A-ANALYTICITY OF SEPARATRICIES OF FOLIATIONS

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ABSTRACT. Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X. We prove that if L is closed under p-th powers for almost all primes p, then any L-invariant smooth formal curve is A-analytic. Building on prior work of Bost we obtain an algebraicity criterion for those curves.

CONTENTS

1. INTRODUCTION

Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X (defined over K).

If L is closed under p-th powers for almost all primes p, then the generalization to foliations by Ekedahl, Shepherd-Barron and Taylor (see [ESBT99, Conjecture F]) of the classical Grothendieck-Katz conjecture predicts that L has algebraic leaves. The conjecture has been shown in some special cases, e.g. if X is a abelian surface and L is induced by a non-zero vector field ([Bos01, Theorem 2.3]), if X is the total space of a line bundle over an algebraic curve and L is induced by a linear connection ([And04, Corollaire 4.3.6]), and if X is a $\mathbb{P}^1$-bundle over an elliptic curve E and L is an Ehresmann connection on $X \to E$ ([Dru21, Proposition 9.3]).

Actually, [Bos01, Theorem 2.3] follows from an algebraicity criterion for smooth formal schemes in algebraic varieties over number fields ([Bos01, Theorem 3.4]), which we recall now. Let $P \in X(K)$, and let $\widehat{V}$ be a smooth formal subscheme of the completion $\widehat{X}_P$ of X at P. If $\widehat{V}$ is A-analytic (we refer to Section 2 for this notion) and $\widehat{V}_{\mathbb{C}}$ satisfies the Liouville condition for some embedding $K \subset \mathbb{C}$, then the formal scheme $\widehat{V}$ is algebraic.

If L is regular at $P \in X(K)$, then the formal leaf $\widehat{V}$ of L through P is smooth but $\widehat{V}$ is not A-analytic in general. Nevertheless, if L is closed under p-th powers for almost all primes p, then $\widehat{V}$ is A-analytic by [Bos01, Proposition 3.9].

Suppose now that $P \in X(K)$ be a singular point of L, and let $K \subset \mathbb{C}$ be an embedding. In [CS82], Camacho and Sad proved that there is at least one separatrix of $L_{\mathbb{C}}$ through $P_{\mathbb{C}} \in X_{\mathbb{C}}$, *i.e.* a (local) irreducible complex curve in $X_{\mathbb{C}}$ passing through $P_{\mathbb{C}}$ and tangent to $L_{\mathbb{C}}$. Suppose in addition that $P_{\mathbb{C}}$ is a non-degenerate reduced singularity of $L_{\mathbb{C}}$. Then, by [MM80, Appendice II], $L_{\mathbb{C}}$ has exactly two separatrices through $P_{\mathbb{C}}$. They are smooth and intersect transversely at $P_{\mathbb{C}}$. In this paper, we extend [Bos01, Proposition 3.9] to this setting.

Theorem 1.1. *Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X. Suppose that L is closed under p-th powers for almost all primes p. Let $P \in X(K)$ be a singular point of L. Suppose furthermore that $P_{\mathbb{C}}$ is a non-degenerate reduced singularity of $L_{\mathbb{C}}$ for some embedding $K \subset \mathbb{C}$. Then the following holds.*

(1) *There exist smooth formal subschemes $\widehat{V}$ and $\widehat{W}$ over K of the completion $\widehat{X}_P$ of X at P such that $\widehat{V}_{\mathbb{C}}$ and $\widehat{W}_{\mathbb{C}}$ are the formal completion at P of the separatrices of $L_{\mathbb{C}}$ through $P_{\mathbb{C}}$ for any embedding $K \subset \mathbb{C}$.*
(2) *The formal curves $\widehat{V}$ and $\widehat{W}$ are A-analytic.*

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In fact, a slightly more general statement is true (see Theorem 4.1 and Corollary 4.2).

The following is an immediate consequence of [Bos01, Theorem 3.4] and Theorem 1.1 together.

Corollary 1.2. *Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X. Suppose that L is closed under p-th powers for almost all primes p. Let $P \in X(K)$ be a singular point of L. Suppose that $P_{\mathbb{C}}$ is a non-degenerate reduced singularity of $L_{\mathbb{C}}$ for some embedding $K \subset \mathbb{C}$. Let $\widehat{X}_P$ be the completion of X at P, and let $\widehat{V} \subset \widehat{X}_P$ be a formal separatrix through P. Suppose in addition that there exists an embedding $K \subset \mathbb{C}$ such that $\widehat{V}_{\mathbb{C}}$ satisfies the Liouville property. Then $\widehat{V}$ is algebraic.*

1.1. Structure of the paper. Section 2 gathers notation, known results and global conventions that will be used throughout the paper. In section 3 we provide technical tools for the proof of the main results. More precisely, we study formal power series solutions of certain p-adic nonlinear differential equation using a Newton iteration procedure. Section 4 is devoted to the proof of Theorem 1.1.

2. Notation, conventions and used facts

2.1. Global convention. Throughout the paper a *variety* is a reduced and irreducible scheme separated and of finite type over a field.

2.2. Foliations. Let K be a number field, and let R be its ring of integers. Let X be a smooth quasi-projective variety over K. A *foliation* on X is a line bundle $L \subseteq T_X$ such that the quotient T_X/L is torsion free.

Let $U \subseteq X$ be the open subset where $L|_U$ is a subbundle of T_U. We say that L is *singular* at $P \in X$ if $P \in X \setminus U$.

Let $Y \subseteq X$ be a closed subvariety, and let D be a derivation on X. Say that Y is *invariant under* D if $D(\mathscr{I}_Y) \subseteq \mathscr{I}_Y$.

Say that Y is *invariant under* L if for any local section D of L over some open subset U of X, $D(\mathscr{I}_{Y\cap U}) \subseteq \mathscr{I}_{Y\cap U}$. To prove that Y is invariant under L it is enough to show that $Y \cap U$ of Y is invariant under $L|_U$ for some open set $U \subseteq X$ such that $Y \cap U$ is dense in Y. If X and Y are smooth and $L \subseteq T_X$ is a subbundle, then Y is invariant under L if and only if $L|_Y \subseteq T_Y \subseteq T_X|_Y$.

Let $N \geqslant 1$ be a sufficiently divisible integer, and let $\mathscr{X}$ (resp. $\mathscr{L}$) be a smooth model of X over $R[1/N]$ (resp. a line bundle on $\mathscr{X}$ contained in $T_{\mathscr{X}/S}$ such that $\mathscr{L} \otimes_R K$ coincides with L and the quotient $T_{\mathscr{X}/S}/\mathscr{L}$ is torsion free), where $S := \operatorname{Spec} R[1/N]$. Let $\mathfrak{p}$ be a maximal ideal of R with $\mathfrak{p} \nmid N$, and let $k_{\mathfrak{p}} := R/\mathfrak{p}$ denote the residue field at $\mathfrak{p}$. Let p denote the characteristic of $k_{\mathfrak{p}}$. The sheaf of derivations $\operatorname{Der}_{k_{\mathfrak{p}}}(\mathscr{O}_{\mathscr{X}_{\mathfrak{p}}}) \cong T_{\mathscr{X}_{\mathfrak{p}}}$ is endowed with the p-th power operation, which maps any local $k_{\mathfrak{p}}$-derivation of $\mathscr{O}_{\mathscr{X}_{\mathfrak{p}}}$ to its p-th iterate.

We say that L *is closed under p-th powers for almost all primes p* if there exists $N \mid N'$ such that, for any $\mathfrak{p} \nmid N'$, $\mathscr{L}|_{\mathscr{X}_{\mathfrak{p}}} \subseteq T_{\mathscr{X}_{\mathfrak{p}}}$ is closed under p-th powers. This condition is independent of the choices of $\mathscr{X}$ and $\mathscr{L}$.

