $\mathbf{B}_-(\xi)$ has codimension at least three by Proposition 6.22. In either case, any irreducible component of $\mathbf{B}_-(\xi)$ has dimension at most 1. Thus, by Lemma 6.17, we must have

$$c_1(\mathscr{E})^2 \cdot H^{n-2} = c_1(\mathscr{E}_S)^2 = 0 \quad \text{and} \quad c_2(\mathscr{E}) \cdot H^{n-2} = c_2(\mathscr{E}_S) = 0.$$

Case 2: ξ *is not movable*. By Proposition 6.16, there exists a line bundle $\mathscr{L}_S$ on S with $\mu_{H_S}(\mathscr{L}_S) = 0$ and a positive integer k such that $h^0(S, S^k\mathscr{E}_S \otimes \mathscr{L}_S) \neq 0$. Since $S^k\mathscr{E}_S$ is polystable, $\mathscr{L}_S^{\otimes -1}$ is a direct summand of $S^k\mathscr{E}_S$, and hence $(\mathscr{L}_S^{\otimes -1})_x \subset (S^k\mathscr{E})_x$ is $\mathrm{Hol}_x(\mathscr{E})$-invariant.

Suppose first that $r = 2$. By [9, 45], $\mathrm{Hol}_x(\mathscr{E}) = \mathrm{SL}(\mathscr{E}_x)$ or $\mathrm{GL}(\mathscr{E}_x)$. In either case, $(S^k\mathscr{E})_x$ is an irreducible $\mathrm{Hol}_x(\mathscr{E})$-module, yielding a contradiction.

Suppose now that $r = 3$. Then $\mathrm{Hol}_x(\mathscr{E}) = \mathrm{SL}(\mathscr{E}_x)$, $\mathrm{GL}(\mathscr{E}_x)$, $\mathrm{SO}(\mathscr{E}_x)$, or $\mathrm{GSO}(\mathscr{E}_x)$ by [9, 45] again, where $\mathrm{GSO}(\mathscr{E}_x)$ denotes the group of proper similarity transformations. Arguing as above, we conclude that $\mathrm{Hol}_x(\mathscr{E}) = \mathrm{SO}(\mathscr{E}_x)$ or $\mathrm{GSO}(\mathscr{E}_x)$. In either case, there exists a rank one reflexive sheaf $\mathscr{L}$ on X with $\mu_H(\mathscr{L}) = 0$ such that $h^0(X, S^2\mathscr{E} \boxtimes \mathscr{L}) \neq 0$. This completes the proof of the theorem. □

7 Holomorphic Riemannian metric and holomorphic connection

7.1 (Bott connection) Let X be a complex manifold, let $\mathscr{G} \subset T_X$ be a regular foliation, and set $\mathscr{N} = T_X/\mathscr{G}$. Let $p\colon T_X \to \mathscr{N}$ denotes the natural projection. For sections U of $\mathscr{N}$, T of T_X, and V of $\mathscr{G}$ over some open subset of X with $U = p(T)$, set $\nabla^{\mathrm{B}}_V U = p([V, U])$. This expression is well-defined, $\mathscr{O}_X$-linear in V and satisfies the Leibniz rule $\nabla^{\mathrm{B}}_V(fU) = f\nabla^{\mathrm{B}}_V U + (V \cdot f)U$ so that ∇^{B} is a $\mathscr{G}$-connection on $\mathscr{N}$ (see [4]). We refer to it as the *Bott* (partial) *connection* on $\mathscr{N}$.

7.2 (Holomorphic Riemannian metric) Given a complex manifold X and a vector bundle $\mathscr{E}$ on X, recall that a holomorphic metric $\mathfrak{g}$ on $\mathscr{E}$ is a global section of $S^2(\mathscr{E}^*)$ such that $\mathfrak{g}(x)$ is non-degenerate for all $x \in X$.

Lemma 7.3 *Let X be a complex manifold, and let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition of T_X into locally free sheaves. Suppose that $\mathscr{E}$ is involutive, and suppose furthermore that $\mathscr{E}$ admits a holomorphic metric $\mathfrak{g}$. Then there exists an $\mathscr{E}$-connection ∇^{LC} on $\mathscr{E}$ such that, for sections U, V, and W of $\mathscr{E}$ over some open subset of X, the following holds:*

1. $\nabla^{\mathrm{LC}}_U V - \nabla^{\mathrm{LC}}_V U = [U, V]$ *(∇^{LC} is torsion-free), and*
2. $W \cdot \mathfrak{g}(U, V) = \mathfrak{g}\big(\nabla^{\mathrm{LC}}_W U, V\big) + \mathfrak{g}\big(U, \nabla^{\mathrm{LC}}_W V\big)$ *(∇^{LC} preserves $\mathfrak{g}$).*

Definition 7.4 We will refer to ∇^{LC} as the *Levi-Civita* (partial) *connection* on $\mathscr{E}$.

Proof of Lemma 7.3 Let U and V be local sections of $\mathscr{E}$ over some open subset of X. Then $\nabla^{\mathrm{LC}}_U V$ is defined by

$$2\mathfrak{g}\big(\nabla^{\mathrm{LC}}_U V, \bullet\big) = U \cdot \mathfrak{g}(V, \bullet) + V \cdot \mathfrak{g}(\bullet, U) - \bullet \cdot \mathfrak{g}(U, V) \\ +\mathfrak{g}\big([U, V], \bullet\big) - \mathfrak{g}\big([V, \bullet], U\big) - \mathfrak{g}\big([U, \bullet], V\big).$$

A straightforward local computation shows that ∇^{LC} is a torsion-free $\mathscr{E}$-connection on $\mathscr{E}$ that preserves the metric. □

Proposition 7.5 *Let X be a complex manifold, and let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition of T_X into locally free involutive subsheaves. Suppose that $\mathscr{E}$ admits a holomorphic metric. Then $\mathscr{E}$ has a holomorphic connection.*

Proof Let $X = U + V$ and W be local sections of $T_X = \mathscr{E} \oplus \mathscr{G}$ and $\mathscr{E}$ respectively. Set

$$\nabla_X W := \nabla^{\mathrm{LC}}_U W + \nabla^{\mathrm{B}}_V W$$

where ∇^{LC} denotes the Levi-Civita connection on $\mathscr{E}$ and where ∇^{B} denotes the Bott connection on $\mathscr{E}$ induced by the foliation $\mathscr{G} \subset T_X$. This expression is $\mathscr{O}_X$-linear in X and satisfies the Leibniz rule $\nabla_X(fW) = f\nabla_X W + (X \cdot f)W$ so that ∇ is a holomorphic connection on $\mathscr{E}$. □

Remark 7.6 In the setup of Proposition 7.5, let $S \subset X$ be a projective subvariety. Then characteristic classes of $\mathscr{E}_{|S}$ vanish. This follows from [3].

8 Bost–Campana–Păun algebraicity criterion: algebraicity of leaves, II

We prove Theorem 1.6 in this section. We first provide a technical tool for the proof of our main result (see [16, Proposition 6.1] for a somewhat related result).

Proposition 8.1 *Let X be a normal complex projective variety, and let H be an ample Cartier divisor. Let $\mathscr{E} \subset T_X$ be a foliation on X. Suppose that $\mathscr{E}$ is H-stable and that $\mu_H(\mathscr{E}) = 0$. Suppose furthermore that through a general point of X, there is a positive-dimensional algebraic subvariety that is tangent to $\mathscr{E}$. Then $\mathscr{E}$ has algebraic leaves.*

Proof There exist a normal projective variety Y, unique up to birational equivalence, a dominant rational map with connected fibers $\varphi\colon X \dashrightarrow Y$, and a holomorphic foliation $\mathscr{H}$ on Y such that the following holds (see [49, Sect. 2.4]):

1. $\mathscr{H}$ is purely transcendental, i.e., there is no positive-dimensional algebraic subvariety through a general point of Y that is tangent to $\mathscr{G}$; and

2. $\mathscr{E}$ is the pullback of $\mathscr{H}$ via φ.

Let $\mathscr{G} \subseteq \mathscr{E}$ be the foliation on X induced by φ. Let $\psi\colon Z \to Y$ be the family of leaves of $\mathscr{G}$, and let $\beta\colon Z \to X$ be the natural morphism, so that $\varphi := \psi \circ \beta^{-1}\colon X \dashrightarrow Y$. By 3.10, there is an effective divisor R on X such that