2.3. A-analyticity of formal smooth schemes. We briefly recall a number of definitions and facts concerning A-analytic formal curves from [Bos01] (see also [BCL09]). We refer to *loc. cit.* for further explanations concerning these notions.

Let K be field equipped with some complete ultrametric absolute value $|\cdot|$ and assume that its valuation ring R is a discrete valuation ring.

Let N be a positive integer, and let r be a positive real number r. For any formal power series $f = \sum_{I\in\mathbb{N}^N} a_I X^I \in \mathbb{C}_p[[X_1,\ldots,X_N]]$, we define $||f||_r$ by the formula

$$||f||_r = \sup_I |a_I| r^{|I|} \in \mathbb{R}_{\geqslant 0} \cup +\infty.$$

The power series f such that $||f||_r < +\infty$ are those that are convergent and bounded on the open N-ball of radius r in K^N.

Fact 2.1. Let $r > 0$ be a real number, and let $f \in \mathbb{C}_p[[X_1,\ldots,X_N]]$ and $g \in \mathbb{C}_p[[X_1,\ldots,X_N]]$ be formal power series with $||f||_r < +\infty$ and $||g||_r < +\infty$. Then $||fg||_r < +\infty$, and $||fg||_r \leqslant ||f||_r ||g||_r$.

For any positive real number r, we denote by $G_{\mathrm{an},r} < \operatorname{Aut}\big(\widehat{\mathbb{A}^N_{K0}}\big)$ the subgroup consisting of all N-tuples $f = (f_1,\ldots,f_N)$ such that $f(0) = 0$, $Df(0) \in \mathrm{GL}_2(R)$, and $||f_i||_r \leqslant r$ for each i. This subgroup may be identified with the group of all analytic automorphisms, preserving the origin, of the open N-dimensional ball of radius r. Set also $G_{\mathrm{an}} := \cup_{r>0} G_{\mathrm{an},r}$.

Notice that a smooth formal subscheme $\widehat{V}$ of dimension d of $\widehat{\mathbb{A}^N_{K}}_0$ is K-analytic if and only if there exists $f \in G_{\mathrm{an}}$ such that $f^{-1}\widehat{V}$ is the formal subscheme $\widehat{\mathbb{A}^d_{K}}_0 \times \{0\}$ of $\widehat{\mathbb{A}^N_{K}}_0$.

Let $\mathscr{X}$ be a quasi-projective R-scheme, and $X =: \mathscr{X} \otimes_R K$ its generic fiber. Let $\mathscr{P} \in \mathscr{X}(R)$, and let $P := \mathscr{P} \otimes_R K$. Let $\widehat{V}$ be a smooth formal subscheme of the completion $\widehat{X}_P$ of X at P. There is a unique way to attach a number $S_{\mathscr{X}}(\widehat{V}) \in [0,1]$ such that the following holds (see [Bos01]).

(1) We have $S_{\mathscr{X}}(\widehat{V}) > 0$ if and only if $\widehat{V}$ is K-analytic.
(2) If $(\mathscr{X}, \mathscr{P}) = (\mathbb{A}^N_R, 0)$ and $\widehat{V}$ is K-analytic, then $S_{\mathscr{X}}(\widehat{V})$ is supremum of the set of real numbers $0 < r \leqslant 1$ for which there exists $f \in G_{\mathrm{an}}$ such that $f^{-1}\widehat{V}$ is the formal subscheme $\widehat{\mathbb{A}^d_{K}}_0 \times \{0\}$ of $\widehat{\mathbb{A}^N_{K}}_0$.
(3) If $\mathscr{X} \to \mathscr{X}'$ is an immersion, then $S_{\mathscr{X}}(\widehat{V}) = S_{\mathscr{X}'}(\widehat{V})$.
(4) For any two triples $(\mathscr{X}, \mathscr{P}, \widehat{V})$ and $(\mathscr{X}', \mathscr{P}', \widehat{V}')$ as above, if there exists an R-morphism $f \colon \mathscr{X} \to \mathscr{X}'$ mapping $\mathscr{P}$ to $\mathscr{P}'$, étale along $\mathscr{P}$, and inducing an isomorphism $\widehat{V} \cong \widehat{V}'$, then $S_{\mathscr{X}}(\widehat{V}) = S_{\mathscr{X}'}(\widehat{V}')$.

We will refer to $S_{\mathscr{X}}(\widehat{V})$ as the *size of $\widehat{V}$ with respect to the model $\mathscr{X}$ of X.*

We will need the following easy observations.

Lemma 2.2 ([BCL09])**.** *Let $\varphi = \sum_{m \geqslant 1} c_m X^m \in K[[X]]$ be formal power series such that $c_1 \in R$, and let $\widehat{V}$ be its graph in $\widehat{\mathbb{A}^2_{K}}_0$. Set $\lambda := \inf_{m \geqslant 1} -\frac{\log|c_{m+1}|}{m} \in \mathbb{R} \cup \{-\infty\}$, and let ρ be the radius of convergence of φ. Then the following holds.*

(1) *The size $S_{\mathbb{A}^2_R}(\widehat{V})$ of $\widehat{V}$ with respect to $(\mathbb{A}^2_R, 0)$ satisfies $S_{\mathbb{A}^2_R}(\widehat{V}) \leqslant \rho$.*
(2) *If $\rho > 0$, then $S_{\mathbb{A}^2_R}(\widehat{V}) \geqslant \min(1, \exp\lambda)$. In particular, $\widehat{V}$ is K-analytic. Moreover, if $\varphi \in R[[X]]$, then $S_{\mathbb{A}^2_R}(\widehat{V}) = 1$.*
(3) *If $\rho > 0$ and $\varphi'(0)$ is a unit in R, then $S_{\mathbb{A}^2_R}(\widehat{V}) = \min(1, \exp\lambda)$.*

Proof. Items (1) and (3) are shown in [BCL09, Proposition 3.5]. To prove Item (2), let

$$f(X_1, X_2) := (X_1 + X_2, \varphi(X_1)) \in \mathrm{Aut}\big(\widehat{\mathbb{A}^2_{K}}_0\big).$$

Then $f^{-1}\widehat{V}$ is the formal subscheme $\widehat{\mathbb{A}^1_{K}}_0 \times \{0\}$ of $\widehat{\mathbb{A}^2_{K}}_0$. If $\rho > 0$, then $\lambda \in \mathbb{R}$. Set $r := \min(1, \exp\lambda) \in [0,1]$. Then $||f||_r \leqslant r$, and hence $f \in G_{\mathrm{an}}$. This finishes the proof the lemma. ☐

Lemma 2.3. *Let $\varphi = \sum_{m \geqslant 2} c_m X^m \in K[[X]]$ be formal power series such that $c_2 \in R$, and set $\varphi := \varphi(X)/X = \sum_{m \geqslant 1} c_{m+1} X^m$. Let $\widehat{V}$ (resp. $\widehat{W}$) be the graph of φ (resp. ψ) in $\widehat{\mathbb{A}^2_{K}}_0$. Then $S_{\mathbb{A}^2_R}(\widehat{V}) \geqslant S_{\mathbb{A}^2_R}(\widehat{W})$.*

Proof. Let be a positive real number such that $r < S_{\mathbb{A}^2_R}(\widehat{W})$. By assumption, there exist formal power series f_1 and f_2 in $K[[X_1, X_2]]$ such that $f = (f_1, f_2) \in G_{\mathrm{an},r}$ and $f^{-1}\widehat{W} = \widehat{A^1_K} \times \{0\}$. This last condition is actually equivalent to the identity

$$f_2(T, 0) = \psi(f_1(T, 0))$$

in $K[[T]]$. Let us write $f_1(T,0) = \sum_{m \geqslant 1} a_m T^m$, and $f_2(T,0) = \sum_{m \geqslant 1} b_m T^m$. We have $b_1 = c_2 a_1$. Since $f \in G_{\mathrm{an},r}$, we must have $Df(0) \in \mathrm{GL}_2(R)$. This immediately implies that a_1 is a unit in R.