$$K_{\mathscr{G}} - K_{\mathscr{E}} = -(\varphi^* K_{\mathscr{H}} + R).$$

Notice that the pull-back $\varphi^* K_{\mathscr{H}}$ is well-defined (see Definition 3.5). Applying [15, Corollary 4.8] to the foliation induced by $\mathscr{H}$ on a desingularization of Y, we see that $K_{\mathscr{H}}$ is pseudo-effective: given an ample divisor A on Y and a positive number $\varepsilon \in \mathbb{Q}$, there exists an effective $\mathbb{Q}$-divisor D_ε such that $K_{\mathscr{H}} + \varepsilon A \sim_{\mathbb{Q}} D_\varepsilon$. This implies that $\mu_H(\phi^* K_{\mathscr{H}}) \geq 0$, and hence $\mu_H(K_{\mathscr{G}}) \leqslant 0$. Since $\mathscr{E}$ is H-stable, we must have $\mathscr{G} = \mathscr{E}$. This proves the lemma. □

The proof of Theorem 1.6 relies on a criterion (see Proposition 8.4) that guarantees that a given foliation has algebraic leaves, which we establish now.

The following observation, due to Bost, will prove to be crucial.

Proposition 8.2 [13, Proposition 2.2] *Let Z be a projective variety over a field k, let x be a k-point, and let H be an ample divisor. Let $\widehat{V} \subset \widehat{Z}$ be a smooth formal subscheme of the formal completion $\widehat{Z}$ of Z at x. Then $\widehat{V}$ is algebraic if and only if there exists $c > 0$ such that, for any positive integer j and any section $s \in H^0\big(Z, \mathscr{O}_Z(jH)\big)$ such that $s_{|\widehat{V}}$ is non-zero, the multiplicity* $\mathrm{mult}_x(s_{|\widehat{V}})$ *of $s_{|\widehat{V}}$ at x is $\leqslant cj$.*

Corollary 8.3 *Let Y° be a smooth complex quasi-projective variety, let Z be a complex projective variety with $Y^\circ \subseteq Z$, and let H be an ample Cartier divisor on Z. Let $V \subset Z$ be a germ of smooth locally closed analytic submanifold along Y° in Z. Then V is algebraic if and only if there exists $c > 0$ such that, for any positive integer j and any section $s \in H^0\big(Z, \mathscr{O}_Z(jH)\big)$ such that $s_{|V}$ is non-zero, the multiplicity* $\mathrm{mult}_{Y^\circ}(s_{|V})$ *of $s_{|V}$ along Y° is $\leqslant cj$.*

Proof Let $\eta \in Z$ be the generic point of Y°. Denote by $k(\eta)$ its residue field. Set $Z_\eta := Z \otimes k(\eta)$, and $H_\eta := H \otimes k(\eta)$. Notice that $H^0\big(Z_\eta, \mathscr{O}_{Z_\eta}(jH_\eta)\big) \cong H^0\big(Z, \mathscr{O}_Z(jH)\big) \otimes k(\eta)$ for any number $j \in \mathbb{Z}$. The point η corresponds to a $k(\eta)$-point on Z_η still denoted by η. Let $\widehat{V}$ be the formal completion of V along Y°. Then $\widehat{V}$ induces a smooth formal subscheme $\widehat{V_\eta}$ of the formal completion $\widehat{Z}_\eta$ of Z_η at η. Observe that $\widehat{V}$ is algebraic if and only if $\widehat{V_\eta}$ is. The lemma now follows from Proposition 8.2 applied to (Z_η, η, H_η) and $\widehat{V_\eta}$ since $\mathrm{mult}_{Y^\circ}(s_{|V}) = \mathrm{mult}_\eta\big(s \otimes k(\eta)_{|\widehat{V_\eta}}\big)$ for any number $j \in \mathbb{Z}$ and any section $s \in H^0\big(Z, \mathscr{O}_Z(jH)\big)$.

The proof of Proposition 8.4 below follows the line of argument given in [15, 4.1] (see also [12, Corollary 3.8]).

Proposition 8.4 *Let X be a normal complex projective variety, let H be an ample Cartier divisor, and let $\mathscr{G} \subseteq T_X$ be a foliation. Suppose that there exists $c > 0$ such that $h^0\big(X, S^{[i]}\mathscr{G}^* \otimes \mathscr{O}_X(jH)\big) = 0$ for any positive integer j and any natural number i satisfying $i > cj$. Then $\mathscr{G}$ has algebraic leaves.*