Let us set

$$g_1(X_1, X_2) = f_1(X_1, X_2), \quad g_2(X_1, X_2) = Y + f_1(X_1, X_2) f_2(X_1, X_2), \quad \text{and} \quad g = (g_1, g_2).$$

Then $||g_1||_r = ||f_1||_r \leqslant r$ and $||g_2||_r \leqslant r$ since $||f_1 f_2||_r \leqslant ||f_1||_r ||f_2||_r \leqslant r^2 \leqslant r$. Moreover,

$$Dg(0) = \begin{pmatrix} a_1 & \partial_{X_2} f(0) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(R),$$

and hence $g \in G_{\mathrm{an},r}$. Finally, notice that $g^{-1}\widehat{V} = \widehat{A^1_K} \times \{0\}$ since $f_2(T,0) f_1(T,0) = \varphi(f_1(T,0))$ in $K[[T]]$. This shows that $S_{\mathbb{A}^2_R}(\widehat{V}) \geqslant S_{\mathbb{A}^2_R}(\widehat{W})$, finishing the proof of the lemma. ☐

Remark 2.4. In the setup of Lemma 2.3, note that the (formal) curve $T \mapsto (T, \psi(T))$ is the proper transform of the (formal) curve $T \mapsto (T, \varphi(T))$ in the blow-up of $\mathbb{A}^1_K$ at 0.

Let K be a number field, and let R be its ring of integers. Let $\mathfrak{p}$ be a maximal ideal of R, let $|\cdot|_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic absolute value, normalized by the condition $|\varpi_{\mathfrak{p}}|_{\mathfrak{p}} = \frac{1}{\sharp(R/\mathfrak{p})}$ for any uniformizing element $\varpi_{\mathfrak{p}}$ at $\mathfrak{p}$. Let $K_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completions of K and R, and $k_{\mathfrak{p}} := R_{\mathfrak{p}}/(\varpi_{\mathfrak{p}})$ the residue field at $\mathfrak{p}$. Let p denote the characteristic of $k_{\mathfrak{p}}$. Let X be a quasi-projective algebraic variety over K, and let $P \in X(K)$. Let $\widehat{V}$ be a smooth formal subscheme (defined over K) of the formal completion $\widehat{X}_P$ of X at P. Let $N \geqslant 1$ be a sufficiently divisible integer, and let $\mathscr{X}$ be a quasi-projective model of X over $R[1/N]$, such that P extends to a point $\mathscr{P} \in \mathscr{X}(R[1/N])$. The smooth formal scheme $\widehat{V}$ is said to be *A-analytic* (see [BCL09, Definition 3.7]) if

1. for any place v of K, the formal scheme $\widehat{V}_{K_v}$ is K_v-analytic, where K_v denotes the completion of K with respect to v, and
2. we have
$$\sum_{\mathfrak{p}\nmid N} \log \frac{1}{S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{V}_{K_{\mathfrak{p}}})} < +\infty.$$

This condition is independent of the choices of $\mathscr{X}$, $\mathscr{P}$, and N.

3. Power series solutions of certain p-adic differential equations

Let p be a prime number. Let $\mathbb{Q}_p$ (resp. $\mathbb{C}_p$) be the field of p-adic rational numbers (resp. complex numbers). We denote by $|\cdot|$ the ultrametric absolute value on $\mathbb{C}_p$ normalized by $|p| = p^{-1}$. We denote by v the p-adic valuation on $\mathbb{C}_p$ normalized by $v(p) = 1$.

In this section, we study formal power series solutions of certain p-adic nonlinear differential equations at a regular singular point. The following is the main result of this section.

Proposition 3.1. *Let p be an odd prime integer, let $1 \leqslant s \leqslant p-1$ and $1 \leqslant t \leqslant p-1$ be relatively prime integers, and set $\alpha := \frac{s}{t}$. Let a, b, and c_m for any integer $m \geqslant 2$ be power series in $\mathbb{C}_p[[X]]$ such that $||a||_r \leqslant \frac{r}{p}$, $||b||_r \leqslant \frac{r}{p}$, and $||c_m||_r \leqslant \frac{1}{p}$ for some real number $r \in]0,1]$. Suppose in addition that $a(0) = a'(0) = 0$ and that $b(0) = 0$. Then the following holds.*

1. *There exists a unique formal power series y in $\mathbb{C}_p[[X]]$ with $y(0) = y'(0) = 0$ solution of the differential equation*

$$xy' + \alpha y = a + by + \sum_{m\geqslant 2} c_m y^m. \tag{3.1}$$

2. *There exists a constant $C > 0$ such that, letting $R := r\exp\Big(-Ct\frac{(\log p)^2}{p^2}\Big)$, we have $||y||_R \leqslant R$.*

We will need the following well-known facts.

Fact 3.2. Let $f = \sum_{m\geqslant 0} a_m X^m \in \mathbb{C}_p[[X]]$ and $g = \sum_{m\geqslant 0} b_m X^m \in \mathbb{C}_p[[X]]$. Suppose that there exists a positive real number r such that both $|a_m|r^m$ and $|b_m|r^m$ goes to 0 as m goes to $+\infty$. Then $||fg||_r = ||f||_r||g||_r$ (see [Rob00, Proposition 2 of Section 6.1.4]).

Fact 3.3. Let $f \in \mathbb{C}_p[[X]]$. If $f(0) = 0$ and $||f||_r \leqslant Cr$ for some positive real numbers r and C, then $||f||_{r_1} \leqslant Cr_1$ for $0 < r_1 \leqslant r$.

Fact 3.4. Let $f = \sum_{m\geqslant 0} a_m X^m \in \mathbb{C}_p[[X]]$. Suppose that there exists a positive real number r such that $|a_m|r^m$ goes to 0 as m goes to $+\infty$. Then $||f||_r = ||f_r||_1 = \sup_{|x|\leqslant 1}|f_r(x)| = \sup_{|x|\leqslant r}|f(x)|$, where $f_r(X) := f(rX) = \sum_{m\geqslant 0} a_m r^m X^m$ (see [Rob00, Proposition 1 of Section 6.1.4]).

Before we give the proof of Proposition 3.1, we need the following auxiliary statements.

Lemma 3.5. *Let k be a positive integer, and let $b = \sum_{m\geqslant 2^k} b_m X^m \in \mathbb{C}_p[[X]]$ be a power series such that $||b||_r \leqslant \frac{r}{p}$ for some real number $r \in]0,1]$. Set $B = \sum_{m\geqslant 2^k} \frac{b_m}{m} X^m$.*

1. *Let $r_1 := r\exp\big(-\frac{2}{p(p-1)}\log p\big)$. Then $||B||_{r_1} \leqslant p^{-\frac{2}{p-1}}$.*
2. *Let $r_2 := r\exp\big(-\frac{k+1}{2^k}\log 2\big)$. Then $||B||_{r_2} \leqslant p^{-\frac{2}{p-1}}$.*

Proof. Let R be a positive real number. Then $||B||_R \leqslant p^{-\frac{2}{p-1}}$ if and only if

$$\log R \leqslant \inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log|b_m| + \log|m| - \frac{2}{p-1}\log p\Big)\right\}.$$

By assumption, we have

$$\log|b_m| + (m-1)\log r \leqslant -\log p$$

for any integer $m \geqslant 2^k$, and hence

$$\begin{aligned}\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log|b_m| + \log|m| - \frac{2}{p-1}\log p\Big)\right\} &\geqslant \inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big((m-1)\log r + \log|m| + \frac{p-3}{p-1}\log p\Big)\right\} \\ &\geqslant \log r + \inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\}\end{aligned}$$

using the fact that $r \leqslant 1$.

Notice that $\log|m| + \frac{p-3}{p-1}\log p \leqslant -\log p + \frac{p-3}{p-1}\log p = -\frac{2}{p-1}\log p < 0$ if $|m| < 1$, and hence

$$\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\} < 0.$$

Then

$$\begin{aligned}\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\} &\geqslant \inf_{m\geqslant 1} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\} \\ &= \inf_{m\geqslant 1 \text{ such that } |m|<1} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\} \\ &= \inf_{t\geqslant 1 \text{ and } u\geqslant 1 \text{ such that } |u|=1} \left\{ \frac{1}{p^t u}\Big(-t + \frac{p-3}{p-1}\Big)\log p\right\} \\ &\geqslant \inf_{t\geqslant 1} \left\{ \frac{1}{p^t}\Big(-t + \frac{p-3}{p-1}\Big)\log p\right\}.\end{aligned}$$

Now observe that $x \mapsto \frac{1}{p^x}\Big(-x + \frac{p-3}{p-1}\Big)$ is an increasing function on $x > \frac{p-3}{p-1} + \frac{1}{\log p}$. On the other hand, we have $\frac{p-3}{p-1} + \frac{1}{\log p} < 2$. Thus

$$\inf_{t\geqslant 1} \left\{ \frac{1}{p^t}\Big(-t + \frac{p-3}{p-1}\Big)\log p\right\} = \min_{t\in\{1,2\}} \left\{ \frac{1}{p^t}\Big(-t + \frac{p-3}{p-1}\Big)\log p\right\} = \frac{-2}{p(p-1)}\log p,$$

and hence

$$\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log|b_m| + \log|m| - \frac{2}{p-1}\log p\Big)\right\} \geqslant \log r - \frac{2}{p(p-1)}\log p.$$

This proves (1).

We proceed to show (2). We have

$$\begin{aligned}\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(\log|m| + \frac{p-3}{p-1}\log p\Big)\right\} &\geqslant \inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log m + \frac{p-3}{p-1}\log p\Big)\right\} \\ &\geqslant \inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log m + \frac{p-3}{p-1}\log p\Big)\right\} \\ &\geqslant \inf_{m\geqslant 2^k} \left\{ -\frac{\log m}{m}\right\} - \frac{1}{2^k}\log 2.\end{aligned}$$

Notice that the function $x \mapsto -\frac{\log x}{x}$ is increasing on $\log x > 1$. Therefore, if $k \geqslant 2$, then

$$\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log m\Big)\right\} = -\frac{k}{2^k}\log 2.$$

If $k = 1$, then

$$\inf_{m\geqslant 2^k} \left\{ \frac{1}{m}\Big(-\log m\Big)\right\} = \min\left\{ -\frac{1}{2}\log 2, -\frac{2}{2^2}\log 2\right\} = -\frac{1}{2}\log 2 \geqslant -\log 2.$$

This proves (2), completing the proof of the lemma. □

Remark 3.6. In the setup of Lemma 3.5, notice that $\exp(B(x))$ is the unique formal power series solution of the differential equation $xy' = by$ such that $y(0) = 1$. We will denote $B(x)$ by $\int_0^x \frac{b(u)}{u} du$.

Lemma 3.7. *Let k be a positive integer, and let $a = \sum_{m \geqslant 2^k} a_m X^m \in \mathbb{C}_p[[X]]$ be a power series such that $||a||_r \leqslant \frac{r}{p}$ for some positive real number r. Let $1 \leqslant s \leqslant p-1$ and $1 \leqslant t \leqslant p-1$ be relatively prime integers, and set $\alpha := \frac{s}{t}$ and $A = \sum_{m \geqslant 2^k} \frac{a_m}{m+\alpha} X^m$.*

(1) *Let $r_1 := r \exp\big(-\frac{t}{(p-1)^2} \log p \big)$. Then $||A||_{r_1} \leqslant r_1$.*
(2) *Suppose that $2^k \geqslant \alpha + 2$, and let $r_2 := r \exp\big(-\frac{kt}{2^{k-1}} \log 2 \big)$. Then $||A||_{r_2} \leqslant r_2$.*

Proof. Let R be a positive real number. Then $||A||_R \leqslant R$ if and only if

$$\log R \leqslant \inf_{m \geqslant 2^k} \left\{ -\frac{1}{m-1} \log \frac{|a_m|}{|m+\alpha|} \right\}.$$

By assumption, we have $\log |a_m| + (m-1) \log r \leqslant -\log p$ for any $m \geqslant 2^k$. Notice also that $|s| = |t| = 1$. Thus

$$\begin{aligned} \inf_{m \geqslant 2^k} \left\{ -\frac{1}{m-1} \log \frac{|a_m|}{|m+\alpha|} \right\} &\geqslant \log r + \inf_{m \geqslant 2^k} \left\{ \frac{1}{m-1} \big(\log p + \log |tm+s| \big) \right\} \\ &\geqslant \log r + \inf_{m \geqslant 2^k t + s} \left\{ \frac{t}{m-s-t} \big(\log p + \log |m| \big) \right\} \\ &\geqslant \log r + \inf_{m \geqslant 2t+s} \left\{ \frac{t}{m-s-t} \big(\log p + \log |m| \big) \right\}. \end{aligned}$$

Note that $\log p + \log |m| < 0$ if $|m| < |p|$, and that $m - s - t \geqslant t \geqslant 1$. Hence

$$\inf_{m \geqslant 2t+s} \left\{ \frac{t}{m-s-t} \big(\log p + \log |m| \big) \right\} < 0,$$

and thus

$$\begin{aligned} \inf_{m \geqslant 2t+s} \left\{ \frac{t}{m-s-t} \big(\log p + \log |m| \big) \right\} &= \inf_{m \geqslant 2t+s \text{ and } |m|<|p|} \left\{ \frac{t}{m-s-t} \big(\log p + \log |m| \big) \right\} \\ &= \inf_{k \geqslant 2 \text{ and } u \geqslant 1 \text{ such that } |u|=1} \left\{ \frac{t}{p^k u - s - t} \big(1-k \big) \log p \right\} \\ &\geqslant \inf_{k \geqslant 2} \left\{ \frac{t}{p^k - 2(p-1)} \big(1-k \big) \log p \right\} \\ &\geqslant \inf_{k \geqslant 2} \left\{ \frac{t}{p^k \Big(1 - \frac{2(p-1)}{p^2} \Big)} \big(1-k \big) \log p \right\} \\ &= \frac{t \log p}{\Big(1 - \frac{2(p-1)}{p^2} \Big)} \inf_{k \geqslant 2} \frac{1-k}{p^k}. \end{aligned}$$

On the other hand, the function $x \mapsto \frac{1-x}{p^x}$ is increasing on $x > 1 + \frac{1}{\log p}$. This immediately implies

$$\inf_{k \geqslant 2} \frac{1-k}{p^k} = -\frac{1}{p^2},$$

and hence

$$\begin{aligned} \frac{t \log p}{\Big(1 - \frac{2(p-1)}{p^2} \Big)} \inf_{k \geqslant 2} \frac{1-k}{p^k} &= -\frac{t}{p^2 \Big(1 - \frac{2(p-1)}{p^2} \Big)} \log p \\ &= -\frac{t}{p^2 - 2p + 2} \log p \\ &\geqslant -\frac{t}{(p-1)^2} \log p. \end{aligned}$$

This proves (1).