Proof Let $X^\circ \subset X_{\text{reg}}$ be the open set where $\mathscr{G}_{|X_{\text{reg}}}$ is a subbundle of $T_{X_{\text{reg}}}$. Set $Z^\circ := X^\circ \times X^\circ$, $Z := X \times X$, and $A := p_1^*H + p_2^*H$ where $p_1, p_2 \colon Z = X \times X \to X$ denote the projections on X. Let $Y^\circ \subset Z^\circ$ be the open subset of the diagonal corresponding to X°. Let now $V \subset Z^\circ$ be the analytic graph of $(X^\circ, \mathscr{G}_{|X^\circ})$. Recall that $Y^\circ \subset V$ and that $\mathscr{N}_{Y^\circ/V} \cong \mathscr{G}_{|X^\circ}$. The closed subset $X \setminus X_\circ$ has codimension ≥ 2, and hence $h^0\big(X^\circ, S^i \mathscr{N}^*_{Y^\circ/V} \otimes \mathscr{O}_{X^\circ}(jH)\big) = 0$ for any positive integer j and any natural number i satisfying $i > cj$ by assumption. This implies that $\text{mult}_{Y^\circ}(s_{|V}) \leqslant cj/2$ for any positive integer j and any section $s \in H^0\big(Z, \mathscr{O}_Z(jA)\big)$ such that $s_{|V}$ is non-zero. The proposition now follows from Corollary 8.3 applied to (Z, Y°, A) and V. □

We end the preparation for the proof of Theorem 1.6 with a flatness criterion (see Remark 8.6 below). We feel that it might be of independent interest.

Proposition 8.5 *Let X be a normal complex projective variety of dimension n, and let H be an ample Cartier divisor. Suppose that X is smooth in codimension two. Let $T_X = \mathscr{E} \oplus \mathscr{G}$ be a decomposition into involutive subsheaves, where $\mathscr{E}$ is H-stable and* $\det(\mathscr{E}) \cong \mathscr{O}_X$. *Suppose furthermore that $h^0\big(X, S^2(\mathscr{E}^*) \boxtimes \mathscr{L}\big) \neq 0$ for some rank one reflexive sheaf $\mathscr{L}$ with $c_1(\mathscr{L}) \cdot H^{n-1} = 0$. Then $c_1(\mathscr{E})^2 \cdot H^{n-2} = c_2(\mathscr{E}) \cdot H^{n-2} = 0$.*

Proof Set $r := \operatorname{rank} \mathscr{E}$. Consider a non-zero section $\mathfrak{g} \in H^0\big(X, S^2(\mathscr{E}^*) \boxtimes \mathscr{L}\big)$. Since $\mathscr{E}$ is H-stable, $\mathfrak{g}$ induces an isomorphism $\mathscr{E} \cong \mathscr{E}^* \boxtimes \mathscr{L}$. Taking determinants and double duals, we obtain $\mathscr{L}^{[\otimes r]} \cong \mathscr{O}_X$. Let $f \colon \widetilde{X} \to X$ be the corresponding cyclic cover (see for instance [40, Lemma 2.53]). Then $f^{[*]}\mathscr{L} \cong \mathscr{O}_{\widetilde{X}}$, and $\mathfrak{g}$ induces a holomorphic metric on $f^{[*]}\mathscr{E}_{|\widetilde{X}_{\text{reg}}}$. Applying Proposition 7.5, we see that $f^{[*]}\mathscr{E}_{|\widetilde{X}_{\text{reg}}}$ admits a holomorphic connection. Notice that $\widetilde{X} \setminus \widetilde{X}_{\text{reg}}$ has codimension at least three. It follows from Remark 7.6 that

$$c_1(\mathscr{E})^2 \cdot H^{n-2} = \frac{1}{\deg(f)} c_1(f^{[*]}\mathscr{E})^2 \cdot (f^*H)^{n-2} = 0$$

and that

$$c_2(\mathscr{E}) \cdot H^{n-2} = \frac{1}{\deg(f)} c_2(f^{[*]}\mathscr{E}) \cdot (f^*H)^{n-2} = 0,$$

proving the proposition. □

Remark 8.6 In the setup of Proposition 8.5, suppose furthermore that X is a $\mathbb{Q}$-factorial variety with only canonical singularities. Then it follows from [27, Theorem 6.5] that there exists a finite surjective morphism $f\colon \widetilde{X} \to X$, étale in codimension one, such that $f^{[*]}\mathscr{E}$ is a locally free, flat sheaf on $\widetilde{X}$.

Proof of Theorem 1.6 Maintaining notation and assumptions of Theorem 1.6, set $r := \operatorname{rank} \mathscr{E}$.