We proceed to show (2). Suppose that $2^k \geqslant \alpha + 2$. If moreover $m \geqslant 2^k t + s$, then

$$\frac{m}{2} - s - t \geqslant \frac{1}{2}(2^k t + s) - s - t \geqslant \frac{1}{2}(s + 2t + s) - s - t = 0,$$

and hence $m - s - t \geqslant \frac{m}{2}$. Notice in addition that we must have $k \geqslant 2$ since $\alpha > 0$ by assumption. Then, we have

$$\begin{aligned}\inf_{m\geqslant 2^k t+s}\left\{\frac{t}{m-s-t}\big(\log p + \log|m|\big)\right\} &\geqslant \inf_{m\geqslant 2^k t+s}\left\{\frac{t}{m-s-t}\big(\log p - \log m\big)\right\}\\ &\geqslant -\inf_{m\geqslant 2^k t+s}\left\{\frac{t}{m-s-t}\log m\right\}\\ &\geqslant -2t\inf_{m\geqslant 2^k t+s}\frac{\log m}{m}\\ &\geqslant -2t\inf_{m\geqslant 2^k}\frac{\log m}{m}\\ &= -\frac{kt}{2^{k-1}}\log 2.\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.8. In the setup of Lemma 3.7, note that A is the unique formal power series solution of the differential equation

$$xy' + \alpha y = a$$

with $y(0) = y'(0) = 0$.

We are now in position to prove Proposition 3.1.

Proof of Proposition 3.1. Notice that $m + \alpha \neq 0$ for any integer $m \geqslant 2$. Item (1) then follows easily.

The proof of Item (2) is subdivided into a number of steps.

Step 1. Let $m \geqslant 2$ be an integer. Observe that $v(m+\alpha) = v(tm+s) \leqslant \frac{\log(tm+s)}{\log p}$. Therefore, [SS81, Theorem 2] applies to show that the formal solution y is convergent.

Set

$$c(X_1, X_2) := \sum_{m\geqslant 2} c_m(X_1)X_2^m \in \mathbb{C}_p[[X_1, X_2]],$$

and consider the formal power series

$$B(x) := \int_0^x \frac{b(u)}{u}du \quad \text{and} \quad z(x) := y(x)\exp(-B(x)).$$

Note that $z(0) = z'(0) = 0$. Then y is solution of Equation (3.1) if and only if z is solution of

$$xz'(x) + \alpha z(x) = \exp(-B(x))\big(a(x) + c(x, z(x)\exp(B(x)))\big).$$

By Lemma 3.5 (1), $B(x)$ converges if $|x| \leqslant r_0 := r\exp\big(-\frac{2}{p(p-1)}\log p\big)$, and $|B(x)| \leqslant p^{-\frac{2}{p-1}} < p^{-\frac{1}{p-1}}$. In particular, $\exp(\pm B(x))$ is well-defined. Set

$$a_0(x) := a(x)\exp(-B(x))) \quad \text{and} \quad c_{0,m}(x) := c_m(x)\exp((m-1)B(x))$$

for all $m \geqslant 2$, so that

$$\exp(-B(x))c(x, z(x)\exp(B(x))) = \sum_{m\geqslant 2} c_{0,m}(x)z(x)^m.$$

Let $r_1 := r\exp\big(-\frac{3}{p(p-1)}\log p\big) < r_0$. Then [Rob00, Theorem of Section 6.1.5] applies to show that $\exp(\pm B)(x)$ converges and that $\exp(\pm B)(x) = \exp(\pm(B(x))$ if $|x| \leqslant r_1$. In addition, we have $||\exp(\pm B)||_{r_1} = \sup_{|x|\leqslant r_1}|\exp(\pm B)(x)| = \sup_{|x|\leqslant r_1}|\exp(\pm(B(x))|$. On the other hand, $\sup_{|x|\leqslant r_1}|\exp(\pm(B(x))| \leqslant 1$ since $|m!| \geqslant p^{-\frac{m}{p-1}}$ for every integer $m \geqslant 1$. It follows that

$$||\exp(\pm B)||_{r_1} = 1$$

since $B(0) = 0$. As a consequence, we have

$$||a_0||_{r_1} = ||a||_{r_1} || \exp(-B)||_{r_1} = ||a||_{r_1} \leqslant \frac{r_1}{p},$$

$$||c_{0,m}||_{r_1} = ||c_m||_{r_1} \leqslant ||c_m||_r \leqslant \frac{1}{p},$$

and

$$||y||_R = ||z||_R$$

for any real number $0 < R \leqslant r_1$.

Replacing r by r_1, if necessary, we may therefore assume without loss of generality that $b = 0$, so that Equation (3.1) reads

$$xy' + \alpha y = a + \sum_{m \geqslant 2} c_m y^m. \tag{3.2}$$

Step 2. The proof of [SS81, Theorem 2] then goes as follows. Let $z_0 \in \mathbb{C}_p[[X]]$ be the unique formal power series solution of

$$xz_0'(x) + \alpha z_0(x) = a(x)$$

with $z_0(x) = O(x^2)$ as x goes to 0. Set $b_0 = 0$. Observe that $a_1(x) := c(x, z_0(x)) = O\big(x^{2^2}\big)$ as x goes to 0. Set $b_1(x) := \partial_{x_2} c(x, z_0(x))$, and let $z_1 \in \mathbb{C}_p[[X]]$ be the unique formal power series solution of

$$xz_1'(x) + \alpha z_1(x) = a_1(x) + b_1(x) z_1(x)$$

with $z_1(x) = O\big(x^{2^2}\big)$ as x goes to 0. Next, one defines inductively $z_k \in \mathbb{C}_p[[X]]$ for all integer $k \geqslant 2$ as follows. Set

$$y_k := \sum_{i=0}^{k} z_i$$

for $k \geqslant 0$. If $k \geqslant 2$, set

$$a_k(x) := c(x, y_{k-1}(x)) - c(x, y_{k-2}(x)) - z_{k-1}(x) \partial_{x_2} c(x, y_{k-2}(x))$$

and

$$b_k(x) := \partial_{x_2} c(x, y_{k-1}(x)).$$

For $k \geqslant 2$, one then proves that there exists a unique formal power series z_k solution of

$$xz_k'(x) + \alpha z_k(x) = a_k(x) + b_k(x) z_k(x) \tag{3.3}$$

with $z_k(x) = O\big(x^{2^{k+1}}\big)$. Finally, one proves that z_k converges for all $k \geqslant 0$ as well as $y := \sum_{k \geqslant 0} z_k$ and that y is the unique solution formal power series solution of Equation (3.2) with $y(x) = O(x^2)$ as x goes to 0.

Step 3. Set $B_k(x) := \int_0^x \frac{b_k(u)}{u} du$. Notice that, for any integer $k \geqslant 1$, the formal power series z_k is solution of Equation (3.3) if and only if

$$w_k(x) := z_k(x) \exp(-B_k(x))$$

is solution of

$$xw_k'(x) + \alpha w_k(x) = \exp(-B_k(x)) a_k(x).$$

Let k_1 be the smallest positive integer such that $\frac{k_1+1}{2^{k_1}} \leqslant \frac{1}{p^2}$. Then $2^{k_1} \geqslant p^2 + 1 \geqslant p + 1 \geqslant \alpha + 2$. Notice that $k_1 \leqslant 5 \log p$. Indeed, let k be any integer such that $k \geqslant \frac{2}{\log 2} \log p + 1$. Then $\frac{k+1}{2^k} \leqslant \frac{1}{2^{k-1}} \leqslant \frac{1}{p^2}$. It follows that $k_1 \leqslant \frac{2}{\log 2} \log p + 2 \leqslant 3 \log p + 2 \leqslant 5 \log p$.

Then, we define inductively a decreasing sequence $(r_k)_{k \geqslant 0}$ of positive real numbers such that the following holds. For any integer $k \geqslant 0$, if $|x| \leqslant r_k$, then $\exp(\pm B_k)(x)$ converges, $\exp(\pm B_k)(x) = \exp(\pm(B_k(x))$, and $z_k(x)$ converges as well. In addition,

$$|| \exp(\pm B_k)||_{r_k} = 1,$$

and

$$||z_k||_{r_k} \leqslant r_k.$$

Finally, we will show that the limit r_∞ of the sequence $(r_k)_{k \geqslant 0}$ satisfies

$$\log r - \log r_\infty \leqslant Ct \frac{(\log p)^2}{p^2}$$

for some constant $C > 0$. In particular, $r_\infty > 0$.

Set $r_0 := r\exp\big(-\frac{t}{(p-1)^2}\log p\big)$. By Lemma 3.7 (1), $||z_0||_{r_0} \leqslant r_0$. Moreover, $B_0 = 0$ since $b_0 = 0$.

Let now k be a positive integer. Suppose $r_{k-1} < r_{k-2} < \cdots < r_0$ have already been defined. For any integer $0 \leqslant j \leqslant k-1$, we have

$$||y_j||_{r_{k-1}} \leqslant \max_{0\leqslant i\leqslant j} ||z_i||_{r_{k-1}} \leqslant r_{k-1},$$

and $y_j(x)$ converges if $|x| \leqslant r_{k-1}$. Moreover,

$$\begin{aligned} ||b_k||_{r_{k-1}} &= ||\partial_{x_2} c\big(x, y_{k-1}(x)\big)||_{r_{k-1}} \\ &= ||\sum_{m\geqslant 2} m c_m y_{k-1}^{m-1}||_{r_{k-1}} \\ &\leqslant \max_{m\geqslant 2}\Big\{|m|||c_m||_{r_{k-1}}||y_{k-1}||_{r_{k-1}}^{m-1}\Big\} \\ &\leqslant \max_{m\geqslant 2}\big\{||c_m||_r r_{k-1}^{m-1}\big\} \\ &\leqslant \frac{r_{k-1}}{p}. \end{aligned}$$

Similarly, we have

$$||a_k||_{r_{k-1}} = ||c(x, y_{k-1}(x)) - c(x, y_{k-2}(x)) - z_{k-1}(x)\partial_{x_2} c(x, y_{k-2}(x))||_{r_{k-1}} \leqslant \frac{r_{k-1}}{p}.$$

Suppose first $k < k_1$. Set

$$r_k := r_{k-1}\exp\Big(-\frac{2t}{(p-1)^2}\log p\Big) \leqslant r_{k-1}\exp\Big(-\frac{2}{p(p-1)}\log p\Big) < r_{k-1}.$$

By Lemma 3.5 (1) applied to b_k and Lemma 3.7 (1) applied to $\exp(-B_k(x))a_k(x)$ and arguing as in Step 1, we see that r_k satisfies all the conditions listed above.

Suppose now that $k \geqslant k_1$, and set

$$r_k := r_{k-1}\exp\Big(-\frac{(k+1)t}{2^{k-1}}\log 2\Big) \leqslant r_{k-1}\exp\Big(-\frac{k+1}{2^k}\log 2\Big) < r_{k-1}.$$

Applying Lemma 3.5 (2) to b_k and Lemma 3.7 (2) to $\exp(-B_k(x))a_k(x)$ and arguing again as in Step 1, we see that r_k also satisfies all the conditions listed above.

Finally, we have

$$\sum_{1\leqslant k\leqslant k_1-1}(\log r_{k-1} - \log r_k) = \sum_{1\leqslant k\leqslant k_1-1}\frac{2t}{(p-1)^2}\log p = \frac{2(k_1-1)t}{(p-1)^2}\log p \leqslant 10t\frac{(\log p)^2}{(p-1)^2},$$

and

$$\begin{aligned} \sum_{k\geqslant k_1}(\log r_{k-1} - \log r_k) &= \sum_{i\geqslant k_1}\frac{(k+1)t}{2^{k-1}}\log 2 \\ &\leqslant t\log 2\int_{k_1}^{+\infty}\frac{x+1}{2^{x-1}}dx \\ &= \frac{t}{2^{k_1-1}}\left(\frac{1}{\log 2} + k_1 + 1\right) \\ &\leqslant 4t\frac{k_1+1}{2^{k_1}} \\ &\leqslant \frac{4t}{p^2}. \end{aligned}$$

Therefore, we have

$$\log r - \log r_\infty = \sum_{k\geqslant 1}(\log r_{k-1} - \log r_k) \leqslant 10t\frac{(\log p)^2}{(p-1)^2} + \frac{4t}{p^2} \leqslant 14t\frac{(\log p)^2}{(p-1)^2}.$$

Set $C := 14$, and $R := r\exp\big(-tC\frac{(\log p)^2}{p^2}\big)$. Then $R \leqslant r_k$ for any $k \geqslant 0$, and $||y||_R \leqslant R$ since $||y_k||_R \leqslant R$ for each $k \geqslant 0$. This completes the proof of the proposition. □

4. Proof of Theorem 1.1

In this section we prove our main result. Note that Theorem 1.1 is an immediate consequence of Theorem 4.1 below.

Let X a smooth complex quasi-projective surface, and let L be a foliation on X. Let P be a singular point of L, and let D be a vector field on some open subset $U \ni P$ such that $L|_U = \mathscr{O}_U D$. Let α_1 and α_2 be the eigenvalues of the linear part of D at P. Recall that P is a *reduced singularity* of L if at least one of the α_i's is non-zero, say α_2, and $\frac{\alpha_1}{\alpha_2}$ is not a positive rational number. A reduced singularity P is called *non-degenerate* if both α_1 and α_2 are non-zero. Then $\left\{\alpha, \frac{1}{\alpha}\right\}$ does not depend on the choice of D, where $\alpha := \frac{\alpha_1}{\alpha_2}$. Finally, recall from [MM80, Appendice II], that if P is a non-degenerate reduced singularity, then there are exactly two analytic curves in X passing through P and invariant under L. They are smooth and intersect transversely at P.