Suppose that there exists a finite cover $f\colon \widetilde{X} \to X$ that is étale in codimension one such that the reflexive pull-back $f^{[*]}\mathscr{E}$ is not stable with respect to f^*H. Applying [34, Lemma 3.2.3], we see that the $f^{[*]}\mathscr{E}$ is polystable, and hence, there exist non-zero reflexive sheaves $(\mathscr{E}_i)_{i\in I}$, f^*H-stable with slopes $\mu_{f^*H}(\mathscr{E}_i) = \mu_{f^*H}(f^{[*]}\mathscr{E}) = 0$ such that

$$f^{[*]}\mathscr{E} \cong \bigoplus_{i\in I} \mathscr{E}_i.$$

Suppose that the number of direct summands is maximal. Then, for any finite cover $g\colon \widehat{X} \to \widetilde{X}$ that is étale in codimension one, the reflexive pull-back $g^{[*]}\mathscr{E}_i$ is obviously stable with respect to $(f \circ g)^*H$. Notice that $\widetilde{X}$ is still smooth in codimension two, since f branches only over the singular set of X. It follows from Proposition 8.1 that if $\mathscr{E}_i$ has algebraic leaves for some $i \in I$, then so does $\mathscr{E}$.

Suppose first that there exists $i_0 \in I$ such that $c_1(\mathscr{E}_{i_0})^2 \cdot (f^*H)^{n-2} \neq 0$ or $c_2(\mathscr{E}_{i_0}) \cdot (f^*H)^{n-2} \neq 0$. Applying Theorem 6.1 to $\mathscr{E}_{i_0}^*$, we see that one of the following holds.

1. Either there exists $c > 0$ such that $h^0\big(\widetilde{X}, S^{[i]}\mathscr{E}_{i_0}^* \otimes \mathscr{O}_{\widetilde{X}}(jf^*H)\big) = 0$ for any positive integer j and any natural number i satisfying $i > cj$,
2. or $r = 3$, and there exists a finite morphism $g\colon \widehat{X} \to \widetilde{X}$ that is étale in codimension one, and a rank 1 reflexive sheaf $\mathscr{L}$ on $\widehat{X}$ with $\mu_{(f\circ g)^*H}(\mathscr{L}) = 0$ such that $h^0\big(\widehat{X}, (S^2(f \circ g)^*\mathscr{E}_{i_0}^*) \boxtimes \mathscr{L}\big) \neq 0$.

If we are in case (1), apply Proposition 8.4 to conclude that $\mathscr{E}_{i_0}$ has algebraic leaves.

Suppose that we are in case (2). Then we may assume that $\widetilde{X} = X$, and that $\mathscr{E}_{i_0} = \mathscr{E}$. Taking pull-backs and double duals, we obtain a decomposition

$$T_{\widehat{X}} \cong g^{[*]}\mathscr{E} \oplus g^{[*]}\mathscr{G}$$

into involutive subsheaves, where $g^{[*]}\mathscr{E}$ is g^*H-stable and $\det(g^{[*]}\mathscr{E}) \cong \mathscr{O}_{\widehat{X}}$. Applying Proposition 8.5 to $g^{[*]}\mathscr{E}$, we see that

$$c_1(\mathscr{E})^2 \cdot H^{n-2} = \frac{1}{\deg(g)} c_1(g^{[*]}\mathscr{E})^2 \cdot (g^*H)^{n-2} = 0$$

and

$$c_2(\mathscr{E}) \cdot H^{n-2} = \frac{1}{\deg(g)} c_2(g^{[*]}\mathscr{E}) \cdot (g^*H)^{n-2} = 0,$$

yielding a contradiction.

Finally, suppose that $c_1(\mathscr{E}_{i_0})^2 \cdot (f^*H)^{n-2} = c_2(\mathscr{E}_i) \cdot (f^*H)^{n-2} = 0$ for each $i \in I$. Let S be a smooth two dimensional complete intersection of general elements in $|mH|$ for a sufficiently large positive integer m. Observe that S is contained in the smooth locus X_{reg} of X, and that $\mathscr{E}$ is locally free along S (see [33, Corollary 1.4]). By the restriction theorem of Mehta and Ramanathan, the locally free sheaf $\mathscr{E}_{i\,|S}$ is stable with respect to $H_{|S}$ with $\mu_{H_{|S}}(\mathscr{E}_{i\,|S}) = 0$, and $c_1(\mathscr{E}_{i\,|S})^2 = c_2(\mathscr{E}_{i\,|S}) = 0$. This implies that $\mathscr{E}_{i\,|S}$ is flat (see Remark 6.4). A straightforward computation then shows that $c_2(\mathscr{E}) \cdot H^{n-2} = c_2(\mathscr{E}_{|S}) = 0$, yielding a contradiction. This completes the proof of the theorem. □

9 Proof of Theorem 1.2

We are now in position to prove our main result.

Proposition 9.1 *Let X be a normal complex projective variety of dimension at most 5, with klt singularities. Assume that $K_X \equiv 0$. Then there exists an abelian variety A as well as a projective variety $\widetilde{X}$ with canonical singularities, a finite cover $f \colon A \times \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$\widetilde{X} \cong \prod_{j \in J} Y_j$$

such that the induced decomposition of $T_{\widetilde{X}}$ agree with the decomposition given by Theorem 1.1.

Proof Notice first that $K_X \sim_{\mathbb{Q}} 0$ by [51, Corollary V 4.9]. Thus, there exists a finite cover $f_1 \colon X_1 \to X$, étale in codimension one, such that $K_{X_1} \sim_{\mathbb{Z}} 0$ (see [40, Lemma 2.53]). By [43, Proposition 3.16], X_1 has klt singularities. It follows that X_1 has canonical singularities since K_{X_1} is a Cartier divisor. Applying Theorem 1.1, we conclude that there exists an abelian variety A as well as a projective variety X_2 with canonical singularities, a finite cover $f_2 \colon A \times X_2 \to X_1$, étale in codimension one, and a decomposition

$$T_{X_2} = \bigoplus_{i \in I} \mathscr{E}_i$$

such that the following holds.

1. The $\mathscr{E}_i$ are integrable subsheaves of T_{X_2}, with $\det(\mathscr{E}_i) \cong \mathscr{O}_{X_2}$.
2. The sheaves $\mathscr{E}_i$ are strongly stable.
3. The augemented irregularity of X_2 is zero.

To prove the theorem, we will show that the foliations $\mathscr{E}_i$ are algebraically integrable. We may obviously assume that T_{X_2} is not strongly stable. Let $\beta\colon \widehat{X}_2 \to X_2$ be a $\mathbb{Q}$-factorial terminalization of X_2. By Lemma 4.9, there is a decomposition

$$T_{\widehat{X}_2} = \bigoplus_{i\in I} \widehat{\mathscr{E}_i}$$

of $T_{\widehat{X}_2}$ into involutive subsheaves with $\det(\widehat{\mathscr{E}_i}) \cong \mathscr{O}_{\widehat{X}_2}$ such that $\mathscr{E}_i \cong (\beta_*\widehat{\mathscr{E}_i})^{**}$. Notice that $\widetilde{q}(\widehat{X}_2) = 0$ by Lemma 4.4. Let

$$T_{\widehat{X}_2} = \bigoplus_{i\in J} \widehat{\mathscr{G}_j}$$

be a decomposition of $T_{\widehat{X}_2}$ into strongly stable involutive subsheaves with $\det(\mathscr{G}_i) \cong \mathscr{O}_{\widehat{X}_2}$ whose existence is guaranteed by Theorem 1.1. For any $j \in J$, $\widehat{\mathscr{G}_j}$ is a direct summand of $\mathscr{E}_{i_j}$ for some $i_j \in I$. To prove the claim, it suffices to prove that the foliations $\widehat{\mathscr{G}_i}$ are algebraically integrable. By Corollary 5.11, we must have $c_2(\widehat{\mathscr{G}_j}) \not\equiv 0$ for each $j \in J$. This implies in particular that $\widehat{\mathscr{G}_j}$ has rank at least 2, and therefore, rank $\widehat{\mathscr{G}_j} \in \{2, 3\}$ since $\dim X \leqslant 5$ and T_{X_2} is not strongly stable by assumption. Now, by Theorem 1.6, we conclude that the sheaves $\mathscr{E}_i$ are algebraically integrable, proving our claim. The proposition then follows from Proposition 4.10. □

Proof of Theorem 1.2 Theorem 1.2 is an immediate consequence of Proposition 9.1 and of the characterization [25, Proposition 8.21] of canonical varieties with trivial canonical class and strongly stable tangent bundle as singular analogues of Calabi–Yau or irreducible holomorphic symplectic manifolds. □

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