Theorem 4.1. *Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X. Suppose that L is closed under p-th powers for almost all primes p. Let $P \in X(K)$ be a singular point of L. Suppose that $P_{\mathbb{C}}$ is a reduced singularity of $L_{\mathbb{C}}$ for some embedding $K \subset \mathbb{C}$. Then the following holds.*

1. *The foliation L has a non-degenerate singularity at P, and $-\alpha_{P_{\mathbb{C}}}(L_{\mathbb{C}}) \in \mathbb{Q}_{>0}$. In particular, $P_{\mathbb{C}}$ is a reduced singularity of $L_{\mathbb{C}}$ for any embedding $K \subset \mathbb{C}$.*
2. *There exist two $\widehat{L}$-invariant smooth formal subschemes $\widehat{V}$ and $\widehat{W}$ of the completion $\widehat{X}_P$ of X at P defined over K, where $\widehat{L}$ denotes the (formal) foliation induced by L on $\widehat{X}_P$. In particular, $\widehat{V}_{\mathbb{C}}$ and $\widehat{W}_{\mathbb{C}}$ are the formal completion at P of the two separatrices of $L_{\mathbb{C}}$ through $P_{\mathbb{C}}$ for any embedding $K \subset \mathbb{C}$.*
3. *The formal curves $\widehat{V}$ and $\widehat{W}$ are A-analytic.*

Proof. Replacing X by a Zariski open neighborhood of P in X, if necessary, we may assume without loss of generality that X is affine, and that $L = \mathscr{O}_X D$ for some vector field $D \in T_X \cong \mathrm{Der}_k(\mathscr{O}_X)$. Let $\overline{K}$ be an algebraic closure of K, and let $\alpha_1 \in \overline{K}$ and $\alpha_2 \in \overline{K}$ be the eigenvalues of the linear part of D at P. Suppose $\alpha_2 \neq 0$. By [McQ08, Proposition II.1.3], D is formally linearisable at P and $\frac{\alpha_1}{\alpha_2} \in \mathbb{Q}$. This immediately implies that $\alpha_1 \neq 0$ since D has isolated zeroes by assumption. In addition, we must have $-\alpha_{P_{\mathbb{C}}}(L_{\mathbb{C}}) = -\frac{\alpha_1}{\alpha_2} \in \mathbb{Q}_{>0}$ for any embedding $K \subset \mathbb{C}$, proving (1).

Let us write $\alpha = -\frac{s}{t}$, where s and t are relatively prime positive integers, and let $K \subset \mathbb{C}$ be an embedding. Let x_1 and x_2 be regular functions on X such that the induced map $X \to \mathbb{A}^2_K$ is étale at P, and maps P to 0. Shrinking X further, we may assume that $X \to \mathbb{A}^2_K$ is étale. Let $\widehat{D}$ denote the K-derivation of $\widehat{\mathscr{O}}_{X,P} \cong K[[X_1, X_2]]$ induced by D. We may assume without loss of generality that

$$\widehat{D} = (X_1 + f_1(X_1, X_2))\partial_{X_1} + (\lambda X_2 + f_2(X_1, X_2))\partial_{X_2},$$

where f_1 and f_2 are formal power series with coefficients in K vanishing to order at least 2 at P. Let φ_1 and φ_2 be formal power series with coefficients in K vanishing to order at least 2 at 0. Then the formal (smooth) curves

$$T \mapsto (\varphi_1(T), T) \quad \text{and} \quad T \mapsto (T, \varphi_2(T))$$

are invariant under $\widehat{D}$ if and only if

$$\widehat{D}(X_1 - \varphi_1(X_2)) \in (X_1 - \varphi_1(X_2)) \quad \text{and} \quad \widehat{D}(X_2 - \varphi_2(X_1)) \in (X_2 - \varphi_2(X_1))$$

if and only if

$$\varphi_1(T) + f_1(\varphi_1(T), T) - \varphi_1'(T)\big(\lambda T + f_2(\varphi_1(T), T)\big) = 0 \tag{4.1}$$

and

$$\lambda\varphi_2(T) + f_2(T, \varphi_2(T)) - \varphi_2'(T)\big(T + f_1(T, \varphi_2(T))\big) = 0. \tag{4.2}$$

One readily checks that there is a unique formal power series $\varphi_1 \in K[[T]]$ (resp. $\varphi_2 \in K[[T]]$) vanishing to order at least 2 at 0 solution of Equation (4.1) (resp. Equation (4.2)) since $-\alpha \in \mathbb{Q}_{>0}$. This proves (2).

We will denote by $\widehat{V}$ (resp. $\widehat{W}$) the formal curve $T \mapsto (\varphi_1(T), T)$ (resp. $T \mapsto (T, \varphi_2(T))$).

The proof of Item (3) is subdivided into a number of steps.

Step 1. Let v be a place of K, and let K_v be the completion of K at v. Notice first that $\widehat{V}_{K_v}$ and $\widehat{W}_{K_v}$ are K_v-analytic by [SS81, Theorem 2] if v is a finite place and [MM80, Appendice II] if v is archimedean.

Step 2. Suppose that the (algebraic) curve $\{x_1\} = 0$ is $\mathscr{L}$-invariant. Shrinking X further, if necessary, we may therefore assume that

$$D = x_1\partial_{x_1} + f(x_1, x_2)\partial_{x_2},$$

where f is a regular function on X.

Let R be the ring of integers of K. If N denotes a sufficiently divisible positive integer, there exists a model $\mathscr{X}$ of X, smooth and quasi-projective over $R[1/N]$, such that P extends to a point $\mathscr{P} \in \mathscr{X}(R[1/N])$. We may also assume that $X \to \mathbb{A}^2_K$ extends to an étale morphism $\mathscr{X} \to \mathbb{A}^2_{R[1/N]}$, and that D extends to a vector field $\mathscr{D} \in H^0(\mathscr{X}, T_{\mathscr{X}/\operatorname{Spec} R[1/N]})$ with isolated zeroes. Let $\mathfrak{p}$ be a maximal ideal of R, and let $|\cdot|_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic absolute value, normalized by the condition $|\varpi_{\mathfrak{p}}|_{\mathfrak{p}} = \frac{1}{\sharp(R/\mathfrak{p})}$ for any uniformizing element $\varpi_{\mathfrak{p}}$ at $\mathfrak{p}$. Let $K_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completions of K and R, and $k_{\mathfrak{p}} := R_{\mathfrak{p}}/(\varpi_{\mathfrak{p}})$ the residue field at $\mathfrak{p}$. Let finally p denote the characteristic of $k_{\mathfrak{p}}$.

Suppose that $\mathfrak{p} \nmid N$, and that $K_{\mathfrak{p}}$ is absolutely unramified. Let $\widehat{\mathscr{D}}_{\mathfrak{p}}$ denote the $K_{\mathfrak{p}}$-derivation of $\widehat{\mathscr{O}}_{\mathscr{X}_{K_{\mathfrak{p}}}, \mathscr{P}_{K_{\mathfrak{p}}}} \cong K_{\mathfrak{p}}[[X_1, X_2]]$ induced by $\mathscr{D}_{K_{\mathfrak{p}}} = D_{K_{\mathfrak{p}}}$. Then

$$\widehat{\mathscr{D}}_{\mathfrak{p}} = X_1\partial_{X_1} + f_{\mathfrak{p}}(X_1, X_2)\partial_{X_2}$$

where $f_{\mathfrak{p}} \in R_{\mathfrak{p}}[[X_1, X_2]]$. Let also $\overline{\mathscr{D}}_{\mathfrak{p}}$ be the $k_{\mathfrak{p}}$-derivation of $k_{\mathfrak{p}}[[X_1, X_2]]$ induced by $\widehat{\mathscr{D}}_{\mathfrak{p}}$. By assumption, $\overline{\mathscr{D}}_{\mathfrak{p}}$ is p-closed. This immediately implies

$$\overline{\mathscr{D}}_{\mathfrak{p}}^{\,p} = \overline{\mathscr{D}}_{\mathfrak{p}}$$

since $\overline{\mathscr{D}}_{\mathfrak{p}}^{\,p}(X_1) = \overline{\mathscr{D}}_{\mathfrak{p}}(X_1) = X_1$. By [AA86, Lemma 6.4] applied to $\overline{\mathscr{D}}_{\mathfrak{p}}$, there exists a formal power series $\overline{Y}_2$ in $k_{\mathfrak{p}}[[X_1, X_2]]$ such that $\overline{\mathscr{D}}_{\mathfrak{p}}(\overline{Y}_2) = \alpha\overline{Y}_2$. Notice that $\overline{Y}_2$ may a priori depend on our choice of $\mathfrak{p}$. Let $Y_2 \in R_{\mathfrak{p}}[[X_1, X_2]]$ be any formal power series whose reduction modulo $\varpi_{\mathfrak{p}}$ is $\overline{Y}_2$. By construction, we have $\widehat{\mathscr{D}}_{\mathfrak{p}}(Y_2) = \lambda Y_2$ modulo $(\varpi_{\mathfrak{p}})$. Set $Y_1 := X_1$. Then, we have

$$\widehat{\mathscr{D}}_{\mathfrak{p}} = Y_1\partial_{Y_1} + (\lambda Y_2 + g_{\mathfrak{p}}(Y_1, Y_2))\partial_{Y_2},$$

where $g_{\mathfrak{p}} \in \varpi_{\mathfrak{p}} R_{\mathfrak{p}}[[Y_1, Y_2]]$ vanishes to order at least 2 at 0. On the other hand, recall that $\widehat{V}_{K_{\mathfrak{p}}}$ is defined by $Y_1 = \varphi(Y_2)$ in $\operatorname{Spf}\widehat{\mathscr{O}}_{\mathscr{X}_{K_{\mathfrak{p}}}, \mathscr{P}_{K_{\mathfrak{p}}}} \cong \operatorname{Spf} K_{\mathfrak{p}}[[Y_1, Y_2]]$, where φ is the unique formal power series in $K_{\mathfrak{p}}[[T]]$ solution of the differential equation

$$T\varphi'(T) - \alpha\varphi(T) = g_{\mathfrak{p}}(X, \varphi(X))$$

such that $\varphi(0) = \varphi'(0) = 0$. Let us write $\varphi = \sum_{m \geqslant 2} a_m T^m$ and $g_{\mathfrak{p}} = \sum_{m \geqslant 0} b_m Y_2^m$, where $b_m \in \varpi_{\mathfrak{p}} R_{\mathfrak{p}}[[Y_1]]$. Then $||c_b||_{\mathfrak{p},1} \leqslant |\omega_{\mathfrak{p}}| = \frac{1}{p^{[K_{\mathfrak{p}}:\mathbb{Q}_p]}}$, so that Proposition 3.1 applies to show that the size $S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{V}_{K_{\mathfrak{p}}})$ satisfies

$$\log \frac{1}{S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{V}_{K_{\mathfrak{p}}})} \leqslant C(\alpha)[K_{\mathfrak{p}} : \mathbb{Q}_p]\frac{(\log p)^2}{p^2},$$

where $C(\alpha) > 0$ depends only on α.

Step 3. In the general setting, let Y be the blow-up of X at P, and let M be the foliation on Y induced by L. The exceptional divisor E is M-invariant and contains exactly two singularities Q_1 and Q_2 (defined over K) of M, both reduced and non-degenerate. In addition, relabeling Q_1 and Q_2 if necessary, we may assume that there is a smooth formal subscheme $\widehat{V}_1$ (resp. $\widehat{W}_1$) of $\widehat{Y}_{Q_1}$ defined over K (resp. $\widehat{Y}_{Q_2}$) which is $\widehat{M}$-invariant and such that the morphism $Y \to X$ induces an isomorphism $\widehat{V_1} \cong \widehat{V}$ (resp. $\widehat{W_1} \cong \widehat{W}$). The blow-up $\mathscr{Y}$ of $\mathscr{X}$ along $\mathscr{P}$ is a smooth and quasi-projective model of Y over $R[1/N]$. In addition, Q_1 and Q_2 extends to points $\mathscr{Q}_1 \in \mathscr{Y}(R[1/N])$ and $\mathscr{Q}_2 \in \mathscr{Y}(R[1/N])$, and M extends to a foliation $\mathscr{M} \subset T_{\mathscr{Y}/\operatorname{Spec} R[1/N]}$. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{Y} & \longrightarrow & \operatorname{Bl}_0\mathbb{A}^2_{R[1/N]} \\ \big\downarrow & & \big\downarrow \\ \mathscr{X} & \longrightarrow & \mathbb{A}^2_{R[1/N]}, \end{array}$$

where the horizontal arrows are étale morphims. Let $U_1 \cong \mathbb{A}^2_{R[1/N]} \subset \operatorname{Bl}_0\mathbb{A}^2_{R[1/N]}$ be the affine charts containing Q_1 with coordinates (y_1, y_2) (centered at Q_1). Then the natural morphism $U_1 \to \mathbb{A}^2_{R[1/N]}$ maps (y_1, y_2) to (y_1y_2, y_2). This immediately implies that the formal subscheme $\widehat{V}_1$ is defined by $Y_1 = \varphi(Y_2)/Y_2$ in $\widehat{Y}_{Q_1} =$

$\operatorname{Spf}\widehat{\mathscr{O}}_{Y,Q_1} \cong \operatorname{Spf} K[[Y_1,Y_2]]$. Finally, recall that the size (of a smooth formal scheme) is invariant by étale localization. Let again $\mathfrak{p}$ be a maximal ideal of R, and suppose that $\mathfrak{p} \nmid N$, and that $K_\mathfrak{p}$ is absolutely unramified. Then Lemma 2.3 and Step 2 imply that $S_{\mathscr{X}_{R_\mathfrak{p}}}(\widehat{V}_{K_\mathfrak{p}})$ satisfies

$$\log \frac{1}{S_{\mathscr{X}_{R_\mathfrak{p}}}(\widehat{V}_{K_\mathfrak{p}})} \leqslant \log \frac{1}{S_{\mathscr{Y}_{R_\mathfrak{p}}}(\widehat{W}_{K_\mathfrak{p}})} \leqslant C'(\alpha)[K_\mathfrak{p} : \mathbb{Q}_p]\frac{(\log p)^2}{p^2}$$

where $C'(\alpha) > 0$ depends only on α. This immediately implies that $\widehat{V}$ is A-analytic, completing the proof of the theorem. ☐

Let L be a foliation on a smooth complex quasi-projective surface X, and let P be a singular point of L. By [CS82, Theorem], there exists a (possibly singular) analytic curve passing through P. The following is an easy consequence of Theorem 4.1 above.

Corollary 4.2. *Let X be a smooth quasi-projective surface over a number field K, and let L be a foliation on X. Suppose that L is closed under p-th powers for almost all primes p. Let $P \in X(K)$ be a singular point of L, and let $\widehat{V}$ be an $\widehat{L}$-invariant smooth formal subcheme of the completion $\widehat{X}_P$ of X at P defined over K, where $\widehat{L}$ denotes the (formal) foliation induced by L on $\widehat{X}_P$. Then $\widehat{V}$ is A-analytic.*

Proof. Let $\overline{K}$ be an algebraic closure of K. By a result proved by Seidenberg ([Sei68]), there exists a composition of a finite number of blow-ups $Y_{\overline{K}} \to X_{\overline{K}}$ of $\overline{K}$-rational points such the foliation $M_{\overline{K}}$ induced by L on $Y_{\overline{K}}$ has reduced singularities. The variety $Y_{\overline{K}}$ is defined over a finite extension F of K. Let M be the foliation on Y indiced by L. Notice that M is closed under p-th powers for almost all primes p. Then Theorem 4.1 applied to the proper transform of $\widehat{V}_F$ in Y together with [BCL09, Proposition 3.4] and Lemma 2.3 imply that $\widehat{V}$ is A-analytic. ☐

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